FILLING ANALYTIC SETS BY THE DERIVATIVES OF C^1 -SMOOTH BUMPS

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ABSTRACT. If X is an infinite-dimensional Banach space, with separable dual, and $M \subset X^*$ is an analytic set such that any point $x^* \in M$ can be reached from 0 by a continuous path contained (except for the point x^*) in the interior of M , then M is the range of the derivative of a C^1 -smooth function on X with bounded nonempty support.

1. INTRODUCTION

A real-valued function f defined on a Banach space X is called a *bump* provided that its *support* $\text{supp } f := \{x \in X : f(x) \neq 0\}$ is bounded and nonempty. We say that f is C^1 -smooth if it is (Fréchet) differentiable at each point of X and its derivative f' is continuous on X . If X is finite-dimensional, then the range $f'(X)$ of f' is automatically closed. If X is separable (infinite-dimensional), then $f'(X)$ is, by the very definition, an analytic set. So a natural question arises, *whether, any analytic set in a separable dual space X^* , satisfying some reasonable additional conditions, can be expressed as the range of the derivative of a C^1 -smooth bump on X .* The aim of this note is to answer this question affirmatively.

This result has the following (pre)history. D. Azagra and R. Deville, encouraged by S. Bates's pioneering paper [6], filled, by the derivatives of a suitable C^1 -smooth bump, all of the dual X^* [1]. This was then further elaborated in [2]. There, higher-order derivatives of mappings between X and another Banach space Y , and also the Gâteaux derivatives were considered. In [7], the problem of exact filling of various closed sets in finite-dimensional spaces by the derivatives of C^1 -smooth bumps was studied. [8] is a continuation of this in infinite dimensions, with focus on open and closed sets. [3] is a kind of a common roof over [2], [7], and [8]. Finally, when the proof of our theorem was done, we learned that T. Gaspari [10] had succeeded to fill in convex analytic sets by derivatives of C^1 -smooth bumps.

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For more information about the ranges of the derivatives of bumps we refer to a survey paper of D. Azagra and M. Jiménez-Sevilla [4].

The main result of this note is the following theorem:

Theorem. *Let X be an infinite-dimensional Banach space with separable dual X^* . Let $M \subset X^*$ be an analytic set such that for every $\xi \in M$ there is a continuous mapping $\phi: [0, 1] \rightarrow X^*$, with $\phi(0) = 0$, $\phi(1) = \xi$, and $\phi([0, 1)) \subset \text{int } M$. Then there exists a C^1 -smooth bump $b: X \rightarrow \mathbb{R}$ such that $b'(X) = M$.*

It is natural to ask whether a converse to our Theorem holds. However, a negative answer follows from [5], where the authors construct a C^1 -smooth bump on ℓ_2 such that the range of its gradient has empty interior. When adding this bump with another suitable C^1 -smooth bump, with support disjoint from the support of the first bump, we get a C^1 -smooth bump $b: X \rightarrow \mathbb{R}$, with $\text{int } b'(X) \neq \emptyset$, and yet satisfying $b'(X) \setminus \overline{\text{int } b'(X)} \neq \emptyset$. In Example 2 below, we show that $b'(X)$ need not satisfy the assumptions of our Theorem even if $b'(X) \subset \overline{\text{int } b'(X)}$ and $\text{int } b'(X)$ is connected. Recently, the third author showed that $\text{int } b'(X)$ need not be connected. This result will be contained in another paper. Nonetheless, it seems to be unknown whether for every C^1 -smooth bump $b: X \rightarrow \mathbb{R}$ the set $M = b'(X)$ satisfies the assumptions of our Theorem provided that M is closed. That our Theorem does not hold (for closed sets) in finite dimensions was shown by T. Gaspari [10]; see Example 3.

2. PROOF OF THEOREM

Before proving our Theorem itself we fix some notation and give three lemmas.

Let $(X, \|\cdot\|)$ be a Banach space. The symbol B_X denotes its closed unit ball centered at the origin. Given $u, v \in X$, we put $[u, v] = \{\alpha u + (1 - \alpha)v : \alpha \in [0, 1]\}$. By a *path* in X we understand any set $C = \varphi([0, 1])$ where $\varphi: [0, 1] \rightarrow X$ is a continuous mapping. We then put $C(0) = \varphi(0)$ and $C(1) = \varphi(1)$.

The first two lemmas are basic bricks for building suitable C^1 smooth bumps.

Lemma 1 ([8, Lemma 2]). *Let X be a Banach space, with separable dual X^* , let $\varepsilon > 0$, and let $\xi \in X^*$ be given. Then there exists a C^1 -smooth bump $f: X \rightarrow [0, 1]$ such that $\text{supp } f \subset B_X$, $f'(X) \subset [0, \xi] + \varepsilon B_{X^*}$, and $(f')^{-1}(\xi)$ is a neighbourhood of 0.*

For another proof of the above Lemma, see [2] or [3, Lemma 2.1].

