APPROXIMATION OF SOLUTIONS OF NONLINEAR EQUATIONS OF HAMMERSTEIN TYPE IN HILBERT SPACE

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ABSTRACT. Let H be a real Hilbert space. Let $F : D(F) \subseteq H \rightarrow H$, $K : D(K) \subseteq H \rightarrow H$ be bounded monotone mappings with $R(F) \subseteq D(K)$, where $D(F)$ and $D(K)$ are closed convex subsets of H satisfying certain conditions. Suppose the equation $0 = u + KF u$ has a solution in $D(F)$. Then explicit iterative methods are constructed that converge strongly to such a solution. No invertibility assumption is imposed on K , and the operators K and F need not be defined on compact subsets of H .

1. INTRODUCTION

Let H be a real Hilbert space. A mapping $A : D(A) \subseteq H \rightarrow H$ is said to be *monotone* if for every $x, y \in D(A)$ we have

$$(1.1) \quad \langle Ax - Ay, x - y \rangle \geq 0.$$

A is called *maximal monotone* if it is monotone and $\mathcal{R}(I + rA)$ (range of $(I + rA)$) = H for each $r > 0$, where I is the identity mapping on H . A is said to satisfy the *range condition* if $cl(D(A)) \subseteq \mathcal{R}(I + rA)$, for each $r > 0$. A is called *uniformly monotone* if for each $x, y \in D(A)$ there exists a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$(1.2) \quad \langle Ax - Ay, x - y \rangle \geq \phi(\|x - y\|).$$

If $\phi(t) = t\psi(t)$ for $\psi : [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$, ψ strictly increasing, then A is called *ψ -strongly monotone*. If there exists $k \in (0, 1)$ such that $\phi(t) = kt^2$, then A is called *strongly monotone*. A close study of monotone mappings shows that every strongly monotone mapping is ψ -strongly monotone. Moreover, every ψ -strongly monotone mapping is uniformly monotone and every uniformly monotone mapping is monotone. The notion of monotone operators was introduced by Zarantonello [13] and Minty [7]. Interest in such mappings stems mainly from their firm connection with equations of evolution. Several problems that arise in differential equations,

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for instance, elliptic boundary value problems whose linear parts possess Green's function, can be put in operator form as

$$(1.3) \quad u + KF u = 0,$$

where K and F are monotone operators (see [2] for more information). Attempts to use the well-known Mann and Ishikawa iteration methods to approximate solutions of equation (1.3) have not provided satisfactory results. In particular, the recursion formulas obtained involved K^{-1} (see, e.g., [1, 10]), and this is not convenient in applications. Part of the difficulty is the fact that the composition of two monotone operators need not be monotone. Recently, in [2], the authors introduced a method that contains an auxiliary operator, defined in an appropriate real Banach space in terms of K and F which, under certain conditions, is ψ -strongly monotone whenever K and F are, and whose zeros are solutions of equation (1.3).

It is our purpose in this paper to use this method to obtain an auxiliary operator that is *monotone* whenever K and F are, and to construct an iterative procedure that converges to the solution of equation (1.3). In our theorems, the operators K and F need not be defined on compact or *angle-bounded* (see, e.g., [9]) subsets of H , and no invertibility of K will be necessary in our iteration process.

2. PRELIMINARIES

Let K be a closed convex subset of a Banach space E , and let Q be a mapping of E onto K . Then Q is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in E$ and $t \geq 0$. A mapping Q of E into E is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Qz = z$ for every $z \in R(Q)$, the *range* of Q . A subset K of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto K , and it is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto K . If $E = H$, the metric projection P_K is a *sunny nonexpansive retraction* from H to any closed convex subset K of H . In the sequel we shall need the following results.

Lemma 2.1 (see, e.g., [4]). *Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative numbers, and let $\{\alpha_n\}$ be a sequence of positive numbers satisfying the conditions $\sum_1^\infty \alpha_n = \infty$ and $\frac{\gamma_n}{\alpha_n} \rightarrow 0$, as $n \rightarrow \infty$. Let the recursive inequality*

$$(2.1) \quad \lambda_{n+1} \leq \lambda_n - \alpha_n \phi(\lambda_{n+1}) + \gamma_n, n \geq 1,$$

be given where ϕ is a nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that it is positive on $\mathbb{R}^+ \setminus \{0\}$, $\phi(0) = 0$, and $\lim_{t \rightarrow \infty} \phi(t) = \infty$. Then $\lambda_n \rightarrow 0$.

