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JACOBI POLYNOMIALS FROM COMPATIBILITY CONDITIONS

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ABSTRACT. We revisit the ladder operators for orthogonal polynomials and re-interpret two supplementary conditions as compatibility conditions of two linear over-determined systems; one involves the variation of the polynomials with respect to the variable z (spectral parameter) and the other a recurrence relation in n (the lattice variable). For the Jacobi weight

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \qquad x \in [-1,1],$$

we show how to use the compatibility conditions to explicitly determine the recurrence coefficients of the monic Jacobi polynomials.

1. Introduction and preliminaries

We begin with some notation. Let $P_n(x)$ be monic polynomials of degree n in x and orthogonal, with respect to a weight, $w(x)$, $x \in [a,b]$;

$$\int_a^b P_m(x)P_n(x)w(x)\,dx = h_n\delta_{m,n}. \tag{1.1}$$

We further assume that $\mathsf{v}'(z) := -w'(z)/w(z)$ exists and that

$$y^n\left[\mathsf{v}'(x) - \mathsf{v}'(y)\right]w(y)/(x-y)$$

is integrable on $[a,b]$ for all n, $n = 0,1,\ldots$. From the orthogonality condition there follows the recurrence relation,

$$zP_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z), \quad n = 0,1,\ldots, \tag{1.2}$$

where $\beta_0 P_{-1}(z) := 0$, α_n, $n = 0,1,2,\ldots$ is real and $\beta_n > 0$, $n = 1,2,\ldots$.

In this paper we describe a formalism which derives properties of orthogonal polynomials, and their recurrence coefficients, from the knowledge of the weight function. We believe this is a new and interesting approach to orthogonal polynomials. In order to keep this work accessible we will only include the example of Jacobi polynomials. We defer to a future publication, the analysis in the case of the generalized Jacobi weights [8, 11]. In the Jacobi case we find the recurrence relations in §2. In §3 we show how our approach leads to the evaluation of monic Jacobi polynomials at $x = \pm 1$. We also show that the evaluation of a Jacobi polynomial at $x = 1$ or $x = -1$ leads to explicit representations of the Jacobi polynomials. Closed form expressions for the normalization constants h_n are also found.

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The actions of the ladder operators on $P_n(z)$ and $P_{n-1}(z)$ are

$$\left(\frac{d}{dz}+B_n(z)\right)P_n(z)=\beta_n A_n(z)P_{n-1}(z), \tag{1.3}$$

$$\left(\frac{d}{dz}-B_n(z)-\mathsf{v}'(z)\right)P_{n-1}(z)=-A_{n-1}(z)P_n(z), \tag{1.4}$$

with

$$A_n(z):=\left.\frac{w(y)\,P_n^2(y)}{h_n(y-z)}\right|_{y=a}^{y=b}+\frac{1}{h_n}\int_a^b\frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y}P_n^2(y)w(y)\,dy, \tag{1.5}$$

$$\begin{aligned}B_n(z):=&\left.\frac{w(y)\,P_n(y)P_{n-1}(y)}{h_{n-1}(y-z)}\right|_{y=a}^{y=b}\\&+\frac{1}{h_{n-1}}\int_a^b\frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y}P_{n-1}(y)P_n(y)w(y)\,dy,\end{aligned} \tag{1.6}$$

where we have used the supplementary condition,

$$B_{n+1}(z)+B_n(z)=(z-\alpha_n)\,A_n(z)-\mathsf{v}'(z), \tag{S$_1$}$$

to arrive at (1.4). The equations (1.3)–(1.6) and the supplementary condition (S_1) were derived by Bonan and Clark [4], Bauldry [3], and Mhaskar [10] for polynomial v, and the authors [7] for general v. Ismail and Wimp [9] identified the additional supplementary condition,

$$B_{n+1}(z)-B_n(z)=\frac{\beta_{n+1}A_{n+1}(z)-\beta_nA_{n-1}(z)-1}{z-\alpha_n}. \tag{S$_2$}$$

Our thesis in this work is that the supplementary conditions, (S_1) and (S_2), being identities in n $(n>0)$ and $z\in\mathbf{C}\cup\{\infty\}$ have the information needed to determine the recurrence coefficients and other auxiliary quantities. We illustrate this by systematically using (S_1) and (S_2) to determine most of the properties of the Jacobi polynomials. See [14], [2], and [12], for detailed information concerning the Jacobi polynomials. In describing our results we shall follow the standard notation for shifted factorials and hypergeometric functions in [2, 12].

