THE CAYLEY TRANSFORM OF LINEAR RELATIONS

MERCEDES FERNANDEZ MIRANDA AND JEAN-PHILIPPE LABROUSSE

(Communicated by Joseph A. Ball)

ABSTRACT. This paper extends the definition of the Cayley transform defined for symmetric linear operators to any linear relation.

INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space with scalar product $(\cdot, \cdot)$ and norm $\|\cdot\|$. $\mathcal{H} \oplus \mathcal{H}$ will denote the direct sum of $\mathcal{H}$ with itself, with its natural Hilbert space structure. The notion of linear relation on a normed space, which was first introduced by R. Arens in [1] (1961), generalizes the notion of linear operator (it is equivalent to the notion of multivalued operator and a general introduction can be found in [3]). The set of such linear relations on $\mathcal{H}$ will be denoted here by $Lr(\mathcal{H})$ while $LR(\mathcal{H})$ will denote the set of closed elements of $Lr(\mathcal{H})$ (i.e. those that have a closed graph in $\mathcal{H} \oplus \mathcal{H}$). The concept of inverse of an element and the operations of multiplication by a scalar, addition and composition can easily be generalized to linear relations.

The concepts of symmetric and selfadjoint operators readily extend to $LR(\mathcal{H})$, and the von Neuman theorem about selfadjoint extensions of symmetric operators can also be extended to $LR(\mathcal{H})$. This was done in [2] by making use of the Cayley transform of linear relations. In that paper, however, the definition of the Cayley transform was extended only to $LR(\mathcal{H})$, via the use of the orthogonal projection on the graph of a closed linear relation.

In this paper we want to show that the Cayley transform of a linear relation can be defined directly (as for operators) by an algebraic formula and can therefore be extended to $Lr(\mathcal{H})$.

The Cayley transform $C(E)$ of an operator E such that $E + iI$ has a trivial kernel is usually defined as follows:

$$C(E) = (E - iI)(E + iI)^{-1} = I - 2i(E + iI)^{-1}.$$

The two expressions in that definition coincide when E is an operator, but this is not in general the case when E is a linear relation as is shown below by Corollary 3.3, and it turns out that it is the second expression that yields the correct definition. In fact, here as in [2], we shall deal with a small variation of the Cayley transform - the Z transform - which yields the additional “desirable” property of being an involution:

$$Z(E) = -iC(E) = -iI - 2(E + iI)^{-1}.$$

Received by the editors March 5, 2003 and, in revised form, October 15, 2003.

2000 *Mathematics Subject Classification*. Primary 47A05, 47A06.

Key words and phrases. Linear relation, operator, Cayley transform.

In § 1 we recall basic definitions and properties of linear relations which are needed for the sequel.

The main result of § 2 is the extension (due to H. de Snoo) of Stone's formula for the orthogonal projection on the graph of a closed linear relation (a more complete presentation can be found in [4]).

Finally in § 3 the Cayley transform of linear relations is defined and is shown to be equivalent to the one given in [2] for closed linear relations. Some of the results of [2] are then generalized to not necessarily closed linear relations.

1. LINEAR RELATIONS

Let J and K denote the symmetric unitary operators on $\mathcal{H} \oplus \mathcal{H}$ described by the matrices:

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad K = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}.$$

We shall also use $U = (J + K)/\sqrt{2}$, $P_1 = (I + J)/2$, and $P_2 = (I - J)/2$. Finally, let $\mathcal{H}_1 = \mathcal{H} \oplus \{0\} = P_1(\mathcal{H} \oplus \mathcal{H})$ and $\mathcal{H}_2 = \{0\} \oplus \mathcal{H} = P_2(\mathcal{H} \oplus \mathcal{H})$.

Proposition 1.1. $JK + KJ = 0$.

Proof. Obvious. □

Definition 1.2. Let $\mathcal{P}(\mathcal{H})$ denote the set of all subsets of $\mathcal{H}$. Then $\mathcal{P}(\mathcal{H})$ can be given a linear space structure on $\mathbf{C}$ as follows: If $M, N \in \mathcal{P}(\mathcal{H})$, set $M + N = \{z \mid z = m + n, m \in M, n \in N\}$ and $\forall \lambda \in \mathbf{C}$, set $\lambda M = \{\lambda z \mid z \in M\}$.