Theorem SR ([11]). *Let H be a real Hilbert space. Let $A \subset H \times H$ be monotone with $cl(D(A))$, the closure of $D(A)$, convex and suppose that A satisfies the range condition: $cl(D(A)) \subset \mathcal{R}(I + rA) \forall r > 0$. Let $J_t x := (I + tA)^{-1}x$, $t > 0$ be the resolvent of A , and assume that $A^{-1}(0)$ is nonempty. Then for each $x \in H$, $\lim_{t \rightarrow \infty} J_t x = P_K x \in A^{-1}(0)$, where P_K is the metric projection from $cl(D(A))$ onto $A^{-1}(0)$.*

Theorem TZ ([12]). *Let H be a Hilbert space, and let $A : D(A) \subseteq H \rightarrow H$ be a maximal monotone mapping. Suppose that for some $x_0 \in D(A)$ and $r > 0$ we have $\|Ax_0\| < r \leq \liminf_{x \in D(A), \|x\| \rightarrow \infty} \|Ax\|$. Then $B_r(0) = \{x \in E : \|x\| < r\} \subseteq \mathcal{R}(A)$.*

Theorem CZ ([3]). Suppose K is a closed convex subset of a real Hilbert space H . Suppose $A : K \rightarrow E$ is a bounded uniformly monotone map and that the equation $Ax = 0$ has a solution. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$(2.2) \quad x_{n+1} := P_K(x_n - \alpha_n Ax_n), \quad n \geq 1,$$

where $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n \leq d_0$, then $\{x_n\}$ converges strongly to the unique solution of $Ax = 0$.

Remark 2.2. Theorems SR, TZ and CZ are, respectively, special cases of theorems proved in more general Banach spaces in Reich [11] (Theorem 1, Remarks 1 and 2), Takahashi and Zhang [12] and Chidume and Zegeye [3] (Theorem 3.8).

The following is elementary.

Lemma 2.3. Let H be a real Hilbert space. Let $E := H \times H$ with norm

$$\|z\|_E := \left(\|u\|_H^2 + \|v\|_H^2 \right)^{\frac{1}{2}}, \quad \text{where } z = [u, v].$$

Then E is a real Hilbert space and for $w_1 = [u_1, v_1], w_2 = [u_2, v_2] \in E$ we have that $\langle w_1, w_2 \rangle = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle$.

3. MAIN RESULTS

We first prove the following lemma.

Lemma 3.1. Let H be a real Hilbert space. Let $F : D(F) \subseteq H \rightarrow H$, $K : D(K) \subseteq H \rightarrow H$ be monotone mappings. Let $E := H \times H$ with norm $\|z\|_E^2 = \|u\|_H^2 + \|v\|_H^2$ for $z := [u, v] \in E$. Define a map $T : D(F) \times D(K) \rightarrow E$ by $Tz = T[u, v] := [Fu - v, u + Kv]$. Then for each $z_1, z_2 \in D(F) \times D(K)$ we have that

$$\langle Tz_1 - Tz_2, z_1 - z_2 \rangle \geq 0,$$

i.e., T is monotone. Moreover, if F and K are bounded, then T is bounded.

Proof. For $z_1 = [u_1, v_1]$ and $z_2 = [u_2, v_2]$ arbitrary elements in $D(F) \times D(K)$, we have $\langle z_1, z_2 \rangle_E = \langle u_1, u_2 \rangle_H + \langle v_1, v_2 \rangle_H$. Thus we get the following estimates:

$$\begin{aligned} \langle Tz_1 - Tz_2, z_1 - z_2 \rangle &= \langle Fu_1 - Fu_2 - (v_1 - v_2), u_1 - u_2 \rangle \\ &\quad + \langle Kv_1 - Kv_2 + (u_1 - u_2), v_1 - v_2 \rangle \\ &= \langle Fu_1 - Fu_2, u_1 - u_2 \rangle - \langle v_1 - v_2, u_1 - u_2 \rangle \\ &\quad + \langle Kv_1 - Kv_2, v_1 - v_2 \rangle + \langle u_1 - u_2, v_1 - v_2 \rangle \\ &\geq 0, \end{aligned}$$

since F and K are monotone. Moreover, the boundedness of T follows from the definition of T . This completes the proof. $\square$