Below, we reinterpret (S_2). We set

$$\Phi_n(z):=\begin{pmatrix}P_n(z)\\P_{n-1}(z)\end{pmatrix}, \tag{1.7}$$

$$\mathsf{M}_n(z):=\begin{pmatrix}-B_n(z)&\beta_nA_n(z)\\-A_{n-1}(z)&B_n(z)+\mathsf{v}'(z)\end{pmatrix}, \tag{1.8}$$

$$\mathsf{U}_n(z):=\begin{pmatrix}z-\alpha_n&-\beta_n\\1&0\end{pmatrix}. \tag{1.9}$$

Now equations (1.3) and (1.4) become,

$$\Phi_n'(z)=\mathsf{M}_n(z)\Phi_n(z), \tag{1.10}$$

and the recurrence relations become

$$\Phi_{n+1}(z)=\mathsf{U}_n(z)\Phi_n(z). \tag{1.11}$$

We find, by requiring (1.10) and (1.11) to be compatible,

$$\begin{aligned}\Phi_{n+1}'(z)&=\mathsf{M}_{n+1}(z)\Phi_{n+1}(z)\\&=\mathsf{M}_{n+1}(z)\mathsf{U}_n(z)\Phi_n(z).\end{aligned}$$

On the other hand,

$$\begin{aligned}\Phi_{n+1}'(z) &= \mathsf{U}_n'(z)\Phi_n(z) + \mathsf{U}_n(z)\Phi_n'(z)\\ &= \mathsf{U}_n'(z)\Phi_n(z) + \mathsf{U}_n(z)\mathsf{M}_n(z)\Phi_n(z).\end{aligned}$$

We now write the above equations in matrix form as

$$\mathsf{S}_n(z)\Phi_n(z) = 0, \tag{1.12}$$

where $\mathsf{S}_n(z)$ is the matrix

$$\mathsf{S}_n(z) := \mathsf{U}_n'(z) + \mathsf{U}_n(z)\mathsf{M}_n(z) - \mathsf{M}_{n+1}(z)\mathsf{U}_n(z),$$

whose entries $S_{i,j}$ are

$$\begin{aligned}\mathsf{S}_n^{11}(z) &= 1 + (z-\alpha_n)\left(B_{n+1}(z) - B_n(z)\right)\\ &\quad +\beta_n A_{n-1}(z) - \beta_{n+1}A_{n+1}(z),\\ \mathsf{S}_n^{12}(z) &= -\beta_n\left(B_{n+1}(z) + B_n(z) + \mathsf{v}'(z) - (z-\alpha_n)A_n(z)\right),\\ \mathsf{S}_n^{21}(z) &= S_n^{12}(z)/\beta_n,\\ \mathsf{S}_n^{22}(z) &= 0.\end{aligned} \tag{1.13}$$

Here $n = 1, 2, \dots$ and $z \in \mathbf{C} \cup \{\infty\}$. Observe that with (S_1), $S_n^{12}(z) = S_n^{21}(z) = 0$. This leaves $S_n^{11}(z)P_n(z) = 0$. Since $P_n(z)$ does not vanish identically, we must have $S_n^{11}(z) = 0$, which is (S_2). It is clear from (1.5) and (1.6) that, if $\mathsf{v}'(z)$ is a rational function, then $A_n(z)$ and $B_n(z)$ are also rational functions. This is particularly useful for our purpose, which is to determine the recurrence coefficients, α_n and β_n. In the next section, we illustrate the method by considering the Jacobi weight $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ for $x \in [-1, 1]$.

Recall that the numerator polynomials [13], [1] are

$$Q_n(z) := \int_{-\infty}^{\infty} \frac{P_n(z) - P_n(y)}{z-y}\, w(y)\, dy, \tag{1.14}$$

and $\{P_n(z)\}$ and $\{Q_n(z)\}$ form a basis of solutions of the recurrence relation. We shall also use the notation

$$F(z) = \int_{-\infty}^{\infty} \frac{w(y)}{z-y}\, dy \tag{1.15}$$

for the Stieltjes transform of the weight function.

2. Jacobi weight

The Jacobi weight is $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$, $x \in [-1, 1]$, and for now we take α and β to be strictly positive. It will become clear, using a real analyticity argument, the results that follow are also valid for $\alpha, \beta > -1$. Let $\left\{\mathcal{P}_n^{(\alpha,\beta)}(x)\right\}$ and $\left\{\mathcal{Q}_n^{(\alpha,\beta)}(x)\right\}$ denote the monic Jacobi polynomials, and their numerators, respectively; see (1.14). Moreover, in the present example, the Stieltjes transform of $w^{\alpha,\beta}$ will be denoted by $F^{(\alpha,\beta)}(z)$.