Definition 1.3. A linear relation E in $\mathcal{H}$ is a mapping of a linear subspace $\tilde{D}(E)$ of $\mathcal{H}$ into $\mathcal{P}(\mathcal{H})$ such that

$$\begin{aligned} \forall x, y \in \tilde{D}(E), E(x + y) &= E(x) + E(y), \\ \forall \lambda \in \mathbf{C}, \forall x \in \tilde{D}(E), E(\lambda x) &= \lambda E(x). \end{aligned}$$

Definition 1.4. Let E be a linear relation in $\mathcal{H}$. We shall define the domain of E by $D(E) = P_1(G(E)) = \tilde{D}(E) \oplus \{0\}$, the range of E by $R(E) = P_2(G(E)) = \{0\} \oplus \tilde{R}(E)$, and the kernel of E by $N(E) = G(E) \cap \mathcal{H}_1 = \tilde{N}(E) \oplus \{0\}$.

Then the graph of E is the linear subspace of $\mathcal{H} \oplus \mathcal{H}$ given by $G(E) = \{(x, y) \mid x \in \tilde{D}(E), y \in E(x)\}$.

Setting $G(E) \cap \mathcal{H}_2 = \{0\} \oplus E(0)$ we shall call $E(0)$ the multivalued part of E . Note that if $E(0) = \{0\}$, then E is an operator. The set of all bounded and everywhere defined linear operators on $\mathcal{H}$ will be denoted by $L(\mathcal{H})$.

When $E \in LR(\mathcal{H})$ we shall write P_E for the orthogonal projection in $\mathcal{H} \oplus \mathcal{H}$ onto $G(E)$.

Example. Let X, Y be two closed subspaces of $\mathcal{H}$ such that $X + Y = \mathcal{H}$. Then every $z \in \mathcal{H}$ can be written as $z = x + y$, with $x \in X$ and $y \in Y$.

Denote by P the linear mapping of $\mathcal{H}$ into $\mathcal{P}(\mathcal{H})$ given by $P : z \mapsto \{x\} + X \cap Y$. Then $P \in LR(\mathcal{H})$.

It is easy to see that $\tilde{N}(P) = Y$ and $P(0) = X \cap Y$.

Definition 1.5. Let $E, F \in LR(\mathcal{H})$. Then

$$g(E, F) = \|P_E - P_F\|.$$

$LR(\mathcal{H})$ equipped with the metric g (the so-called gap metric) is complete.

Definition 1.6. Let $E \in Lr(\mathcal{H})$ and $\lambda \in \mathbf{C}$. Then λE is the linear relation such that its graph is $G(\lambda E) = \{\{x, \lambda y\} \mid \{x, y\} \in G(E)\}$.

Definition 1.7. Let $E, F \in Lr(\mathcal{H})$. Then $E + F$, the sum of E and F , is the linear relation with graph $G = \{\{x, u\} \mid \exists y, \exists z \text{ such that } \{x, y\} \in G(E); \{x, z\} \in G(F) \text{ and } u = y + z\}$. Clearly $D(E + F) = D(E) \cap D(F)$ and $(E + F)(0) = E(0) + F(0)$.

Remark 1.8. The sum is associative and commutative. If 0 is the linear relation whose graph is $\mathcal{H}_1$, then clearly $\forall E \in Lr(\mathcal{H})$, $0 + E = E + 0 = E$ so that 0 is a neutral element for the sum. Finally note that unless $E \in L(\mathcal{H})$, $E + (-E) \neq 0$ although $E + (-E) + E = E$ and $(-E) + E + (-E) = -E$ so that $-E$ is a generalized inverse of E for the sum (for a survey of the theory of generalized inverses see [5]).

Definition 1.9. Let $E \in Lr(\mathcal{H})$. Then the inverse E^{-1} is the linear relation with graph $G(E^{-1}) = KJ(G(E)) = JK(G(E))$ (since $JK = -KJ$ by Proposition 1.1). Clearly $(E^{-1})^{-1} = E$.