Remark 3.2. The method of proof of Lemma 3.1 also yields that if F and K are uniformly monotone, then T is uniformly monotone with $\phi := \min\{\phi_1, \phi_2\}$, where ϕ_1 and ϕ_2 are the strictly increasing functions corresponding to F and K , respectively.

We shall need the following concept. Let D_1 and D_2 be subsets of a real Hilbert space, H . Let $F : D_1 \rightarrow H, K : D_2 \rightarrow H$ be monotone mappings. Then D_1 and D_2 are said to satisfy *property (P)* if $D_1 \subseteq (I + rF)(D_1) - D_2$ and $D_2 \subseteq (I + rK)(D_2) + D_1$, for each $r > 0$.

Remark 3.3. If F and K satisfy the range condition and if $D(F)$ and $D(K)$ contain the origin, then condition (P) is satisfied. Moreover, if $D(F) = D(K) = H$ and if K, F satisfy the range condition, then condition (P) is clearly satisfied.

Now we state and prove the following theorem.

Theorem 3.4. *Let H be a real Hilbert space. Let $F : D(F) \subseteq H \rightarrow H$, $K : D(K) \subseteq H \rightarrow H$ be bounded monotone mappings with $\mathcal{R}(F) \subseteq D(K)$, where $D(F)$ and $D(K)$ are closed convex subsets of H satisfying property (P). Suppose that the equation $0 = u + KF u$ has a solution in $D(F)$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in $(0, 1]$ satisfying the following conditions:*

- (i) $\lim_{n \rightarrow \infty} \theta_n = 0$;
- (ii) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$, $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\theta_n} = 0$;
- (iii) $\lim_{n \rightarrow \infty} \frac{(\frac{\theta_{n-1}}{\theta_n} - 1)}{\lambda_n \theta_n} = 0$. Let sequences $\{u_n\} \subseteq D(F)$ and $\{v_n\} \subseteq D(K)$ be generated from $u_0 \in D(F)$ and $v_0 \in D(K)$, respectively, by

$$\begin{aligned} u_{n+1} &= P_{D(F)}(u_n - \lambda_n(Fu_n - v_n + \theta_n(u_n - w_1))), \\ v_{n+1} &= P_{D(K)}(v_n - \lambda_n(Kv_n + u_n + \theta_n(v_n - w_2))), \end{aligned}$$

where $w_1 \in D(F)$, $w_2 \in D(K)$ are arbitrary but fixed. Then there exists $d > 0$ such that if $\lambda_n \leq d$ and $\frac{\lambda_n}{\theta_n} \leq d^2$ for all $n \geq 0$, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^* , respectively, in H , where u^* is a solution of the equation $0 = u + KF u$ and $v^* = Fu^*$.

Proof. Let $E := H \times H$ be endowed with norm $\|z\|_E^2 = \|u\|_H^2 + \|v\|_H^2$ for $z = [u, v] \in E$, and define the map $T : D(F) \times D(K) \rightarrow E$ by $Tz := T[u, v] = [Fu - v, Kv + u]$. Since K and F are bounded monotone mappings, we have by Lemma 3.1 that T is a bounded monotone mapping. Moreover, since $D(F)$ and $D(K)$ satisfy property (P), we have that T satisfies the range condition. Furthermore, we observe that u^* is a solution of $0 = u + KF u$ if and only if $z^* = [u^*, v^*]$ is a solution of $0 = Tz$, where $v^* = Fu^*$. Hence, $N(T) := \{z \in D(F) \times D(K) : Tz = 0\} \neq \emptyset$. Let $x^* \in N(T)$. For initial point $z_0 := [u_0, v_0]$, define the sequence $\{z_n\}$ by