Lemma 2. *Let X be a Banach space, with separable dual, let $C \subset X^*$ be a path, with $C(0) = 0$, and let $\varepsilon > 0$ be given. Then there exists a C^1 -smooth bump $f: X \rightarrow [0, 1]$ such that $\text{supp } f \subset B_X$, $f'(X) \subset C + \varepsilon B_{X^*}$ and $(f')^{-1}(C(1))$ is a neighbourhood of 0.*

Proof. First, we shall show the following statement: *For every $n \in \mathbb{N}$, every $\Delta > 0$, and every $\xi_1, \dots, \xi_n \in X^*$ there exists a C^1 -smooth bump $f: X \rightarrow [0, 1]$ such that $\text{supp } f \subset B_X$, $f'(X) \subset ([0, \xi_1] \cup [\xi_1, \xi_2] \cup \dots \cup [\xi_{n-1}, \xi_n]) + \Delta B_{X^*}$ and $(f')^{-1}(\xi_n)$ is a neighbourhood of 0.* For $n = 1$ the statement is reduced to Lemma 1. Assume that we verified the statement for some $n \in \mathbb{N}$. Let $\Delta > 0$ and $\xi_1, \dots, \xi_n, \xi_{n+1} \in X^*$ be given. Find a C^1 -smooth bump $g: X \rightarrow [0, +\infty)$ such that $\text{supp } g \subset B_X$, $g'(X) \subset ([0, \xi_1] \cup [\xi_1, \xi_2] \cup \dots \cup [\xi_{n-1}, \xi_n]) + \Delta B_{X^*}$, and $(g')^{-1}(\xi_n)$ is a neighbourhood of 0. By Lemma 1, find a C^1 -smooth bump $h: X \rightarrow [0, +\infty)$ such that $\text{supp } h \subset (g')^{-1}(\xi_n) \cap$

B_X , $h'(X) \subset [0, \xi_{n+1} - \xi_n] + \Delta B_{X^*}$, and $(h')^{-1}(\xi_{n+1} - \xi_n)$ is a neighbourhood of 0. Then put $f = g + h$; this is a C^1 -smooth bump with $\text{supp } f \subset B_X$. Take any $x \in X$. If $x \notin \text{supp } h$, then $f'(x) = g'(x) \in ([0, \xi_1] \cup \dots \cup [\xi_{n-1}, \xi_n] \cup [\xi_n, \xi_{n+1}]) + \Delta B_{X^*}$. If $x \in \text{supp } h$, then $f'(x) = \xi_n + h'(x) \subset \xi_n + [0, \xi_{n+1} - \xi_n] + \Delta B_{X^*} = [\xi_n, \xi_{n+1}] + \Delta B_{X^*} \subset ([0, \xi_1] \cup \dots \cup [\xi_n, \xi_{n+1}]) + \Delta B_{X^*}$. Also, if $\|x\|$ is small enough, then $f'(x) = g'(x) + h'(x) = \xi_n + \xi_{n+1} - \xi_n = \xi_{n+1}$. Finally, replacing f by $\frac{1}{\lambda}f(\lambda \cdot)$, with a suitable $\lambda > 0$, we can force that $f(X) \subset [0, 1]$.

Having the above statement verified, we can complete the proof. Write $C = \varphi([0, 1])$, with $\varphi: [0, 1] \rightarrow X$ continuous. Then φ is uniformly continuous, and hence there is $n \in \mathbb{N}$ so large that $\|\varphi(t) - \varphi(t')\| < \frac{\varepsilon}{2}$ whenever $t, t' \in [0, 1]$ and $|t - t'| < \frac{1}{n}$. By the statement, find the bump f for $\Delta = \frac{\varepsilon}{2}$ and for $\xi_1 = \varphi(\frac{1}{n}), \dots, \xi_{n-1} = \varphi(\frac{n}{n-1}), \xi_n = \varphi(1)$. Then

$$f'(X) \subset ([\varphi(0), \varphi(\frac{1}{n})] \cup \dots \cup [\varphi(\frac{n-1}{n}), \varphi(1)]) + \frac{\varepsilon}{2} B_{X^*} \subset C + \varepsilon B_{X^*}. \quad \square$$

The lemma below is a key statement. It enables us to use the analytic structure of M . To formulate it we need some more notation.

The set of positive integers is denoted by $\mathbb{N}$. We denote by $\mathbb{N}^{<\mathbb{N}}$ the set of all finite sequences of elements of $\mathbb{N}$. We assume that $\mathbb{N} \subset \mathbb{N}^{<\mathbb{N}}$. The empty sequence is denoted by $\emptyset$; thus $\emptyset \in \mathbb{N}^{<\mathbb{N}}$. The symbol $|s|$ means the “length” of a sequence $s \in \mathbb{N}^{<\mathbb{N}}$. If $s \in \mathbb{N}^{<\mathbb{N}}$ and $n < |s|$, or $s \in \mathbb{N}^{\mathbb{N}}$, and $n = 0, 1, 2, \dots$, then $s|n$ denotes the finite sequence formed by the first n elements of the sequence s . For $s \in \mathbb{N}^{<\mathbb{N}}$ and $t \in \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}$ the relation $s \prec t$ means that s is a “beginning” of t .

