SELF-ADJOINTNESS OF THE PERTURBED WAVE OPERATOR ON $L^2(\mathbf{R}^n)$, $n \geq 2$

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(Communicated by Joseph A. Ball)

ABSTRACT. We give classes of unbounded real-valued V for which $\square + V$ is self-adjoint on $\mathcal{D}(\square) \subset L^2(\mathbf{R}^n)$, $n \geq 2$, where $\square$ is the wave operator defined on $\mathbf{R}^n$.

1. INTRODUCTION

There are many classes of V for which $-\Delta + V$ is self-adjoint on $\mathcal{D}(-\Delta)$. In this paper we investigate the self-adjointness of $\square + V$ where $\square = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \dots - \frac{\partial^2}{\partial x_n^2}$ is the usual wave operator and V is an unbounded real-valued function defined on $\mathbf{R}^n$. By using Fourier transforms on $L^2(\mathbf{R}^n)$ one can show that $\square$ is self-adjoint on $\mathcal{D}(\square) = \{f \in L^2(\mathbf{R}^n) : \square f \in L^2(\mathbf{R}^n)\}$ (we understand that the partial derivative is taken in the distributional sense). In the sequel $\mathcal{D}(\square)$ will be denoted by M^n where n stands for the dimension. We assume definitions and notions about unbounded operators, Fourier transforms, *BMO* and interpolation theory, and for references one may consult [2], [3] or [4]. We also recall without proof the Kato-Rellich perturbation theorem (for a proof see [3], Theorem X.12), but before that we have

Definition 1. Let A and B be densely defined linear operators on a Hilbert space H . Suppose that:

- i) $D(A) \subset D(B)$;
- ii) for some a and b in $\mathbf{R}$ and all $\varphi \in D(A)$,

$$\|B\varphi\| \leq a\|A\varphi\| + b\|\varphi\|.$$

Then B is said to be **A -bounded**. The infimum of such an a is called the **relative bound** of B with respect to A .

Theorem 1 (Kato-Rellich theorem). *Suppose that A is self-adjoint, B is symmetric, and B is A -bounded with relative bound $a < 1$. Then $A + B$ is self-adjoint on $D(A)$.*

Remark 1. Throughout this paper, $\tilde{c}$ will denote any arbitrary constant that need not be the same each time.

Received by the editors July 8, 2003 and, in revised form, October 1, 2003.

2000 *Mathematics Subject Classification.* Primary 47B25, 47A55, 46B70; Secondary 35L05, 32A37, 46E35.

2. M^2 AND THE SPACE BMO

It is known that by a simple change of variables we can change $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ into $\frac{\partial^2}{\partial \eta \partial \xi}$. So we will use $\frac{\partial^2}{\partial x \partial y}$ to mean $\square$. We will prove that $M^2 \subset BMO(\mathbf{R}^2)$, but first we have

Lemma 1. Fix $y_1 \in \mathbf{R}$, and put $f_1(x) = \varphi(x, y_1)$ where $\varphi \in M^2$. Then $f_1 \in BMO(\mathbf{R})$ with uniform BMO bound. Also $y \mapsto g_1(y) = \varphi(x_1, y)$ is in $BMO(\mathbf{R})$.

Proof. We have to prove $\frac{1}{|I|} \int_I |f_1(x) - \overline{f_1}| dx \leq c$ where c does not depend on y_1 , and where $\overline{f_1} = \frac{1}{|I|} \int_I f_1(x) dx$. By putting $\rho = \frac{\partial^2 \varphi}{\partial x \partial y}$ we have

$$(f_1 - f)'(x) = \int_y^{y_1} \rho(x, z) dz \text{ where } f(x) = \varphi(x, y),$$

and where we have the freedom to choose the y that is convenient to us. Then we have

$$\|(f_1 - f)'\|_2^2 \leq (y_1 - y) \|\square \varphi\|_2^2 < \infty \text{ with the assumption } y < y_1.$$

So $(f_1 - f)' \in L^2(\mathbf{R})$. Hence, $f_1 - f$ is continuous and so there exists a $c \in I$ such that

$$(f_1 - f)(x) - \overline{f_1 - f} = (f_1 - f)(x) - (f_1 - f)(c),$$

and since $(f_1 - f)' \in L^1_{loc}(\mathbf{R})$ we have

$$(f_1 - f)(x) - (f_1 - f)(c) = \int_c^x (f_1 - f)'(t) dt.$$

By taking $I = (a, b)$ and since both c and x are in I , one has

$$|(f_1 - f)(x) - (f_1 - f)(c)| \leq (b - a)^{\frac{1}{2}} \|(f_1 - f)'\|_2 = |I|^{\frac{1}{2}} \|(f_1 - f)'\|_2.$$

Hence

$$\frac{1}{|I|} \int_I |(f_1 - f)(x) - (f_1 - f)(c)| dx \leq |I|^{\frac{1}{2}} (y_1 - y)^{\frac{1}{2}} \|\square \varphi\|_2.$$

So in order to find a uniform bound for $\|f_1 - f\|_{BMO}$ for this particular I , it suffices to take y such that $(y_1 - y)^{\frac{1}{2}} \leq \frac{1}{|I|^{\frac{1}{2}}}$, and in such a case we will have

$$\frac{1}{|I|} \int_I |(f_1 - f)(x) - \overline{(f_1 - f)}| dx \leq \|\square \varphi\|_2.$$