Definition 1.10. Let $E, F \in Lr(\mathcal{H})$. Then FE , the product of E on the left by F , is the linear relation with graph $G = \{\{x, y\} \mid \exists z \in \mathcal{H} \text{ such that } \{x, z\} \in G(E), \{z, y\} \in G(F)\}$.

Remark 1.11. The product is associative and noncommutative. If I is the identity mapping on $\mathcal{H}$, then clearly $\forall E \in Lr(\mathcal{H})$, $IE = EI = E$ so that I is a neutral element for the product. Finally note that in general $EE^{-1} \neq I$ although $EE^{-1}E = E$ and $E^{-1}EE^{-1} = E^{-1}$ so that E^{-1} is a generalized inverse of E for the product.

Definition 1.12. Let $E \in Lr(\mathcal{H})$. Then the adjoint E^* is the closed linear relation with graph $G(E^*) = K(G(E)^\perp) = (KG(E))^\perp$.

Proposition 1.13. Let $E \in Lr(\mathcal{H})$. Then $\tilde{D}(E^*) = E(0)^\perp$.

Proof. Let $u \in \tilde{D}(E^*)$. Then $\exists v \in \mathcal{H}$ such that $\{u, v\} \in G(E^*)$. Moreover if $w \in E(0)$, then $\{0, w\} \in G(E) \Leftrightarrow \{w, 0\} \perp G(E^*)$ so that: $(u, w) = (\{u, v\}, \{w, 0\}) = 0$ so that $\tilde{D}(E^*) \subseteq E(0)^\perp$. Conversely if $u \perp D(E^*)$, then $\{u, 0\} \perp G(E^*)$, which implies that $\{0, u\} \in G(E) \Rightarrow u \in E(0)$ and the proposition follows. $\square$

Proposition 1.14. Let $E \in LR(\mathcal{H})$. Then $(E^*)^{-1} = (E^{-1})^*$.

Proof. $G((E^*)^{-1}) = JK(G(E^*)) = JKK(G(E)^\perp) = K([KJ(G(E))]^\perp) = G((E^{-1})^*)$. $\square$

Proposition 1.15. Let $E \in LR(\mathcal{H})$ and $F \in L(\mathcal{H})$. Then

$$E^* + F^* = (E + F)^*.$$

Proof. Let $x \in \tilde{D}(E^*)$, and let $\{x, m\} \in G(E^*)$. Then $\forall u \in \tilde{D}(E), \forall s$ with $\{u, s\} \in G(E)$, $(\{u, s\}, \{m, -x\}) = 0 \Leftrightarrow (u, m) = (s, x)$. Also, $\forall u, \forall x \in \mathcal{H}$, $(\{u, Fu\}, \{F^*x, -x\}) = 0 \Leftrightarrow (u, F^*x) = (Fu, x)$. Therefore $\forall u \in \tilde{D}(E) \forall s$, such that $\{u, s\} \in G(E) \Leftrightarrow \{u, s + Fu\} \in G(E + F)$, $(\{u, s + Fu\}, \{m + F^*x, -x\}) = 0 \Leftrightarrow (u, m + F^*x) = (s + Fu, x)$ so that $\{x, m + F^*x\} \in G((E + F)^*)$ and consequently $G(E^* + F^*) \subseteq G((E + F)^*)$. Hence, since $E + F - F = E$, it follows that $G((E + F)^* - F^*) \subseteq G((E + F - F)^*) = G(E^*)$ so that $G((E + F)^*) = G((E + F)^* - F^* + F^*) \subseteq G(E^* + F^*)$ Hence $(E + F)^* = E^* + F^*$. $\square$

2. THE ORTHOGONAL PROJECTION ON THE GRAPH OF A CLOSED LINEAR RELATION

Proposition 2.1 (Stone-de Snoo formula). *Let $E \in LR(\mathcal{H})$, and let*

$$(2.1) \quad \begin{pmatrix} \Delta_E & \nabla_{E^*} \\ \nabla_E & I - \Delta_{E^*} \end{pmatrix}$$

be the matrix representation of the orthogonal projection P_E in $\mathcal{H} \oplus \mathcal{H}$ onto the graph of E . Then

- (i) $\Delta_E = (I + E^*E)^{-1} \in L(\mathcal{H})$.
- (ii) $\nabla_E = (I - P_{E(0)})E\Delta_E \in L(\mathcal{H})$.
- (iii) $\Delta_{E^*} = (I + EE^*)^{-1} \in L(\mathcal{H})$.
- (iv) $\nabla_{E^*} = (I - P_{E^*(0)})E^*\Delta_{E^*} \in L(\mathcal{H})$.