Lemma 3. *Let X and M be as in our Theorem. Then there are points $y_s \in \text{int } M$ and paths $C_s \subset \text{int } M$, $s \in \mathbb{N}^{<\mathbb{N}}$, such that the following conditions are fulfilled:*

- (1) $y_\emptyset = 0$ and $C_\emptyset = \{0\}$;
- (2) $C_s(0) = y_{s|(|s|-1)}$, $C_s(1) = y_s$ for every $s \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$;
- (3) for every $\xi \in M$ there is $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that $\|y_{\sigma|n} - \xi\| \rightarrow 0$;
- (4) for every $\sigma \in \mathbb{N}^{\mathbb{N}}$ there is $\xi \in M$ such that $\|y_{\sigma|n} - \xi\| \rightarrow 0$ (this is a crucial property);
- (5) the diameter of C_s is less than $2^{-|s|+1}$ for every $s \in \mathbb{N}^{<\mathbb{N}} \setminus \mathbb{N}$.

Proof. From the analyticity of M , we find a (countable) system F_s , $s \in \mathbb{N}^{<\mathbb{N}}$, of nonempty closed subsets of $\overline{M}$ such that $M = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} F_{\sigma|n}$, and such that $F_s \subset F_t$ whenever $s, t \in \mathbb{N}^{<\mathbb{N}}$ and $s \succ t$; see, e.g., [9, Chapter III, §35.II].

By induction, we shall construct index sets $\mathcal{S}_k \subset \mathbb{N}^k \times \mathbb{N}^k$, $k = 0, 1, 2, \dots$, and open sets $G_{s,t} \subset X^*$, $(s, t) \in \mathcal{S} := \bigcup_{k=0}^{\infty} \mathcal{S}_k$ in such a way that they satisfy:

- (i) $\mathcal{S}_0 = \{(\emptyset, \emptyset)\}$, $G_{\emptyset, \emptyset} = X^*$;
- (ii) $\mathcal{S}_k = \{(s^\wedge m, t^\wedge n) : (s, t) \in \mathcal{S}_{k-1}, F_{s^\wedge m} \cap G_{s,t} \neq \emptyset, m, n \in \mathbb{N}\}$ for every $k \in \mathbb{N}$;
- (iii) the diameter of $G_{s,t}$ is less than $2^{-|s|}$ for every $(s, t) \in \mathcal{S} \setminus \{(\emptyset, \emptyset)\}$;
- (iv) $G_{s,t} \cap F_s \neq \emptyset$ for every $(s, t) \in \mathcal{S}$;
- (v) $\overline{G_{s^\wedge m, t^\wedge n}} \subset G_{s,t}$ whenever $(s, t) \in \mathcal{S}$, $m, n \in \mathbb{N}$, and $(s^\wedge m, t^\wedge n) \in \mathcal{S}$;
- (vi) if $(s, t) \in \mathcal{S}$, $m \in \mathbb{N}$, and $F_{s^\wedge m} \cap G_{s,t} \neq \emptyset$, then

$$F_{s^\wedge m} \cap G_{s,t} \subset \bigcup_{n=1}^{\infty} G_{s^\wedge m, t^\wedge n}.$$

The construction starts by fulfilling (i) and then continues by induction. Let $k \in \mathbb{N}$ be arbitrary fixed and assume that we have already constructed $\mathcal{S}_{k-1}$ and $G_{s,t}$ for every $(s, t) \in \mathcal{S}_{k-1}$. Define $\mathcal{S}_k$ as in (ii). Now fix any $(s, t) \in \mathcal{S}_{k-1}$ and any $m \in \mathbb{N}$ such that $F_{s^\wedge m} \cap G_{s,t} \neq \emptyset$. As X^* is a separable metric space (and hence hereditarily Lindelöf), we can find a sequence of open sets $G_{s^\wedge m, t^\wedge n} \subset X^*$, $n \in \mathbb{N}$, so that $\text{diam } G_{s^\wedge m, t^\wedge n} < 2^{-k}$, $F_{s^\wedge m} \cap G_{s^\wedge m, t^\wedge n} \neq \emptyset$, $\overline{G_{s^\wedge m, t^\wedge n}} \subset G_{s,t}$ for every $n \in \mathbb{N}$, and $F_{s^\wedge m} \cap G_{s,t} \subset \bigcup_{n=1}^{\infty} G_{s^\wedge m, t^\wedge n}$. This completes the induction step.