But our interest is in the function f_1 itself, not in $f_1 - f$. We can find a BMO bound for f_1 for this particular I if we come to show that f is BMO . We have

$$\frac{1}{|I|} \int_I |f(x) - \overline{f}| dx \leq \frac{1}{|I|} \int_I |f(x)| dx + \frac{1}{|I|} \int_I \left(\frac{1}{|I|} \int_I |f(t)| dt \right) dx.$$

Then

$$\frac{1}{|I|} \int_I |f(x) - \overline{f}| dx \leq \frac{2\|f\|_2}{|I|^{\frac{1}{2}}}.$$

So far we have not used the fact that $\varphi \in L^2(\mathbf{R}^2)$! To have a uniform BMO bound for f we need for instance $\|f\|_2 \leq c|I|^{\frac{1}{2}}$ ($c > 0$, to be determined). By

applying a simple calculus argument, one can say that there exists a $y \in \mathbf{R}$, $|y_1 - y| \leq \frac{1}{(b-a)}$ such that $\|f\|_2 \leq 2\|\varphi\|_2|I|^{\frac{1}{2}}$. So

$$\frac{1}{|I|} \int_I |f_1(x) - \overline{f_1}| dx \leq \tilde{c}\|\square\varphi\|_2 + \tilde{c}\|\varphi\|_2.$$

The same proof works for g_1 . $\square$

Theorem 2. *Let $\varphi \in M^2$. Then $\varphi \in BMO(\mathbf{R}^2)$ and*

$$\|\varphi\|_{BMO(\mathbf{R}^2)} \leq a\|\square\varphi\|_2 + b\|\varphi\|_2.$$

Proof. The proof follows from Lemma 1 and Fubini's Theorem. $\square$

Hence we have the following corollary whose proof is an immediate consequence of an interpolation theorem ([4], Theorem 6.8).

Corollary 1. *Let $2 \leq p < \infty$. Then any $\varphi \in M^2$ will be in $L^p(\mathbf{R}^2)$, and we have*

$$(1) \quad \|\varphi\|_p \leq a\|\square\varphi\|_2 + b\|\varphi\|_2$$

where the constants a and b depend on p .

Remark 2. The case $p = \infty$ is excluded, as will be shown below.

Lemma 2. *The constant a in Corollary 1 may be made as small as we want.*

Proof. Take $\varphi_\lambda(x, y) = \varphi(\lambda x, \lambda y) : \lambda > 0$. We get

$$\|\square\varphi_\lambda\|_2 = \lambda\|\square\varphi\|_2, \quad \|\varphi_\lambda\|_2 = \frac{1}{\lambda}\|\varphi\|_2 \quad \text{and} \quad \|\varphi_\lambda\|_p = \frac{1}{\lambda^{\frac{2}{p}}}\|\varphi\|_p.$$

Thus the estimate (1) applied to φ_λ instead of φ becomes

$$\|\varphi\|_p \leq a\lambda^{\frac{2}{p}+1}\|\square\varphi\|_2 + b\lambda^{\frac{2}{p}-1}\|\varphi\|_2, \quad \lambda > 0, \quad p \geq 2.$$

By taking λ small enough, the constant in front of $\|\square\varphi\|_2$ will be arbitrarily small. $\square$

Theorem 3. *Let $\square$ be the wave operator on $L^2(\mathbf{R}^2)$. Let $\epsilon > 0$, and let $V : \mathbf{R}^2 \rightarrow \mathbf{R}$ be such that $V \in L^{2+\epsilon}(\mathbf{R}^2)$. Then $\square + V$ is self-adjoint on $\mathcal{D}(\square)$.*

Proof. We have by Corollary 1 that

$$\|\varphi\|_p \leq a\|\square\varphi\|_2 + b\|\varphi\|_2, \quad \text{for } 2 \leq p < \infty.$$

Then by Hölder's inequality,

$$\|V\varphi\|_2 \leq \|V\|_q\|\varphi\|_p \leq a\|V\|_q\|\square\varphi\|_2 + b\|V\|_q\|\varphi\|_2,$$

for $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$ or $q = \frac{2p}{p-2}$. Since the constant in front of $\|\square\varphi\|_2$ can be made small enough so that we have $a\|V\|_q < 1$, we conclude by the Kato-Rellich perturbation theorem that $\square + V$ is self-adjoint on $\mathcal{D}(\square) = M^2$. $\square$

Proposition 1. *Let $V \in L^{2+\epsilon}(\mathbf{R}^2) + L^\infty(\mathbf{R}^2)$ be a real-valued function, $\epsilon > 0$. Then $\square + V$ is self-adjoint on $\mathcal{D}(\square)$.*

Proof. The same as for Theorem 3. $\square$

3. A COUNTEREXAMPLE

Proposition 2. *Let $\varphi \in L^2(\mathbf{R}^2)$ be such that $\frac{\partial^2 \varphi}{\partial x \partial y} \in L^2(\mathbf{R}^2)$. Then φ need not be essentially bounded on $\mathbf{R}^2$.*

Proof. We are going to build up the counterexample by using a linear interpolation. We define $(x, y) \mapsto \varphi(x, y)$ on $\mathbf{R} \times (y_n, y_{n+1})$ by

$$\varphi(x, y) = \frac{1}{y_{n+1} - y_n} [(y - y_n)f_{n+1}(x) - (y - y_{n+1})f_n(x)] \text{ where } f_n(x) = \varphi(x, y_n),$$

and the f_n and y_n are to be defined below.