Proof. (i) Let $u \in \mathcal{H}$. Then by definition $\{u, 0\} = \{\Delta_E u, \nabla_E u\} + \{y, -x\}$ where $\{y, -x\} \in G(E)^\perp \Leftrightarrow \{x, y\} \in G(E^*)$. Hence $\nabla_E u = x \in \tilde{D}(E^*) = E(0)^\perp$ (Proposition 1.13) and $\{\Delta_E u, y\} \in G(E^*E) \Leftrightarrow \{\Delta_E u, y + \Delta_E u\} = \{\Delta_E u, u\} \in G(I + E^*E)$ so that finally $\{u, \Delta_E u\} \in G((I + E^*E)^{-1}) \Rightarrow G(\Delta_E) \subseteq G((I + E^*E)^{-1})$. Now let $w \in \tilde{N}(I + E^*E)$. Then $\{w, 0\} \in G(I + E^*E) \Leftrightarrow \{w, -w\} \in G(E^*E)$ so that $\exists z \in \mathcal{H}$ such that $\{w, z\} \in G(E)$ and $\{z, -w\} \in G(E^*) \Leftrightarrow \{w, z\} \perp G(E)$ and therefore $\|w\|^2 + \|z\|^2 = 0 \Rightarrow w = 0$. Hence $\tilde{N}(I + E^*E) = \{0\}$ so that $(I + E^*E)^{-1}$ is an operator and consequently since $\tilde{D}(E) = \mathcal{H}$, $\Delta_E = (I + E^*E)^{-1}$. Since clearly Δ_E is closed, (i) is proved.

(ii) It is clear that $\{u, \nabla_E u\} \in G(E\Delta_E)$ and since $\nabla_E u \perp E(0)$ it follows that $G(\nabla_E) \subseteq G((I - P_{E(0)})E\Delta_E)$. As above, the fact that $(I - P_{E(0)})E\Delta_E$ is an operator and $\tilde{D}((I - P_{E(0)})E\Delta_E) = \mathcal{H}$ implies that $\nabla_E = (I - P_{E(0)})E\Delta_E$. Since clearly ∇_E is closed, (ii) is proved.

(iii) Let $v \in \mathcal{H}$. Then by definition $\{0, v\} = \{\nabla_{E^*} v, v - \Delta_{E^*} v\} + \{-t, s\}$ where $\{-t, s\} \in G(E)^\perp \Leftrightarrow \{s, t\} \in G(E^*)$. Hence $s = \Delta_{E^*} v$ and $t = \nabla_{E^*} v$. Also $\{\nabla_{E^*} v, v - \Delta_{E^*} v\} \in G(E) \Leftrightarrow \{v - \Delta_{E^*} v, -\nabla_{E^*} v\} \in G(E^*)^\perp$. Hence $\{v, 0\} = \{s, t\} + \{v - \Delta_{E^*} v, -\nabla_{E^*} v\}$ so that (iii) and (iv) follow from (i) and (ii), exchanging E with E^* . $\square$

Corollary 2.2. *Let $E \in LR(\mathcal{H})$. Then $\nabla_E^* = \nabla_{E^*}$ and Δ_E and Δ_{E^*} are selfadjoint.*

Proof. Immediate consequence of the fact that P_E is orthogonal. $\square$

Corollary 2.3. *Let $E \in LR(\mathcal{H})$. Then $\Delta_E^2 + \nabla_{E^*} \nabla_E = \Delta_E$, $\Delta_E \nabla_{E^*} - \nabla_{E^*} \Delta_{E^*} = 0$, $\nabla_E \Delta_E - \Delta_{E^*} \nabla_E = 0$, and $\nabla_E \nabla_{E^*} + \Delta_{E^*}^2 = \Delta_{E^*}$.*