Let $0 \in D \subset \text{int } M$ be a countable set, dense in $\text{int } M$. We shall construct index sets $\mathcal{T}_k \subset \mathbb{N}^k \times \mathbb{N}^k \times \mathbb{N}^k$, $k = 0, 1, \dots$, and for every $(s, t, u) \in \mathcal{T} := \bigcup_{k=0}^{\infty} \mathcal{T}_k$ points $\tilde{y}_{s,t,u} \in D$ and paths $\tilde{C}_{s,t,u}$ in $\text{int } M$. Put $\mathcal{T}_0 = \{(\emptyset, \emptyset, \emptyset)\}$, $\tilde{y}_{\emptyset, \emptyset, \emptyset} = 0 \in X^*$, and $\tilde{C}_{\emptyset, \emptyset, \emptyset} = \{0\} \subset X^*$. Let $k \in \mathbb{N}$ be arbitrary fixed, and suppose we have defined $\mathcal{T}_{k-1}$ and $\tilde{y}_{s,t,u}$, $\tilde{C}_{s,t,u}$ for every $(s, t, u) \in \mathcal{T}_{k-1}$. Moreover, suppose that $(s, t) \in \mathcal{S}_{k-1}$ whenever $(s, t, u) \in \mathcal{T}_{k-1}$. For $(s, t, u) \in \mathcal{T}_{k-1}$ and for $m, n \in \mathbb{N}$ such that $(s^\wedge m, t^\wedge n) \in \mathcal{S}_k$, put

$$D_{s,t,u}^{m,n} = \{\xi \in D \cap G_{s^\wedge m, t^\wedge n} : \\ \tilde{y}_{s,t,u} = C(0), \xi = C(1) \text{ for a path } C \subset \text{int } M \cap G_{s,t}\}.$$

Clearly, this set is either empty or countable. In the second case, we enumerate it as $D_{s,t,u}^{m,n} = \{\tilde{y}_{s^\wedge m, t^\wedge n, u^\wedge l} : l \in \mathbb{N}\}$. Let $\tilde{C}_{s^\wedge m, t^\wedge n, u^\wedge l}$ denote a path C corresponding to the point $\tilde{y}_{s^\wedge m, t^\wedge n, u^\wedge l}$. Further, put

$$\mathcal{T}_k = \{(s^\wedge m, t^\wedge n, u^\wedge l) : (s, t, u) \in \mathcal{T}_{k-1}, (s^\wedge m, t^\wedge n) \in \mathcal{S}_k, D_{s,t,u}^{m,n} \neq \emptyset, m, n, l \in \mathbb{N}\},$$

$$[\mathcal{T}] = \{(\sigma, \tau, v) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} : \forall k \in \mathbb{N} (\sigma|k, \tau|k, v|k) \in \mathcal{T}_k\},$$

and

$$\mathcal{U} = \{(\sigma|k, \tau|k, v|k) : (\sigma, \tau, v) \in [\mathcal{T}], k \in \mathbb{N}\} \cup \{(\emptyset, \emptyset, \emptyset)\}.$$

We claim that *the set $[\mathcal{T}]$ is large enough, i.e., for every $\xi \in M$ there is $(\sigma, \tau, v) \in [\mathcal{T}]$ such that $\tilde{y}_{\sigma|k, \tau|k, v|k}$ converge to ξ as $k \rightarrow \infty$* . Indeed, let $\xi \in M$ be arbitrary. Then there is some $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that $\xi \in \bigcap_{k=1}^{\infty} F_{\sigma|k}$. Then $(\emptyset, \emptyset) \in \mathcal{S}_0$, $\xi \in F_{\sigma|1} \cap G_{\emptyset, \emptyset}$, and (vi) yields $t_1 \in \mathbb{N}$ such that $(\sigma|1, (t_1)) \in \mathcal{S}_1$ and $\xi \in G_{\sigma|1, (t_1)}$. Thus $\xi \in F_{\sigma|2} \cap G_{\sigma|1, (t_1)}$ and (vi) yields $t_2 \in \mathbb{N}$ such that $(\sigma|2, (t_1, t_2)) \in \mathcal{S}_2$. For a general $k \in \mathbb{N}$, if we already found $t_1, \dots, t_k \in \mathbb{N}$ such that $(\sigma|k, (t_1, \dots, t_k)) \in \mathcal{S}_k$ and $\xi \in F_{\sigma|(k+1)} \cap G_{\sigma|k, (t_1, \dots, t_k)}$, (vi) yields $t_{k+1} \in \mathbb{N}$ such that $(\sigma|(k+1), (t_1, \dots, t_k, t_{k+1})) \in \mathcal{S}_{k+1}$ and $\xi \in G_{\sigma|(k+1), (t_1, \dots, t_k, t_{k+1})}$. Performing this for every $k \in \mathbb{N}$, it is clear that there is $\tau \in \mathbb{N}^{\mathbb{N}}$ such that for every $k \in \mathbb{N}$ we have $(\sigma|k, \tau|k) \in \mathcal{S}_k$ and $\xi \in G_{\sigma|k, \tau|k}$. Let $\varphi: [0, 1] \rightarrow M$ be a continuous mapping such that $\varphi(0) = 0$, $\varphi(1) = \xi$, and $\varphi([0, 1]) \subset \text{int } M$. For every $k \in \mathbb{N}$ find $r_k \in (0, 1)$ so that $\varphi([r_k, 1]) \subset G_{\sigma|k, \tau|k}$. By a microscopic change of φ , we can reach that $\varphi(r_k) \in D$ for every $k \in \mathbb{N}$. Thus, by the definition of $\mathcal{T}$, there is $v \in \mathbb{N}^{\mathbb{N}}$ such that for every $k \in \mathbb{N}$ we have $(\sigma|k, \tau|k, v|k) \in \mathcal{T}_k$ and $\varphi(r_k) = \tilde{y}_{\sigma|k, \tau|k, v|k}$. This proves the Claim.