Hence on $\mathbf{R} \times (y_1, \infty)$ we have

$$\|\varphi\|_2^2 = \iint_{\mathbf{R} \times (y_1, \infty)} |\varphi(x, y)|^2 dx dy = \sum_1^\infty \iint_{\mathbf{R} \times (y_n, y_{n+1})} |\varphi(x, y)|^2 dx dy.$$

After some calculations the condition that makes φ belong to L^2 is

$$\sum_1^\infty (y_{n+1} - y_n) (\|f_n\|_2^2 + \|f_{n+1}\|_2^2) < \infty.$$

We also have

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{1}{y_{n+1} - y_n} (f'_{n+1} - f'_n) \in L^2(\mathbf{R} \times (y_1, \infty)) \text{ if}$$

$$\sum_1^\infty \frac{1}{y_{n+1} - y_n} \|\psi'_n\|_2^2 < \infty \text{ where } \psi_n(x) = f_{n+1}(x) - f_n(x).$$

Since $\psi_n(x) = f_{n+1}(x) - f_n(x)$, then $f_n(x) = -\sum_n^\infty \psi_k(x)$. We also want $f_n \notin L^\infty(\mathbf{R})$ so that $\varphi \notin L^\infty(\mathbf{R}^2)$.

$$\text{Take } \psi_n(x) = \begin{cases} \frac{e^n}{n}x + \frac{1}{n} & \text{if } -e^{-n} \leq x \leq 0, \\ -\frac{e^n}{n}x + \frac{1}{n} & \text{if } 0 \leq x \leq e^{-n}, \\ 0 & \text{if } |x| \geq e^{-n}. \end{cases}$$

Hence $\|\psi'_n\|_2^2 \sim \frac{e^n}{n^2}$ and $\|\psi_n\|_2^2 \sim \frac{e^{-n}}{n^2}$. We also have

$$\|f_n\|_2 \leq \sum_{k=n}^\infty \|\psi_k\|_2 = a \sum_{k=n}^\infty \frac{e^{-\frac{k}{2}}}{n} \simeq \int_n^\infty \frac{e^{-\frac{x}{2}}}{x} dx \leq \frac{1}{n} \int_n^\infty e^{-\frac{x}{2}} dx \sim \frac{e^{-\frac{n}{2}}}{n}.$$

Now if we choose $y_{n+1} - y_n = e^n$, then the series

$$\sum_1^\infty \frac{1}{y_{n+1} - y_n} \|\psi'_n\|_2^2 = \sum_1^\infty \frac{1}{e^n} \times \frac{e^n}{n^2} = \sum_1^\infty \frac{1}{n^2}$$

converges and so does

$$\sum_1^\infty (y_{n+1} - y_n) (\|f_n\|_2^2 + \|f_{n+1}\|_2^2) \leq \sum_1^\infty e^n \times \left[\frac{e^{-(n+1)}}{(n+1)^2} + \frac{e^{-n}}{n^2} \right] \sim \sum_1^\infty \frac{1}{n^2}.$$

Now the φ defined on $\mathbf{R} \times (y_n, y_{n+1})$ is given by

$$\varphi(x, y) = e^{-n} \left[(y - y_n) \left(-\sum_{n+1}^\infty \psi_k(x) \right) - (y - y_{n+1}) \left(-\sum_n^\infty \psi_k(x) \right) \right].$$

This φ is actually defined only for $x \in \mathbf{R}$ and $y \geq y_1$. To extend it to the case $y < y_1$ we define φ for $x \in \mathbf{R}$ and $y_1 - y_{n+1} < y < y_1 - y_n$ as follows:

$$\varphi(x, y) = \frac{1}{y_{n+1} - y_n} [(y - y_1 + y_n)f_{n+1}(x) - (y - y_1 + y_{n+1})f_n(x)].$$

This φ is in M^2 for sure. Now we need to show that φ is not in $L^\infty(\mathbf{R}^2)$. Let $x > 0$ and $x \leq e^{-k}$. Then $\ln x \leq -k$ or $\ln \frac{1}{x} \geq k$. So

$$-f_n(x) = x \sum_n^{\lfloor \ln \frac{1}{x} \rfloor} \frac{e^k}{k} - \sum_n^{\lfloor \ln \frac{1}{x} \rfloor} \frac{1}{k} \geq \sum_n^{\lfloor \ln \frac{1}{x} \rfloor} \frac{1}{k}.$$

Now as $x \rightarrow 0$, then $\lfloor \ln \frac{1}{x} \rfloor \rightarrow \infty$. Hence $\ln \lfloor \ln \frac{1}{x} \rfloor \rightarrow \infty$. Thus $f_n(x) \rightarrow \infty$, which implies that $\varphi(x, y) \rightarrow \infty$. So $\varphi \notin L^\infty(\mathbf{R}^2)$. $\square$

Remark 3. This counterexample found is a *BMO* function by Theorem 2.