Proof. Immediate consequence of the fact that P_E is a projection. $\square$

Proposition 2.4. *Let $E \in LR(\mathcal{H})$. Then*

$$(2.2) \quad \forall u \in \mathcal{H}, \|\Delta_E u\|^2 + \|\nabla_E u\|^2 \leq \|\Delta_E u\| \|u\|;$$

$$(2.3) \quad \|\Delta_E\| \leq 1;$$

$$(2.4) \quad \|\nabla_E\| \leq \frac{1}{2}.$$

Proof. Let $u \in \mathcal{H}$. Then $\{u, 0\} = \{s, t\} + \{x, -y\}$ with $\{s, t\} \in G(E)$ and $\{y, x\} \in G(E^*)$ and $u = s + x$, $0 = t - y$. Then $t = y = \nabla_E u$, $s = \Delta_E u$ and $(s, u) - \|s\|^2 -$

$\|t\|^2 = (\{s, t\}, \{x, -y\}) = (P_E\{s, t\}, (I - P_E)\{x, -y\}) = 0$. Therefore,

$$(2.5) \quad \|s\|^2 + \|t\|^2 = (s, u)$$

so that $\|s\|^2 + \|t\|^2 \leq \|s\| \|u\|$, which establishes (2.2). Then $\forall u \in \mathcal{H}$, $\|\Delta_E u\|^2 \leq \|\Delta_E u\| \|u\|$ so that $\forall u \in \mathcal{H}$, $\|\Delta_E u\| \leq \|u\|$, which implies (2.3) and $\forall u \in \mathcal{H}$, $\|\Delta_E u\|^2 + \|\nabla_E u\|^2 \leq \|\Delta_E u\|^2 + (1/4)\|u\|^2$ so that $\forall u \in \mathcal{H}$, $\|\nabla_E u\|^2 \leq (1/4)\|u\|^2$, which implies (2.4). $\square$

Corollary 2.5. *Let $E \in LR(\mathcal{H})$. Then*

- (i) Δ_E is positive;
- (ii) $\tilde{N}(\Delta_E) = E^*(0) = \tilde{D}(E)^\perp$.

Proof. (i) From Corollary 2.2 we know that Δ_E is selfadjoint and from (2.5) it follows that $\forall u \in \mathcal{H}$, $(\Delta_E u, u) \geq \|\Delta_E u\|^2 \geq 0$.

(ii) $u \in \tilde{N}(\Delta_E) \Leftrightarrow \{0, u\} \in G(I + E^*E) \Leftrightarrow \{0, u\} \in G(E^*E) \Rightarrow \exists z \in \mathcal{H}$ such that $\{0, z\} \in G(E)$ and $\{z, u\} \in G(E^*) \Rightarrow (\{z, 0\}, \{z, u\}) = 0 \Rightarrow z = 0$ so that $u \in E^*(0)$. Note that $E^*(0) = E^*E(0)$. Indeed it is clear that $E^*(0) \subseteq E^*E(0)$ and conversely if $v \in E^*E(0)$, then $\{0, v\} \in G(E^*E)$ so that $\exists w \in \tilde{D}(E^*)$ such that $\{0, w\} \in G(E)$ and $\{w, v\} \in G(E^*)$. But then $\{w, 0\} \in G(E^*)^\perp$ so that $\|w\|^2 = (\{w, 0\}, \{w, v\}) = 0$ and consequently $\{0, v\} = \{w, v\} \in G(E^*)$ so that $v \in E^*(0)$. Hence $\{0, u\} \in G(E^*) \Leftrightarrow \{0, u\} \in G(E^*E) \Leftrightarrow \{0, u\} \in G(I + E^*E) \Leftrightarrow u \in \tilde{N}(\Delta_E)$, and therefore the first equality is proved. The second equality follows from Proposition 1.13. $\square$

3. THE EXTENDED CAYLEY TRANSFORM

Definition 3.1. Let E be a linear relation. Then $Z(E) = -iI - 2(E + iI)^{-1}$ is by definition the Cayley transform of E .