From (1.5)–(1.6) we find

$$h_n A_n(z) = \frac{\alpha}{1-z}\int_{-1}^{1}\left[\mathcal{P}_n^{(\alpha.\beta)}(y)\right]^2(1-y)^{\alpha-1}(1+y)^{\beta}\,dy$$
$$+\frac{\beta}{1+z}\int_{-1}^{1}\left[\mathcal{P}_n^{(\alpha.\beta)}(y)\right]^2(1-y)^{\alpha}(1+y)^{\beta-1}\,dy.$$

Through integration by parts, it readily follows that

$$A_n(z) = -\frac{R_n}{z-1}+\frac{R_n}{z+1}, \tag{2.1}$$

for some constant R_n. Similarly we find

$$B_n(z) = -\frac{n+r_n}{z-1}+\frac{r_n}{z+1}. \tag{2.2}$$

Here R_n and r_n are given by

$$R_n = R_n(\alpha,\beta) := \frac{\beta}{h_n}\int_{-1}^{1}\frac{\left[\mathcal{P}_n^{(\alpha,\beta)}(y)\right]^2}{1+y}\,w^{(\alpha,\beta)}(y)\,dy, \tag{2.3}$$

$$r_n = r_n(\alpha,\beta) := \frac{\beta}{h_{n-1}}\int_{-1}^{1}\frac{\mathcal{P}_n^{(\alpha,\beta)}(y)\mathcal{P}_{n-1}^{(\alpha,\beta)}(y)}{1+y}\,w^{(\alpha,\beta)}(y)\,dy. \tag{2.4}$$

It is easy to see that

$$R_n(\alpha,\beta) = \frac{\beta}{h_n}\,\mathcal{P}_n^{(\alpha,\beta)}(-1)\left[\mathcal{Q}_n^{(\alpha,\beta)}(-1)-F^{(\alpha,\beta)}(-1)\mathcal{P}_n^{(\alpha,\beta)}(-1)\right],$$
$$r_n(\alpha,\beta) = \frac{\beta}{h_{n-1}}\,\mathcal{P}_{n-1}^{(\alpha,\beta)}(-1)\left[\mathcal{Q}_n^{(\alpha,\beta)}(-1)-F^{(\alpha,\beta)}(-1)\mathcal{P}_n^{(\alpha,\beta)}(-1)\right].$$

The reader may ask, "What is the point of this formalism, since in the attempt to find α_n and β_n, two new unknown quantities, R_n and r_n, have been introduced?" However, when v' is a rational function, both sides of (S_1) and (S_2) are rational functions, and by equating coefficients and residues of both sides of (S_1) and (S_2), we shall arrive at four equations, which should be sufficient for the determination of R_n and r_n as well as α_n and β_n. Equating residues of both sides of (S_1), at $z=-1$ and $z=+1$, gives

$$-2n-1-r_n-r_{n+1} = \alpha - R_n\left(1-\alpha_n\right), \tag{2.5}$$
$$r_n+r_{n+1} = \beta - R_n\left(1+\alpha_n\right). \tag{2.6}$$

Similarly, from (S_2), we obtain

$$\left(r_n-r_{n+1}-1\right)\left(1-\alpha_n\right) = \beta_n R_{-1}-\beta_{n+1}R_{n+1}, \tag{2.7}$$
$$\left(r_n-r_{n+1}\right)\left(1+\alpha_n\right) = \beta_{n+1}R_{n+1}-\beta_n R_{n-1}. \tag{2.8}$$

Observe that R_n can be obtained immediately by adding (2.5) and (2.6):

$$R_n = \frac{1}{2}(\alpha+\beta+2n+1). \tag{2.9}$$

The sum of (2.7) and (2.8) gives

$$1-\alpha_n = 2\left(r_n-r_{n+1}\right), \tag{2.10}$$

while (2.8) minus (2.7) and with (2.9) gives

$$\beta-\alpha-2n-1-\left(\alpha+\beta+2n+1\right)\alpha_n = 2\left(r_n+r_{n+1}\right). \tag{2.11}$$