One can prove Proposition 2 by using Fourier transforms, but it is only an existence proof. One may also wonder under what additional conditions M^2 will be in $L^\infty(\mathbf{R}^2)$. The answer is given by

Proposition 3. *Let $\varphi \in L^2(\mathbf{R}^2)$ be such that $\frac{\partial \varphi}{\partial x}$, $\frac{\partial \varphi}{\partial y}$ and $\frac{\partial^2 \varphi}{\partial x \partial y}$ are all in $L^2(\mathbf{R}^2)$. Then $\hat{\varphi} \in L^1(\mathbf{R}^2)$ and hence $\varphi \in L^\infty(\mathbf{R}^2)$.*

4. CLASSES OF SELF-ADJOINT $\square + V$ ON $L^2(\mathbf{R}^n)$, $n \geq 3$

Remark 4. We will state theorems and propositions in the n -dimensional case, but we will only prove them for $n = 3$ since the proofs are the same for $n > 3$.

Let us first discuss the following Cauchy problem:

$$(I) \quad \begin{cases} u_{tt} - u_{xx} - u_{yy} = f(x, y, t), & (x, y, t) \in \mathbf{R}^2 \times \mathbf{R}^+, \\ u(x, y, 0) = \varphi(x, y); u_t(x, y, 0) = \psi(x, y). \end{cases}$$

Now let us take the Fourier transform of (I) in the (x, y) -plane only. We get

$$(\hat{I}) \quad \begin{cases} \hat{u}_{tt} + (\eta^2 + \xi^2)\hat{u} = \hat{f}(\eta, \xi, t), \\ \hat{u}(\eta, \xi, 0) = \hat{\varphi}(\eta, \xi); \hat{u}_t(\eta, \xi, 0) = \hat{\psi}(\eta, \xi). \end{cases}$$

$(\hat{I})$ is a second-order ODE in t that has the following solution:

$$(2) \quad \begin{aligned} \hat{u}_{\eta, \xi}(t) = & \hat{\varphi}(\eta, \xi) \cos(t\sqrt{\eta^2 + \xi^2}) + \frac{\hat{\psi}(\eta, \xi)}{\sqrt{\eta^2 + \xi^2}} \sin(t\sqrt{\eta^2 + \xi^2}) \\ & + \int_0^t \hat{f}(\eta, \xi, s) \frac{\sin \sqrt{\eta^2 + \xi^2}(t-s)}{\sqrt{\eta^2 + \xi^2}} ds. \end{aligned}$$

The previous holds for $t \geq 0$. For $t < 0$, (I) becomes, after setting $\tilde{u}(x, y, t) = u(x, y, -t)$,

$$(\tilde{I}) \quad \begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} - \tilde{u}_{yy} = f(x, y, -t), & (x, y, t) \in \mathbf{R}^2 \times \mathbf{R}^-, \\ \tilde{u}(x, y, 0) = \varphi(x, y); \tilde{u}_t(x, y, 0) = -\psi(x, y). \end{cases}$$

Now we “fourier” everything in the (x, y) -plane to obtain

$$(\hat{\tilde{I}}) \quad \begin{cases} \hat{\tilde{u}}_{tt} + (\eta^2 + \xi^2)\hat{\tilde{u}} = \hat{f}(\eta, \xi, -t), \\ \hat{\tilde{u}}(\eta, \xi, 0) = \hat{\varphi}(\eta, \xi); \hat{\tilde{u}}_t(\eta, \xi, 0) = -\hat{\psi}(\eta, \xi), \end{cases}$$

which has the following solution:

$$(3) \quad \begin{aligned} \hat{u}_{\eta,\xi}(-t) = \hat{u}_{\eta,\xi}(t) = & \hat{\varphi}(\eta, \xi) \cos(t\sqrt{\eta^2 + \xi^2}) - \frac{\hat{\psi}(\eta, \xi)}{\sqrt{\eta^2 + \xi^2}} \sin(t\sqrt{\eta^2 + \xi^2}) \\ & + \int_0^{-t} \hat{f}(\eta, \xi, s) \frac{\sin \sqrt{\eta^2 + \xi^2}(-t-s)}{\sqrt{\eta^2 + \xi^2}} ds, \end{aligned}$$

and this holds for $t < 0$. After adding up (2) and (3), one gets for $t \in \mathbf{R}$:

$$(4) \quad \begin{aligned} \hat{u}_{\eta,\xi}(t) = & 2\hat{\varphi}(\eta, \xi) \cos(t\sqrt{\eta^2 + \xi^2}) + \int_0^t \hat{f}(\eta, \xi, s) \frac{\sin \sqrt{\eta^2 + \xi^2}(t-s)}{\sqrt{\eta^2 + \xi^2}} ds \\ & + \int_0^{-t} \hat{f}(\eta, \xi, s) \frac{\sin \sqrt{\eta^2 + \xi^2}(-t-s)}{\sqrt{\eta^2 + \xi^2}} ds - \hat{u}_{\eta,\xi}(-t). \end{aligned}$$