Proposition 3.2. *Let $E \in Lr(\mathcal{H})$ and let $F \in L(\mathcal{H})$. Then*

$$G[(E + F^*)(E + F)^{-1}] = G[I + (F^* - F)(E + F)^{-1}] + \{0\} \oplus E(0).$$

Proof. Let $\{u, v\} \in G[(E + F^*)(E + F)^{-1}]$. Then $\exists w \in \tilde{D}(E)$ such that $\{u, w\} \in G[(E + F)^{-1}] \Leftrightarrow \{w, u\} \in G(E + F)$ and $\{w, v\} \in G(E + F^*)$. Then $\{w, u - Fw\} \in G(E)$ and $\{w, v - F^*w\} \in G(E)$ so that $\{0, F^*w - Fw + u - v\} \in G(E)$. Also $\{u, F^*w - f\} \in G[(F^* - F)(E + F)^{-1}]$ so that $\{u, v\} = \{u, (F^* - F)w + u\} - \{0, (F^* - F)w + u - v\} \in G[I + (F^* - F)(E + F)^{-1}] + \{0\} \oplus E(0)$. Conversely let $\{r, s\} \in G[I + (F^* - F)(E + F)^{-1}] + \{0\} \oplus E(0)$. Then $\exists x \in E(0)$ such that $\{u, v - u - x\} \in G[(F^* - F)(E + F)^{-1}]$ so that $\exists t \in \tilde{D}(E)$ such that $\{r, t\} \in G[(E + F)^{-1}] \Leftrightarrow \{t, r\} \in G(E + F)$ and $s - r - x = (F^* - F)t$. Hence $\{t, r - Ft\} \in G(E) \Rightarrow \{t, s - F^*t\} = \{t, r - Ft + x\} \in G(E)$, so that $\{t, s\} \in G(E + F^*)$ and consequently $\{r, s\} \in G[(E + F^*)(E + F)^{-1}]$. $\square$

Corollary 3.3. *Let $E \in Lr(\mathcal{H})$, and let $F \in L(\mathcal{H})$. Then if $E + F$ is injective and E is not an operator,*

$$G[(E + F^*)(E + F)^{-1}] \neq G[I + (F^* - F)(E + F)^{-1}].$$

Proof. Under the hypothesis, $I + (F^* - F)(E + F)^{-1}$ is an operator (i.e. has a trivial multivalued part) while $E(0)$ is the nontrivial multivalued part of $(E + F^*)(E + F)^{-1}$. $\square$

Proposition 3.4. *Let $E \in LR(\mathcal{H})$ and $F = Z(E)$. Then $P_F = UP_EU$, which is how the Cayley transform of closed linear relations was defined in [2].*

Proof. Let $u \in \mathcal{H}$. Then $\{\Delta_E u, \nabla_E u\} \in G(E)$ so that $\{-i\Delta_E u, -i\nabla_E u + \Delta_E u\} \in G(E + iI)$. Hence

$$(3.1) \quad -i\nabla_E u + \Delta_E u \in \tilde{R}(E + iI).$$

Similarly $\{\nabla_{E^*} u, u - \Delta_{E^*} u\} \in G(E)$, which implies that $\{\nabla_{E^*} u, u - \Delta_{E^*} u + i\nabla_{E^*} u\} \in G(E + iI)$. Hence

$$(3.2) \quad u - \Delta_{E^*} u + i\nabla_{E^*} u \in \tilde{R}(E + iI).$$

Then $v = (1/2)(-i\nabla_E u + \Delta_E u + u - \Delta_{E^*} u + i\nabla_{E^*} u) \in \tilde{R}(E + iI) = \tilde{D}(F)$ so that $\exists w \in \mathcal{H}$ such that $\{v, w\} \in G(F)$. Then $w = -iv + r$ with $\{v, r\} \in G(-2(E + iI)^{-1})$ or equivalently $\{-2v, r\} \in G((E + iI)^{-1}) \Leftrightarrow \{r, -2v\} \in G(E + iI)$. Then $\{r, -2v\} + \{-i\Delta_E u, -i\nabla_E u + \Delta_E u\} + \{\nabla_{E^*} u, u - \Delta_{E^*} u + i\nabla_{E^*} u\} = \{r - i\Delta_E u + \nabla_{E^*} u, 0\} \in G(E + iI)$. Set $s = r - i\Delta_E u + \nabla_{E^*} u \in \tilde{N}(E + iI) = F(0)$. Hence $w = s + (1/2)(2i\Delta_E u - 2\nabla_{E^*} u - i\Delta_E u + \nabla_{E^*} u - \nabla_E u - iu + i\Delta_{E^*} u) = s + (1/2)(i\Delta_E u - \nabla_{E^*} u - \nabla_E u - iu + i\Delta_{E^*} u)$ and $\{v, w - s\} \in G(F)$. Now $\{\Delta_{E^*} u, \nabla_{E^*} u\} \in G(E^*)$ so that $\{\Delta_{E^*} u, \nabla_{E^*} u - i\Delta_{E^*} u\} \in G(E^* - iI)$ and therefore