Now, (2.10) $\pm$ (2.11) implies

$$4r_n = \beta - \alpha - 2n - (\alpha + \beta + 2n + 2)\,\alpha_n, \tag{2.12}$$

$$4r_{n+1} = \beta - \alpha - 2n - 2 - (\alpha + \beta + 2n)\,\alpha_n. \tag{2.13}$$

When (2.12) and (2.13) are made compatible, we obtain a first-order difference equation satisfied by α_n:

$$\alpha_{n+1}(\alpha + \beta + 2n + 4) - \alpha_n(\alpha + \beta + 2n) = 0, \tag{2.14}$$

which has a very simple "integrating factor," $\alpha + \beta + 2n + 2$. Using this, we find

$$\alpha_n = \frac{C_1}{(2R_n - 1)\,(2R_n + 1)},$$

where C_1 is an "integration" constant, determined by the initial condition

$$\alpha_0 = \frac{\mu_1}{\mu_0} = \frac{\beta - \alpha}{\alpha + \beta + 2}, \qquad C_1 = \beta^2 - \alpha^2.$$

Here $\mu_j := \int\limits_{-1}^{1} t^j w(t)\,dt$, $j = 0, 1, \ldots$ are the moments. Therefore we have established

$$\alpha_n = \frac{\beta^2 - \alpha^2}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 2)}. \tag{2.15}$$

Going back to (2.8) and using (2.10), we see that β_n satisfies the linear difference equation:

$$\beta_{n+1}R_{n+1} - \beta_n R_{n-1} = \frac{1 - \alpha_n^2}{2}, \tag{2.16}$$

which has the "integrating factor" R_n. Therefore,

$$\begin{aligned} \beta_n R_n R_{n-1} &= C_2 + \frac{1}{2}\sum_{j=0}^{n-1}\left(1 - \alpha_j^2\right) R_j \\ &= C_2 + \frac{1}{2}\sum_{j=0}^{n-1}\left(1 - \frac{C_1^2}{\left(4R_j^2 - 1\right)^2}\right) R_j, \end{aligned} \tag{2.17}$$

where C_2 is another integration constant to be determined by the initial condition

$$\beta_1 = \frac{h_1}{h_0} = \frac{h_1}{\mu_0} = \frac{\mu_2}{\mu_0} - \left(\frac{\mu_1}{\mu_0}\right)^2 = \frac{4(\alpha + 1)(\beta + 1)}{(\alpha + \beta + 2)^2(\alpha + \beta + 3)}.$$

After some computations,

$$C_2 = \beta_1 R_0 R_1 - \frac{1}{2}\left(1 - \alpha_0^2\right) R_0 = 0.$$

Now the sum (2.17) may look complicated; however, with a partial fraction expansion, the sum can be taken and leads to

$$\beta_n = \frac{n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2 R_n R_{n-1}}.$$

Therefore, after some simplifications we establish

$$\beta_n = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)(2n + \alpha + \beta - 1)}. \tag{2.18}$$

3. Explicit formulas

We first determine $\mathcal{P}_n^{(\alpha,\beta)}(\pm 1)$. Write (1.3) as

$$\frac{d}{dz}\,\mathcal{P}_n^{(\alpha,\beta)}(z) = \frac{(n+r_n)\,\mathcal{P}_n^{(\alpha,\beta)}(z) - \beta_n R_n \mathcal{P}_{n-1}^{(\alpha,\beta)}(z)}{z-1} + \frac{\beta_n R_n \mathcal{P}_{n-1}^{(\alpha,\beta)}(z) - r_n \mathcal{P}_n^{(\alpha,\beta)}(z)}{z+1},$$

and since $\frac{d}{dz}\,\mathcal{P}_n^{(\alpha,\beta)}(z)$ is regular at $z = \pm 1$, we arrive at

$$(n+r_n)\,\mathcal{P}_n^{(\alpha,\beta)}(1) - \beta_n \mathcal{R}_n P_{n-1}^{(\alpha,\beta)}(1) = 0,$$
$$\beta_n R_n \mathcal{P}_{n-1}^{(\alpha,\beta)}(-1) - r_n \mathcal{P}_n^{(\alpha,\beta)}(-1) = 0.$$

Thus we find

$$\mathcal{P}_n^{(\alpha,\beta)}(1) = \mathcal{P}_0^{(\alpha,\beta)}(1) \prod_{j=1}^{n} \frac{\beta_j R_j}{r_j + j} \tag{3.1}$$

and

$$\mathcal{P}_n^{(\alpha,\beta)}(-1) = \mathcal{P}_0^{(\alpha,\beta)}(-1) \prod_{j=1}^{n} \frac{\beta_j R_j}{r_j}.$$