Now we can change our Cauchy problem by introducing different initial conditions mainly for the t variable; i.e., instead of working on the intervals $(0, t)$ and $(-t, 0)$, we will be working on $(\frac{t+\alpha}{2}, t)$ and $(\alpha, \frac{t+\alpha}{2})$ where α is to be chosen freely. Now given $u \in C_0^\infty$, then we can regard it as a solution of the Cauchy problem with $u \in C_0^\infty(\mathbf{R}^3)$ where $f = \square u$, $u(x, y, 0) = \varphi(x, y)$ and $u_t(x, y, 0) = \psi(x, y)$. Then (4) will be:

$$(5) \quad \begin{aligned} \hat{u}(\eta, \xi, t) = & 2\hat{u}(\eta, \xi, \frac{t+\alpha}{2}) \cos[\sqrt{\eta^2 + \xi^2}(t)] - \hat{u}(\eta, \xi, \alpha) \\ & + \int_{\frac{t+\alpha}{2}}^t \hat{f}(\eta, \xi, s) \frac{\sin \sqrt{\eta^2 + \xi^2}(t-s)}{\sqrt{\eta^2 + \xi^2}} ds + \int_{\frac{t+\alpha}{2}}^\alpha \hat{f}(\eta, \xi, s) \frac{\sin \sqrt{\eta^2 + \xi^2}(\alpha-s)}{\sqrt{\eta^2 + \xi^2}} ds. \end{aligned}$$

Proposition 4. Let $n \geq 1$. Let $u \in M^{n+1}$. Then $\|u(\dots, t)\|_{L^2(\mathbf{R}^n)} \in L^\infty(\mathbf{R})$ and we have

$$\sup_{t \in \mathbf{R}} \|u(\dots, t)\|_{L^2(\mathbf{R}^n)} \leq a \|\square u\|_{L^2(\mathbf{R}^{n+1})} + b \|u\|_{L^2(\mathbf{R}^{n+1})}.$$

Proof. We shall prove this proposition for functions in $C_0^\infty(\mathbf{R}^3)$ first.¹ Then the result follows for functions in M^3 since $C_0^\infty(\mathbf{R}^3)$ is dense in M^3 in the graph norm of $\square$.

We choose α such that $|t - \alpha| \leq 1$, and then we obtain:

$$|\hat{u}(\eta, \xi, t)| \leq 2|\hat{u}(\eta, \xi, \frac{t+\alpha}{2})| + \tilde{c} \int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta, \xi, s)| ds + \tilde{c} \int_{\frac{t+\alpha}{2}}^\alpha |\hat{f}(\eta, \xi, s)| ds + |\hat{u}(\eta, \xi, \alpha)|,$$

where we have used the fact that $|\frac{\sin X}{X}| \leq 1$. Using the Cauchy-Schwarz inequality and taking the squares of both sides, we will have

$$(6) \quad \begin{aligned} |\hat{u}(\eta, \xi, t)|^2 \leq & \tilde{c} |\hat{u}(\eta, \xi, \frac{t+\alpha}{2})|^2 + \tilde{c} \int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta, \xi, s)|^2 ds + \tilde{c} \int_{\frac{t+\alpha}{2}}^\alpha |\hat{f}(\eta, \xi, s)|^2 ds \\ & + \tilde{c} |\hat{u}(\eta, \xi, \alpha)|^2. \end{aligned}$$

¹ $C_0^\infty(\mathbf{R}^3)$ is the space of infinitely differentiable functions with compact support.

Then integrate (6) with respect to η and ξ in $\mathbf{R}^2$, and bound the integral of $|\hat{f}|^2$ by the one over $\mathbf{R}^3$ to obtain

$$\begin{aligned} \iint_{\mathbf{R}^2} |\hat{u}(\eta, \xi, t)|^2 d\eta d\xi &\leq \tilde{c} \iint_{\mathbf{R}^2} |\hat{u}(\eta, \xi, \frac{t+\alpha}{2})|^2 d\eta d\xi + \tilde{c} \iiint_{\mathbf{R}^3} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds \\ &\quad + \tilde{c} \iiint_{\mathbf{R}^3} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds + \tilde{c} \iint_{\mathbf{R}^2} |\hat{u}(\eta, \xi, \alpha)|^2 d\eta d\xi. \end{aligned}$$

Now integrate everything w.r.t. α in the segment $|t - \alpha| \leq 1$ and bound again by integrals over $\mathbf{R}^3$ to get

$$\begin{aligned} \iint_{\mathbf{R}^2} |\hat{u}(\eta, \xi, t)|^2 d\eta d\xi &\leq \tilde{c} \iiint_{\mathbf{R}^3} |\hat{u}(\eta, \xi, \frac{t+\alpha}{2})|^2 d\eta d\xi d\alpha + \tilde{c} \iiint_{\mathbf{R}^3} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds \\ &\quad + \tilde{c} \iiint_{\mathbf{R}^3} |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds + \tilde{c} \iiint_{\mathbf{R}^3} |\hat{u}(\eta, \xi, \alpha)|^2 d\eta d\xi d\alpha. \end{aligned}$$

Thus by taking the supremum in t over $\mathbf{R}$ and by using the Plancherel theorem, one has

$$(7) \quad \sup_{t \in \mathbf{R}} \|u(\cdot, \cdot, t)\|_{L^2(\mathbf{R}^2)} \leq \tilde{c} \|\square u\|_{L^2(\mathbf{R}^3)} + \tilde{c} \|u\|_{L^2(\mathbf{R}^3)}.$$

The estimate for functions in M^3 then follows by the denseness of C_0^∞ in M^3 . $\square$

Theorem 4. *Let $\square$ be the wave operator defined on $L^2(\mathbf{R}^{n+1})$. Let V be a real-valued function such that $\int_{-\infty}^{\infty} \|V(\cdot, t)\|_{L^\infty(\mathbf{R}^n)}^2 dt < \infty$. Then $\square + V$ is self-adjoint on $\mathcal{D}(\square)$.*