Further, fix any $(\sigma, \tau, v) \in [\mathcal{T}]$. For every $k \in \mathbb{N}$, $k \geq 2$, the set

$$\bigcup \left\{ \tilde{C}_{s,t,u} : (s, t, u) \in \mathcal{U}, \sigma|k \prec s, \tau|k \prec t, v|k \prec u \right\}$$

is contained in $G_{\sigma|(k-1), \tau|(k-1)}$, and hence has diameter less than 2^{-k+1} . In particular, using this and the completeness of X^* , we have that $\tilde{y}_{\sigma|k, \tau|k, v|k}$ converges to some $\xi \in X^*$ whenever $(\sigma, \tau, v) \in [\mathcal{T}]$. Further, $\tilde{y}_{\sigma|k, \tau|k, v|k} \in G_{\sigma|k, \tau|k}$; hence, by (iii) and (iv), the distance of $\tilde{y}_{\sigma|k, \tau|k, v|k}$ from $F_{\sigma|l}$ is at most 2^{-k} for every $k, l \in \mathbb{N}$, $k \geq l$.

It follows that $\xi \in F_{\sigma|l}$ for every $l \in \mathbb{N}$, and therefore $\xi \in \bigcap_{l=1}^{\infty} F_{\sigma|l} \subset M$. Thus we proved that *for every* $(\sigma, \tau, v) \in [T]$ *the sequence* $(\tilde{y}_{\sigma|k, \tau|k, v|k})_{k \in \mathbb{N}}$ *converges to an element of the set* M .

Now, we are going to relabel the points $\tilde{y}_{s,t,u}$ and the paths $\tilde{C}_{s,t,u}$, and thus we shall obtain the points $y_s \in D$ and paths $C_s \subset \text{int } M$, $s \in \mathbb{N}^{<\mathbb{N}}$. We define a bijection $\psi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{U}$ as follows. Put $\psi(\emptyset) = (\emptyset, \emptyset, \emptyset)$. Take any $r \in \mathbb{N}^{<\mathbb{N}}$ and assume that $\psi(r) \in \mathcal{U}$ was already defined. Write $\psi(r) = (s, t, u)$. From the above construction, it follows that the set $\{(s^\wedge m, t^\wedge n, u^\wedge l) : m, n, l \in \mathbb{N}\} \cap \mathcal{U}$ is infinite. We enumerate it as $\{\psi(r^\wedge i) : i \in \mathbb{N}\}$. Having ψ thus defined, for every $r \in \mathbb{N}^{<\mathbb{N}}$ we put $y_r = \tilde{y}_{\psi(r)}$ and $C_r = \tilde{C}_{\psi(r)}$ for $r \in \mathbb{N}^{<\mathbb{N}}$. From the above paragraphs we can easily deduce that these families have all the required properties (1) – (5). $\square$

Remark. If for every $(s, t) \in \mathcal{S}$ we can choose the set $G_{s,t}$ in such a way that $G_{s,t} \cap \text{int } M$ is connected, then the argument can be simplified. In particular, considering the “third” index u is not necessary, and joining the points $y_{\sigma|k}$, $k \in \mathbb{N}$, by paths causes no problems. This happens, for instance, if $\text{int } M$ is convex.

Proof of the Theorem. Let $(y_s : s \in \mathbb{N}^{<\mathbb{N}})$ and $(C_s : s \in \mathbb{N}^{<\mathbb{N}})$ be the families found in Lemma 3. For every $s \in \mathbb{N}^{<\mathbb{N}}$ and every $n \in \mathbb{N}$ choose $0 < \varepsilon_{s^\wedge n} < 2^{-|s|+1}$ such that $C_{s^\wedge n} + \varepsilon_{s^\wedge n} B_{X^*} \subset \text{int } M$. By Lemma 2, there is a C^1 -smooth bump $h_{s^\wedge n}: X \rightarrow [0, 1]$ satisfying the following conditions:

- (a) $h'_{s^\wedge n}(X) \subset C_{s^\wedge n} - y_s + \varepsilon_{s^\wedge n} B_{X^*}$;
- (b) $(h'_{s^\wedge n})^{-1}(y_{s^\wedge n} - y_s)$ is a neighborhood of 0;
- (c) $\text{supp } h_{s^\wedge n} \subset B_X$.