$$(3.3) \quad -\nabla_{E^*} u + i\Delta_{E^*} u \in \tilde{R}(E^* - iI).$$

Similarly $\{\nabla_E u, u - \Delta_E u\} \in G(E^*)$ so that $\{\nabla_E u, u - \Delta_E u - i\nabla_E u\} \in G(E^* - iI)$ and hence

$$(3.4) \quad -iu + i\Delta_E u - \nabla_E u \in \tilde{R}(E^* - iI).$$

Using Propositions (1.14) and (1.15) it is easy to see that $F^* = iI - 2(E^* - iI)^{-1}$. Then $w - s = -\nabla_{E^*} u + i\Delta_{E^*} u - iu + i\Delta_E u - \nabla_E u \in \tilde{R}(E^* - iI) \in \tilde{D}(F^*)$, so that $\exists x \in \mathcal{H}$ such that $\{w - s, x\} \in G(F^*)$. Then $x = i(w - s) + t$ with $\{w - s, t\} \in G(-2(E^* - iI)^{-1})$ or equivalently $\{-2(w - s), t\} \in G((E^* - iI)^{-1}) \Leftrightarrow \{t, -2(w - s)\} \in G(E^* - iI)$. So $\{t, -2(w - s)\} - \{\Delta_{E^*} u, \nabla_{E^*} u - i\Delta_{E^*} u\} + \{-i\nabla_E u, -iu + i\Delta_E u - \nabla_E u\} = \{t - \Delta_{E^*} u - i\nabla_E u, 0\} \in G(E^* - iI)$. Set $y = t - \Delta_{E^*} u - i\nabla_E u \in \tilde{N}(E^* - iI) = F^*(0)$. Then

$$\begin{aligned} x &= y + (1/2)(2\Delta_{E^*} u + 2i\nabla_E u - \Delta_E u - i\nabla_{E^*} u - i\nabla_E u + u - \Delta_{E^*} u) \\ &= y + (1/2)(\Delta_{E^*} u + i\nabla_E u - \Delta_E u - i\nabla_{E^*} u + u) = y + u - v. \end{aligned}$$

Since $\{v, x\} \in G(F^*F)$, it follows that $\{v, x + v\} = \{v, u + y\} \in G(I + F^*F)$, which shows that $\{v, u\} \in G(I + F^*F) \Leftrightarrow v = \Delta_F u$ so that

$$\Delta_F = (1/2)(\Delta_E + i\nabla_{E^*} - i\nabla_E + I - \Delta_{E^*}).$$

Quite analogously it can be shown that

$$\Delta_{F^*} = (1/2)(\Delta_{E^*} - i\nabla_E + i\nabla_{E^*} + I - \Delta_E).$$

Finally let $\{u, z\} \in G(F\Delta_F)$. Then $\exists v$ such that $v = \Delta_F u$ and $\{v, z\} \in G(F)$. As above, it follows that $z = -iv + m$ with $\{v, m\} \in G(-2(E + iI)^{-1})$, which is equivalent to $\{m, -2v\} \in G(E + iI)$. Hence, using (3.1) and (3.2), we get $\{n, 0\} = \{m - i\nabla_E u + \nabla_{E^*} u, 0\} \in G(E + iI)$ so that $\{0, n\} \in F(0) = F\Delta_F(0)$ and consequently $\{u, z - n\} \in G(F\Delta_F)$. Therefore, since $z = -iv + m = (1/2)(i\Delta_E - \nabla_{E^*} - \nabla_E - iI + i\Delta_{E^*})u + n$ and using (3.3) and (3.4) we see that $z - n \in \tilde{R}(E^* - iI) = \tilde{D}(F^*) = F(0)^\perp$. It follows that $z - n = \nabla_F u$. Hence $\nabla_F = (1/2)(i\Delta_E - \nabla_{E^*} - \nabla_E - iI + i\Delta_{E^*})$ and taking adjoints $\nabla_{F^*} = (\nabla_F)^* =$