Proof. The proof is very similar to the one of Theorem 3. $\square$

Proposition 5. *Let $V = V_1 + V_2$ be real-valued such that V_1 is as in Theorem 4 and $V_2 \in L^\infty(\mathbf{R}^{n+1})$. Then $\square + V$ is self-adjoint on $\mathcal{D}(\square)$.*

By going back to equation (4), we can have a better estimate than the one in Proposition 4 and we have

Proposition 6. *Let $n \geq 2$, and let $u \in M^{n+1}$. Then $\|u(\cdot, \cdot, t)\|_{L^r(\mathbf{R}^n)} \in L^\infty(\mathbf{R})$, $2 < r < \frac{2n}{n-1}$ and we have*

$$\sup_{t \in \mathbf{R}} \|u(\cdot, \cdot, t)\|_{L^r(\mathbf{R}^n)} \leq a \|\square u\|_{L^2(\mathbf{R}^{n+1})} + b \|u\|_{L^2(\mathbf{R}^{n+1})}.$$

In order to do that we will need some lemmas.

Lemma 3 (Sobolev's inequality). *Let $n \geq 2$. Let $f \in H^1(\mathbf{R}^n)$.² Then $f \in L^q(\mathbf{R}^n)$ for $2 \leq q < \frac{2n}{n-2}$ (with $n = 2$ giving $q = \infty$) and we have*

$$\|f\|_q \leq a \|\nabla f\|_2 + b \|f\|_2.$$

We recall the following interpolation result:

Proposition 7. *Let $u(t) \in L^2(\mathbf{R}^n)$ be such that also $u(t) \in L_w^p(\mathbf{R}^n)$.³ Then $u(t) \in L^r(\mathbf{R}^n)$ for $2 < r < p$.*

² $H^1(\mathbf{R}^n)$ is the Sobolev space $\{f \in L^2(\mathbf{R}^n) : \nabla f \in L^2(\mathbf{R}^n)\}$.

³ The L^p -weak norm of u will be denoted by $\|\cdot\|_{p,w}$.

Lemma 4. Assume for all $\epsilon > 0$, there exist V, g such that $u(t) = V + g$, $\|V\|_2 \leq d\epsilon$ and $\|g\|_q \leq \frac{\epsilon}{c}$ where c, d are two constants. Then $u(t) \in L_w^p(\mathbf{R}^n)$ where $p = \frac{4q}{2+q}$ and $2 \leq q < \frac{2n}{n-2}$.

Proof. We only prove the lemma for $n = 2$. Let $E_\lambda = \{x : |u(x)| \geq \lambda\}$. Then, since $u = V + g$,

$$E_\lambda \subset \{x : |V(x)| \geq \frac{\lambda}{2}\} \cup \{x : |g(x)| \geq \frac{\lambda}{2}\},$$

so that

$$|E_\lambda| \subset |\{x : |V(x)| \geq \frac{\lambda}{2}\}| + |\{x : |g(x)| \geq \frac{\lambda}{2}\}|.$$

Then by using the Chebyshev inequality, we get $|E_\lambda| \leq 4\lambda^{-2}\|V\|_2^2 + 2^q\lambda^{-q}\|g\|_q^q$. But we have the freedom to choose any $\epsilon(\lambda) > 0$. So take $\epsilon(\lambda) = \lambda^b$ (b to be determined). With this choice we obtain

$$|E_\lambda| \leq 4d^2\lambda^{2b-2} + 2^q c^q \lambda^{-bq-q} \text{ or } \lambda^p |E_\lambda| \leq 4d^2\lambda^{p+2b-2} + 2^q c^q \lambda^{p-bq-q}.$$

Then $u(t)$ belongs to $L_w^p(\mathbf{R}^2)$ if and only if $\sup_{\lambda>0} [4d^2\lambda^{p+2b-2} + 2^q c^q \lambda^{p-bq-q}] < \infty$, which is only possible if $b = \frac{2-p}{2}$ and $p = \frac{4q}{2+q}$. $\square$

Now we prove Proposition 6.

Proof. Let $u \in C_0^\infty$. We use (5) to show that $u(t)$ is in $L^r(\mathbf{R}^2)$, $2 \leq r < 4$ ($n = 2$ gives $q < \infty$ and hence $r < 4$). We set $u(t) = V + g$. So $\hat{u}(t) = \hat{V} + \hat{g}$,

$$\hat{V}(\eta, \xi) = 2\hat{u}(\eta, \xi, \frac{t+\alpha}{2}) \cos[\sqrt{\eta^2 + \xi^2}(t)] - \hat{u}(\eta, \xi, \alpha)$$

and

$$\begin{aligned} \hat{g}(\eta, \xi) &= \int_{\frac{t+\alpha}{2}}^t \hat{f}(\eta, \xi, s) \frac{\sin \sqrt{\eta^2 + \xi^2}(t-s)}{\sqrt{\eta^2 + \xi^2}} ds \\ &\quad + \int_{\frac{t+\alpha}{2}}^t \hat{f}(\eta, \xi, s) \frac{\sin \sqrt{\eta^2 + \xi^2}(\alpha-s)}{\sqrt{\eta^2 + \xi^2}} ds \end{aligned}$$

where $\hat{f} \in L^2(\mathbf{R}^3)$.