The desired bump b will be a sum of suitably translated and scaled functions $h_{s^\wedge n}$. We shall define the translation and the scaling by induction.

We start by putting $r_\emptyset = 1$, $z_\emptyset = 0 \in X$ and finding an infinite $\frac{1}{4}$ -discrete set $\{z_n : n \in \mathbb{N}\}$ in $\frac{1}{2}B_X$. Then put $g_n(x) = \frac{1}{16}h_n(16(x - z_n))$, $x \in X$, for every $n \in \mathbb{N}$.

Next, suppose that for some $s \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ we have already defined a C^1 -smooth bump $g_s: X \rightarrow [0, 1]$ such that $(g'_s)^{-1}(y_s - y_{s|(|s|-1)})$ has nonempty interior. Then the latter set contains an open ball $B(u_s, r_s)$ with centre $u_s \in X$ and radius $0 < r_s < 2^{-|s|}$. Choose an infinite $\frac{1}{4}r_s$ -discrete set $\{z_{s^\wedge n} : n \in \mathbb{N}\}$ in $B(u_s, \frac{1}{2}r_s)$, and put

$$g_{s^\wedge n}(x) = \frac{r_s}{16}h_{s^\wedge n}\left(\frac{16}{r_s}(x - z_{s^\wedge n})\right), \quad x \in X,$$

for every $n \in \mathbb{N}$. By the construction, $g_{s^\wedge n}$ is a C^1 -smooth bump, $|g_{s^\wedge n}(x)| \leq \frac{1}{16 \cdot 2^{|s|}}$ for every $x \in X$, and $g'_{s^\wedge n}(X) = h'_{s^\wedge n}(X)$. Also, by (5) and (a), $\|g'_{s^\wedge n}(x)\| < 2^{-|s|+2}$ whenever $|s| \geq 1$ and $x \in X$. Moreover,

$$\text{supp } g_{s^\wedge n} \subset B\left(z_{s^\wedge n}, \frac{r_s}{16}\right) \subset B\left(u_s, \frac{9}{16}r_s\right) \subset (g'_s)^{-1}(y_s - y_{s|(|s|-1)}).$$

Hence, for every $s \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$, the family of sets $\{\text{supp } g_{s^\wedge n} : n \in \mathbb{N}\}$ is discrete, i.e., every $x \in X$ has a neighbourhood that intersects $\text{supp } g_{s^\wedge n}$ for at most one $n \in \mathbb{N}$.

Having this information and the fact that $\text{supp } g_{s^\wedge n} \subset \text{supp } g_s$ at hand, it follows easily by induction that $\{\text{supp } g_s : s \in \mathbb{N}^{<\mathbb{N}}, |s| = n\}$ is a discrete family for every

$n \in \mathbb{N}$. Therefore, if for $n \in \mathbb{N}$ we define

$$f_n(x) = \sum_{|s|=n} g_s(x), \quad x \in X,$$

we obtain a C^1 -smooth bump satisfying $f'_n(x) = \sum_{|s|=n} g'_s(x)$ for every $x \in X$ and $f'_n(x) = g'_s(x) = y_s - y_{s|(|s|-1)}$ whenever $s \in \mathbb{N}^n$ and $x \in B(u_s, r_s)$.

Finally, put

$$b(x) = \sum_{n=1}^{\infty} f_n(x), \quad x \in X.$$

As $|f_n(x)| \leq 2^{-n-4}$ and $\|f'_n(x)\| < 2^{-n+2}$ for every $x \in X$ and every $n \geq 2$, the function b is a well-defined C^1 -smooth bump and $b'(x) = \sum_{n=1}^{\infty} f'_n(x)$ for every $x \in X$.

We observe that $b'(X) \subset M$. Indeed, by the construction, given any $x \in X$, then: either there is $K \in \mathbb{N}$ so that $b'(x) = \sum_{n=1}^K f'_n(x)$ and there is $s \in \mathbb{N}^{<\mathbb{N}}$ such that $|s| = K$, $x \in \text{supp } g_s$, and $b'(x) = \sum_{n=1}^{K-1} (y_{s|n} - y_{s|(n-1)}) + g'_s(x) \in \text{int } M$, by the choice of ε_s ; or there is $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that $x \in \bigcap_{n \in \mathbb{N}} \text{supp } g_{\sigma|n} \subset \bigcap_{n \in \mathbb{N}} B(u_{\sigma|n}, r_{\sigma|n})$ and hence $b'(x) = \sum_{n=1}^{\infty} (y_{\sigma|n} - y_{\sigma|(n-1)}) = \lim_{n \rightarrow \infty} y_{\sigma|n} \in M$, by (4).