$(1/2)(-i\Delta_E - \nabla_E - \nabla_{E^*} + iI - i\Delta_{E^*})$. To establish the proposition it now suffices to observe that the coefficients in the matrices representing P_F and UP_EU are the same. $\square$

Corollary 3.5. *The transform Z is an isometry on $LR(\mathcal{H})$, i.e. $\forall E, F \in LR(\mathcal{H})$, $g(Z(E), Z(F)) = g(E, F)$.*

Proof. $g(Z(E), Z(F)) = \|P_{Z(E)} - P_{Z(F)}\| = \|U(P_E - P_F)U\| = \|P_E - P_F\| = g(E, F)$. $\square$

Proposition 3.6. *Let E be a linear relation (not necessarily closed). Then $Z(Z(E)) = E$. This property is the reason for our modification (by a factor $-i$) of the usual definition of the Cayley transform.*

Proof. Let $F = Z(E)$. Then

$$\begin{aligned} F + iI &= -2(E + iI)^{-1} \Leftrightarrow (F + iI)^{-1} = -(1/2)(E + iI) \\ &\Leftrightarrow E + iI = -2(F + iI)^{-1} \Leftrightarrow E = Z(F) \end{aligned}$$

so that $E = Z(Z(E))$. $\square$

Proposition 3.7. *Let E be a linear relation (not necessarily closed). Then $Z(E^{-1}) = (Z(E))^{-1}$.*

Proof. Let $F = Z(E)$. Then we have $(u, v) \in G(F) \Leftrightarrow v = -iu + r$ and $\{u, r\} \in G(-2(E + iI)^{-1}) \Leftrightarrow v = -iu + r$ and $\{-2u, r\} \in G((E + iI)^{-1}) \Leftrightarrow v = -iu + r$ and $\{r, -2u\} \in G(E + iI) \Leftrightarrow v = -iu + r$ and $\{r, -2u - ir\} \in G(E) \Leftrightarrow v = -iu + r$ and $\{-2u - ir, r\} \in G(E^{-1}) \Leftrightarrow v = -iu + r$ and $\{-2u - ir, 2r - 2iu\} \in G(E^{-1} + iI) \Leftrightarrow v = -iu + r$ and $\{2v, -2u - ir\} \in G((E^{-1} + iI)^{-1}) \Leftrightarrow u = 2u + ir - iv$ and $\{v, 2u + ir\} \in G(-2(E^{-1} + iI)^{-1}) \Leftrightarrow \{v, u\} = \{v, 2u + ir - iv\} \in G(Z(E^{-1}))$ so that $\{v, u\} \in G((Z(E))^{-1}) \Leftrightarrow \{v, u\} \in G(Z(E^{-1}))$, which establishes the proposition. $\square$

REFERENCES

- [1] R. ARENS, *Operational calculus of linear relations*, Pacific J. Math. **11**, (1961), pp. 9-23. MR 23:A517
- [2] Z. BOULMAAROUF and J-Ph. LABROUSSE, *The Cayley transform of linear relations*, J. Egyptian Math. Soc. **2**, (1994), pp. 53-65. MR 96a:47034
- [3] R. CROSS, *Multivalued linear operators*, Marcel Dekker, Inc., New York (1998). MR 99j:47003
- [4] Y. MEZROUI, *Projection orthogonale sur le graphe d'une relation linéaire fermée*, Trans. Amer. Math. Soc. **352**(6), (2000), pp. 2789-2800. MR 2000j:47003
- [5] M. Z. NASHED (ed.), *General inverses and applications*, Academic Press, New York (1976). MR 56:9943

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE ANTOFAGASTA, ANGAMOS 601, ANTOFAGASTA, CHILE

E-mail address: mfernandez@uantof.cl

UNIVERSITÉ DE NICE-SOPHIA ANTIPOLIS, LABORATOIRE J. A. DIEUDONNÉ, UMR DU CNRS N° 6621, PARC VALROSE, NICE CEDEX 02, FRANCE

E-mail address: labro@math.unice.fr