Substituting for β_n, r_n and R_n from (2.9), (2.12), and (2.18), and applying (2.15) we prove that

$$\mathcal{P}_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n}{2^n} \prod_{j=1}^{n} \frac{(j+\beta)(j+\alpha+\beta)}{[j+(\alpha+\beta/2)]\,[j+(\alpha+\beta-1/2)]}.$$

Using the facts $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$, $(2\lambda)_{2n} = 4^n(\lambda)_n(\lambda+1/2)_n$ we rewrite the above equation as

$$\mathcal{P}_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n\, 2^n\, (\beta+1)_n}{(\alpha+\beta+n+1)_n}. \tag{3.2}$$

Similarly

$$\mathcal{P}_n^{(\alpha,\beta)}(1) = \frac{2^n\,(\alpha+1)_n}{(\alpha+\beta+n+1)_n}. \tag{3.3}$$

We next evaluate h_n, the squares of the L^2 norms. In general (1.1) and (1.2) yield ([12])

$$h_n = h_0 \beta_1 \beta_2 \cdots \beta_n. \tag{3.4}$$

The beta integral evaluation gives

$$h_0 = 2^{\alpha+\beta+1}\,\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \tag{3.5}$$

Thus

$$\int_{-1}^{1} \mathcal{P}_m^{(\alpha,\beta)}(x)\mathcal{P}_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\,dx = h_n \delta_{m,n}, \tag{3.6}$$

with

$$h_n = \frac{2^{\alpha+\beta+n+1}\,\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(\alpha+\beta+n+1)_n\,\Gamma(\alpha+\beta+2n+2)}\; n!. \tag{3.7}$$

We now prove that

$$\frac{d}{dz}\,\mathcal{P}_n^{(\alpha,\beta)}(z) = n\mathcal{P}_n^{(\alpha+1,\beta+1)}(z). \tag{3.8}$$

For $\alpha > -1$, $\beta > -1$, and $m < n-1$, integration by parts gives

$$\begin{aligned}\int_{-1}^{1} x^m \left(\frac{d}{dx}\,\mathcal{P}_n^{(\alpha,\beta)}(x)\right)(1-x)^{\alpha+1}(1+x)^{\beta+1}\,dx \\ = -\int_{-1}^{1} \mathcal{P}_n^{(\alpha,\beta)}(x)\,f(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx\end{aligned}$$

where $f(x) = x^{m-1}\left[m + x(\beta-\alpha) - x^2(\alpha+\beta+m+2)\right]$. Since f has degree at most $n-1$, the above integral must vanish, and we conclude that $\frac{d}{dx}\,\mathcal{P}_n^{(\alpha,\beta)}(x)$ is orthogonal to all polynomials of degree less than $n-1$ with respect to $w^{(\alpha+1,\beta+1)}(x)$. The uniqueness of the orthogonal polynomials and the fact that $\mathcal{P}_n^{(\alpha,\beta)}(x)$, $n \geq 0$ are monic establish (3.8). Clearly (3.8) and (3.2) give

$$\begin{aligned}\frac{d^k}{dx^k}\,\mathcal{P}_n^{(\alpha,\beta)}(x)\bigg|_{x=-1} &= \frac{n!}{(n-k)!}\,\mathcal{P}_{n-k}^{(k+\alpha,k+\beta)}(-1) \\ &= \frac{n!}{(n-k)!}\,\frac{(-2)^{n-k}(\beta+k+1)_{n-k}}{(\alpha+\beta+n+k+1)_{n-k}}.\end{aligned} \tag{3.9}$$

The Taylor series about $x=-1$ now gives the representation

$$\begin{aligned}\mathcal{P}_n^{(\alpha,\beta)}(x) &= \frac{(-2)^n(\beta+1)_n}{(\alpha+\beta+n+1)_n} \\ &\times {}_2F_1(-n, n+\alpha+\beta+1; \beta+1; (1+x)/2),\end{aligned} \tag{3.10}$$

which we recognize to be the monic Jacobi polynomials. Similarly (3.3) and (3.8) give the alternate representation

$$\begin{aligned}\mathcal{P}_n^{(\alpha,\beta)}(x) &= \frac{(2)^n(\alpha+1)_n}{(\alpha+\beta+n+1)_n} \\ &\times {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2).\end{aligned} \tag{3.11}$$

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