In order to show that $M \subset \overline{b'(X)}$, let $\xi \in M$ be arbitrary. Find $\sigma \in \mathbb{N}^{\mathbb{N}}$ by (3). Then, since $B(u_{\sigma|n}, r_{\sigma|n}) \supset \overline{B(u_{\sigma|(n+1)}, r_{\sigma|(n+1)})}$ and $r_{\sigma|n} < 2^{-n}$ for every $n \in \mathbb{N}$, these balls have a one-point intersection, $x \in X$, say. Thus

$$b'(x) = \sum_{n=1}^{\infty} (y_{\sigma|n} - y_{\sigma|(n-1)}) = \lim_{n \rightarrow \infty} y_{\sigma|n} = \xi.$$

This completes the proof of our Theorem. $\square$

3. EXAMPLES AND REMARKS

In the first example we shall show that our Theorem is false if the property of the set M is weakened a little. In fact, the construction works in any at least two-dimensional Banach space.

Example 1 (Comb). *If X is any Banach space, of dimension at least two, then there exists a closed pathwise connected subset $M \subset X^*$ such that $0 \in \text{int } M$ and $\text{int } M$ is connected and dense in M , but M is the range of the derivative of no C^1 -smooth bump on X .*

Proof. Put

$$N = [-1, 0] \times [-1, 1] \cup [0, 1] \times \left(\left[-1, \frac{1}{2}\right] \cup \left[\frac{2}{3}, \frac{3}{4}\right] \cup \left[\frac{4}{5}, \frac{5}{6}\right] \cup \cdots \cup \{1\} \right).$$

This set is equal to the closure of its interior and $(0, 0)$ belongs to its interior. Moreover, it is easy to check that N is pathwise connected and $\text{int } N$ is connected. Let X be any Banach space, with dimension at least 2. Then we can write $X = \mathbb{R}^2 \times Y$, with a suitable (possibly trivial) subspace $Y \subset X$. Then put

$$M = N \times B_{Y^*};$$

this set is equal to the closure of its interior and the origin belongs to its interior. Clearly, $\text{int } M$ is connected and every point of M can be reached by a continuous path in M beginning at the origin (i.e., M is pathwise connected). However, the paths corresponding to the points of the form $((t, 1), \eta)$, $t \in (0, 1]$, $\eta \in B_{Y^*}$, do not always go through $\text{int } M$, as is required in our Theorem. We shall show that

$M = f'(X)$ for no C^1 -smooth bump $f: X \rightarrow \mathbb{R}$. Assume that there exists such an f . Consider $\xi = ((1, 1), 0) \in M$ and take the direction $h = ((0, 1), 0) \in X$. By the proof of [7, Theorem 2], we find $x \in X$ and a sequence $(z_n)_{n \in \mathbb{N}}$ in X so that $f'(x) = \xi$, $z_n \rightarrow x$, and $f'(z_n)(h) \neq \xi(h) = 1$ for every $n \in \mathbb{N}$. Consider the neighbourhood $U = (\frac{2}{3}, \frac{4}{3}) \times (\frac{2}{3}, \frac{4}{3}) \times B_{Y^*}$ of ξ . From the continuity of f' , we find a convex neighbourhood V of x so that $f'(V) \subset U$. From the above, we find $z \in V$ such that $f'(z)(h) \neq 1$. Then, as $f'(z) \in U \cap M$, there is $m \in \mathbb{N}$ such that $f'(z) \in [0, 1] \times [\frac{m}{m+1}, \frac{m+1}{m+2}] \times B_{Y^*}$. Define

$$\varphi(t) = f'(tx + (1-t)z), \quad t \in [0, 1];$$

this is a continuous function, and $\varphi(t) \in f'(V) \subset U \cap M$ for every $t \in [0, 1]$. Put

$$t_0 = \sup \{t \in [0, 1] : \varphi(t) \in [0, 1] \times [\frac{m}{m+1}, \frac{m+1}{m+2}] \times B_{Y^*}\}.$$

Clearly, $t_0 < 1$. Now, for every $t \in (t_0, 1]$ we have $\varphi(t) \notin [0, 1] \times [\frac{m}{m+1}, \frac{m+1}{m+2}] \times B_{Y^*}$. Hence, $\varphi(t)$, with such t , belongs either to $(\frac{2}{3}, 1] \times \{1\} \times B_{Y^*}$ or to $(\frac{2}{3}, 1] \times [\frac{k}{k+1}, \frac{k+1}{k+2}] \times B_{Y^*}$ for some $k \neq m$. Thus $\|\varphi(t_0) - \varphi(t)\| \geq \frac{m+2}{m+3} - \frac{m+1}{m+2} > 0$ for every $t \in (t_0, 1]$, which contradicts the continuity of φ at $t = t_0$. $\square$

The next example shows that even a weak converse to our Theorem is false in ℓ_2 .