We observe that both $\hat{V}$ and $\hat{g}$ depend on α . We will first show that $\nabla g \in L^2(\mathbf{R}^2)$ so that $g \in L^q(\mathbf{R}^2)$, $2 \leq q < \infty$, by Lemma 3 since $g \in L^2(\mathbf{R}^2)$. Then we show that $V \in L^2(\mathbf{R}^2)$. We only do the cases $\hat{u}(\eta, \xi, \frac{t+\alpha}{2}) \cos[\sqrt{\eta^2 + \xi^2}(t)]$ and $\int_{\frac{t+\alpha}{2}}^t \hat{f}(\eta, \xi, s) \frac{\sin \sqrt{\eta^2 + \xi^2}(t-s)}{\sqrt{\eta^2 + \xi^2}} ds$, which we also denote by $\hat{V}(\eta, \xi)$ and $\hat{g}(\eta, \xi)$. The proof of the other two cases is simply the same.

Also, the assumptions we made in Lemma 4 suggested that $\|V\|_2 \leq d\epsilon$ and $\|g\|_q \leq \frac{\epsilon}{c}$. So here we find the constants c and d explicitly and, with no surprise, they depend on $\|u\|_{L^2(\mathbf{R}^3)}$ and $\|\square u\|_{L^2(\mathbf{R}^3)}$.

We also bear in mind that we still have the freedom to choose α . We also give details of why we have $\nabla g \in L^2(\mathbf{R}^2)$. We have $\hat{u} = \hat{V} + \hat{g}$ where

$$\hat{V}(\eta, \xi) = 2\hat{u}(\eta, \xi, \frac{t+\alpha}{2}) \cos[\sqrt{\eta^2 + \xi^2}(t)]$$

and

$$\hat{g}(\eta, \xi) = \int_{\frac{t+\alpha}{2}}^t \hat{f}(\eta, \xi, s) \frac{\sin \sqrt{\eta^2 + \xi^2}(t-s)}{\sqrt{\eta^2 + \xi^2}} ds \text{ where } \hat{f} \in L^2(\mathbf{R}^3).$$

Also, $\hat{g} \in L^2(\mathbf{R}^2)$ and since

$$-i\eta \frac{\sin \sqrt{\eta^2 + \xi^2}(t-s)}{\sqrt{\eta^2 + \xi^2}}, -i\xi \frac{\sin \sqrt{\eta^2 + \xi^2}(t-s)}{\sqrt{\eta^2 + \xi^2}} \in L^\infty(\mathbf{R}^2),$$

then $-i\eta\hat{g}, -i\xi\hat{g} \in L^2(\mathbf{R}^2)$, i.e., $\widehat{\nabla}g \in L^2(\mathbf{R}^2)$. So by Sobolev's inequality we have $g \in L^q(\mathbf{R}^2)$, $2 \leq q < \infty$. We also have

$$\|\nabla g\|_{L^2(\mathbf{R}^2)}^2 = \|\widehat{\nabla}g\|_{L^2(\mathbf{R}^2)}^2 = \iint_{\mathbf{R}^2} |\eta\hat{g}(\eta, \xi)|^2 d\eta d\xi + \iint_{\mathbf{R}^2} |\xi\hat{g}(\eta, \xi)|^2 d\eta d\xi.$$

But

$$\eta^2 |\hat{g}(\eta, \xi)|^2 \leq a^2 |t - \alpha| \int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta, \xi, s)|^2 ds$$

and

$$\xi^2 |\hat{g}(\eta, \xi)|^2 \leq a^2 |t - \alpha| \int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta, \xi, s)|^2 ds.$$

So by choosing α such that $|t - \alpha| \leq \frac{1}{\epsilon^2}$ we get

$$\|\nabla g\|_{L^2(\mathbf{R}^2)}^2 \leq \frac{2a^2}{\epsilon^2} \iint_{\mathbf{R}^2} \int_{\frac{t+\alpha}{2}}^t |\hat{f}(\eta, \xi, s)|^2 d\eta d\xi ds.$$

Thus

$$\|\nabla g\|_{L^2(\mathbf{R}^2)} \leq \frac{c}{\epsilon} \|\square u\|_{L^2(\mathbf{R}^3)}.$$

In a similar way, one gets $\|g\|_{L^2(\mathbf{R}^2)} \leq \frac{c}{\epsilon^{\frac{3}{2}}} \|\square u\|_{L^2(\mathbf{R}^3)}$. So for $\epsilon \geq 1$ we have $\|g\|_{L^2(\mathbf{R}^2)} \leq \frac{c}{\epsilon} \|\square u\|_{L^2(\mathbf{R}^3)}$. Also, for $\epsilon \leq 1$ one has to use $\hat{g} = \hat{u} - \hat{V}$ and the estimate we will get for $\|V\|_{L^2(\mathbf{R}^2)}$ below and the fact that u belongs to $L^2(\mathbf{R}^3)$.