Example 2. *There is a C^1 -smooth bump $b: \ell_2 \rightarrow \mathbb{R}$ such that the set $M = b'(\ell_2)$ does not satisfy the assumptions of our Theorem and yet $0 \in \text{int } M$, $M \subset \overline{\text{int } M}$ and $\text{int } M$ is connected.*

Proof. In [5], a C^1 -smooth bump $b_1: \ell_2 \rightarrow \mathbb{R}$ is constructed such that

$$b'_1(\ell_2) \subset Z := \{0\} \cup \{(z_n) \in \ell_2 : z_n \neq 0 \text{ for infinitely many } n \in \mathbb{N}\}$$

(see the last two paragraphs of the proof of Theorem 2.2 and Corollary 2.3 in [5]).

Fix any nonzero $\xi \in b'_1(\ell_2)$, put $r = \|\xi\|/2$, and choose $n_0 \in \mathbb{N}$, with $n_0 > \frac{1}{r}$. Put

$$F_0 = (\{0\} \cup \{\pm \frac{1}{n} : n > n_0\}) \times [-r, r],$$

$$N_0 = \mathbb{R}^2 \setminus F_0,$$

$$N' = N_0 \times \ell_2 \subset \ell_2.$$

Then N' is an open connected subset of ℓ_2 , dense in ℓ_2 such that $0 \notin N'$ and there is no continuous $\varphi: [0, 1] \rightarrow \ell_2$ with $\varphi(1) = 0$ and $\varphi([0, 1)) \subset N'$. Put $N = N' + \xi$. Then N is an open connected set in ℓ_2 containing 0, and hence by our Theorem (in fact already by [8] or [10]) there is a C^1 -smooth bump $b_2: \ell_2 \rightarrow \mathbb{R}$ with $b'_2(\ell_2) = N$. Up to a translation, we may assume that $\text{supp } b_1 \cap \text{supp } b_2 = \emptyset$. Then put $b = b_1 + b_2$. Then b is a C^1 -smooth bump such that $b'(\ell_2) = N \cup b'_1(\ell_2)$. We claim that $\text{int } b'(\ell_2) = N$. The inclusion $\supset$ is clear. Let us show the reverse one. Suppose that there is some $x = (x_n) \in \text{int } b'(\ell_2) \setminus N$. Then necessarily x is a relative interior point in H of $b'_1(\ell_2) \cap H$, where $H = \{(z_n) \in \ell_2 : z_1 = x_1, |z_2| \leq r\}$. However, $Z \cap H$ has empty interior in H , a contradiction.

So, $\text{int } b'(\ell_2) = N$ is connected and dense in ℓ_2 and, moreover, there is no path $\varphi: [0, 1] \rightarrow \ell_2$ with $\varphi(1) = \xi$ and $\varphi([0, 1)) \subset \text{int } b'(\ell_2)$. This completes the proof. $\square$

The last example is from [10] and shows that our Theorem does not hold in finite dimensions, even for closed sets. (Note that the range of the gradient of any

C^1 -smooth bump on a finite-dimensional space is closed, due to the compactness of closed balls.)

Example 3. If $n \in \mathbb{N}$, $n \geq 2$, then there exists a closed subset $M \subset \mathbb{R}^n$ that is the range of the gradient of no C^1 -smooth bump on $\mathbb{R}^n$, and yet for every $\xi \in M$ there is a continuous path from 0 to ξ through the points of $\text{int } M$. Moreover, these paths can have uniformly bounded lengths.

Proof. Put

$$N_1 = N \cup [-1, 1] \times [1, 2], \quad \text{and} \quad M = N_1 \times [-1, 1]^{n-2},$$

where N is from Example 1. (If $n = 2$, put $M = N_1$.) This set has all the required properties. Yet there is no C^1 -smooth bump $f: \mathbb{R}^n \rightarrow \mathbb{R}$ with $f'(X) = M$. Indeed, if there were such an f , then M would be a continuous image of a compact locally connected space (a sufficiently large closed ball in $\mathbb{R}^n$), and hence M itself would be locally connected. However, it is obvious that M is not locally connected. $\square$

Next we shall formulate an open problem which seems to be essential for understanding what kind of property a set needs to have “to be the range of the derivative of a C^1 -smooth bump”.

Problem. Let X be a Banach space with separable dual and $M \subset X^*$ the range of the derivative of a C^1 -smooth bump on X . Suppose that $h: X^* \rightarrow X^*$ is a homeomorphism (or even a linear isomorphism) of X^* onto X^* and $0 \in \text{int } \overline{h(M)}$. Is then $h(M)$ also the range of the derivative of a C^1 -smooth bump?

Let us remark that the answer is positive if $h = L^*$ when L is any linear isomorphism of X onto X . It can also be easily checked that $M - \xi$ is the range of the derivative of a C^1 -smooth bump whenever M has this property, $0, \xi \in \text{int } M$ and ξ can be reached from 0 by a path contained in $\text{int } M$.

Remark. We are convinced that our method (Lemma 3) can be used for several “analytic” statements which may hold. We refer the reader to [3] for more fantasy on how to formulate and prove such statements.

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