For V we have

$$\hat{V}(\eta, \xi) = 2\hat{u}(\eta, \xi, \frac{t+\alpha}{2}) \cos[\sqrt{\eta^2 + \xi^2}(t)].$$

Then

$$\|\hat{V}\|_{L^2(\mathbf{R}^2)}^2 \leq 4 \iint_{\mathbf{R}^2} |\hat{u}(\eta, \xi, \frac{t+\alpha}{2})|^2 d\eta d\xi,$$

and then by integrating w.r.t. $\alpha \in \{\alpha : |t - \alpha| \leq \frac{1}{\epsilon^2}\}$ one obtains

$$\int_{t-\frac{1}{\epsilon^2}}^{t+\frac{1}{\epsilon^2}} \|\hat{V}\|_{L^2(\mathbf{R}^2)}^2 d\alpha \leq 4 \iint_{\mathbf{R}^2} \int_{t-\frac{1}{\epsilon^2}}^{t+\frac{1}{\epsilon^2}} |\hat{u}(\eta, \xi, \frac{t+\alpha}{2})|^2 d\eta d\xi d\alpha \leq 4 \|\hat{u}\|_{L^2(\mathbf{R}^3)}^2.$$

Then there exists $\alpha \in (t - \frac{1}{\epsilon^2}, t + \frac{1}{\epsilon^2})$ s.t. $\|\hat{V}\|_{L^2(\mathbf{R}^2)}^2 \leq \frac{4\|\hat{u}\|_{L^2(\mathbf{R}^3)}^2}{\epsilon^2}$. So $\|V\|_{L^2(\mathbf{R}^2)} \leq d\epsilon \|u\|_{L^2(\mathbf{R}^3)}$. So one has

$$(8) \quad \sup_{\lambda > 0} [4d^2 \lambda^{p-2+2b} + 2^q c^q \lambda^{p-q-bq}] = \tilde{c} \|u\|_{L^2(\mathbf{R}^3)}^2 + \tilde{c} \|\square u\|_{L^2(\mathbf{R}^3)}^q.$$

We have by Proposition 7 that $\|u(t)\|_{L^r(\mathbf{R}^2)} \leq \tilde{c} \|u(t)\|_2 + \tilde{c} \|u(t)\|_{p,w}$. So by using (8), Proposition 4, scaling by a suitable constant and taking the supremum in t over $\mathbf{R}$, one obtains

$$(9) \quad \sup_{t \in \mathbf{R}} \|u(\cdot, \cdot, t)\|_{L^r(\mathbf{R}^2)} \leq a \|\square u\|_{L^2(\mathbf{R}^3)} + b \|u\|_{L^2(\mathbf{R}^3)}.$$

□

Remark 5. The constant a in (9) may be made as small as we would like.

Theorem 5. Let $n \geq 2$. Let $\square$ be the wave operator defined on $L^2(\mathbf{R}^{n+1})$. Let V be a real-valued function such that $\int_{-\infty}^{\infty} \|V(\cdot, \cdot, t)\|_{L^s(\mathbf{R}^n)}^2 dt < \infty$ for $\frac{1}{s} = \frac{1}{2} - \frac{1}{r}$ and $2 \leq r < \frac{2n}{n-1}$. Then $\square + V$ is self-adjoint on $\mathcal{D}(\square)$.

Proof. Again we only prove the theorem for $n = 2$. We have by Hölder's inequality, for $\frac{1}{2} = \frac{1}{r} + \frac{1}{s}$, $2 \leq r < 4$:

$$\begin{aligned} \iint_{\mathbf{R}^2} |V(x, y, t)u(x, y, t)|^2 dx dy &\leq \|V(\cdot, \cdot, t)\|_{L^s(\mathbf{R}^2)}^2 \cdot \|u(\cdot, \cdot, t)\|_{L^r(\mathbf{R}^2)}^2 \\ &\leq \sup_{t \in \mathbf{R}} \|u(\cdot, \cdot, t)\|_{L^r(\mathbf{R}^2)}^2 (\|V(\cdot, \cdot, t)\|_{L^s(\mathbf{R}^2)}^2). \end{aligned}$$

Then by using Proposition 6 and by integrating w.r.t. t over $\mathbf{R}$ one gets

$$\|Vu\|_2^2 \leq a \left(\int_{\mathbf{R}} \|V(\cdot, \cdot, t)\|_{L^s(\mathbf{R}^2)}^2 dt \right) \|\square u\|_2^2 + b \left(\int_{\mathbf{R}} \|V(\cdot, \cdot, t)\|_{L^s(\mathbf{R}^2)}^2 dt \right) \|u\|_2^2.$$

Since we can make a small enough to have $a \int_{\mathbf{R}} \|V(\cdot, \cdot, t)\|_{L^\infty(\mathbf{R}^2)}^2 dt < 1$ we conclude by the Kato-Rellich perturbation theorem that $\square + V$ is self-adjoint on $\mathcal{D}(\square)$. $\square$

5. OPEN PROBLEMS

This paper contains results from my Ph.D. thesis. I have a series of questions that I would like to know the answers to:

- 1) If V is real-valued and $V \in L^2(\mathbf{R}^n)$, then will $\square + V$ be self-adjoint?
- 2) If $V \geq 0$ is real-valued and $V \in L_{loc}^2(\mathbf{R}^n)$, then will $\square + V$ be self-adjoint? (We insist on V being positive since if it has no sign.) Then there are examples that show that this may fail, and they exploit the non-self-adjointness of the one-dimensional Hamiltonian for some classes of potentials [3].
- 3) In Section 4, can we have larger spaces than the ones found in Section 2 and 4 and which include M^n , $n \geq 3$?

ACKNOWLEDGMENT

I should be very grateful to my supervisor Prof. Alexander M. Davie for the helpful suggestions, comments and hints. I also would like to thank the Algerian Government for the full scholarship it has provided me with.

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