$$(3.1) \quad z_{n+1} := P_{D(F) \times D(K)}(z_n - \lambda_n(Tz_n + \theta_n(z_n - w))),$$

for arbitrary but fixed $w = [w_1, w_2]$. One can show that $z_{n+1} = [u_{n+1}, v_{n+1}]$. In fact, it suffices to show that $P_{D(F) \times D(K)}[u, v] = [P_{D(F)}u, P_{D(K)}v]$. Observe that

$$\begin{aligned} \|P_{D(F) \times D(K)}[u, v] - [u, v]\|^2 &= \min_{[x, y] \in D(F) \times D(K)} \{\|[u, v] - [x, y]\|^2\} \\ &= \min_{[x, y] \in D(F) \times D(K)} \{\|[u - x, v - y]\|^2\} \\ &= \min_{[x, y] \in D(F) \times D(K)} \{\|u - x\|^2 + \|v - y\|^2\} \\ (3.2) \quad &= \min_{x \in D(F)} \{\|u - x\|^2\} + \min_{y \in D(K)} \{\|v - y\|^2\} \end{aligned}$$

$$\begin{aligned}
&= \|P_{D(F)}u - u\|^2 + \|P_{D(K)}v - v\|^2 \\
&= \|[P_{D(F)}u - u, P_{D(K)}v - v]\|^2 \\
&= \|[P_{D(F)}u, P_{D(K)}v] - [u, v]\|^2,
\end{aligned}$$

equality (3.2) following since $\|\cdot\|^2$ is a continuous and convex function. This implies that $\|P_{D(F) \times D(K)}[u, v] - [u, v]\| = \|[P_{D(F)}u, P_{D(K)}v] - [u, v]\|$. Then by the uniqueness of the projection we obtain that $P_{D(F) \times D(K)}[u, v] = [P_{D(F)}u, P_{D(K)}v]$. Let $r > 1$ be sufficiently large such that $z_0 \in B_r(x^*)$ and $w \in B_{\frac{r}{2}}(x^*)$. Set $M := 2r + \sup\{\|Tz_n\| : z_n \in \overline{B_r(x^*)}\}$.

Claim: $\{z_n\}$ is bounded. It suffices to show that $\{z_n\}$ belongs to $B = \overline{B_r(x^*)}$ for all positive integers. We do this by induction. Now, $z_0 \in B$ by assumption. Hence we may assume $z_n \in B$ and prove that $z_{n+1} \in B$. Suppose z_{n+1} is not in B . Then $\|z_{n+1} - x^*\| > r$, and thus from (3.1) we have that $\|z_{n+1} - x^*\| \leq \|z_n - x^* - \lambda_n(Tz_n + \theta_n(z_n - w))\| \leq \|z_n - x^*\| + \lambda_n\|Tz_n + \theta_n(z_n - w)\| \leq r + M$. Moreover, from (3.1) and the fact that $\theta_n \leq 1$, we get that

$$\begin{aligned}
\|z_{n+1} - x^*\|^2 &\leq \|z_n - x^* - \lambda_n(Tz_n + \theta_n(z_n - w))\|^2 \\
&\leq \|z_n - x^*\|^2 - 2\lambda_n\langle Tz_n, z_n - x^* \rangle - 2\lambda_n\theta_n\langle z_n - w, z_n - x^* \rangle \\
&\quad + \lambda_n^2\|Tz_n + \theta_n(z_n - w)\|^2 \\
&\leq \|z_n - x^*\|^2 - 2\lambda_n\langle Tz_n, z_n - x^* \rangle \\
(3.3) \quad &\quad - 2\lambda_n\theta_n\langle z_n - w, z_n - x^* \rangle + \lambda_n^2M.
\end{aligned}$$

Since T is monotone and $x^* \in N(T)$, we have $\langle Tz_n, z_n - x^* \rangle \geq 0$. Hence (3.3) gives

$$(3.4) \quad \|z_{n+1} - x^*\|^2 \leq \|z_n - x^*\|^2 - 2\lambda_n\theta_n\langle z_n - w, z_n - x^* \rangle + \lambda_n^2M.$$

Let $K > 0$ be sufficiently small such that $K \leq \frac{r^2}{4} \left(\frac{\sqrt{2}}{\sqrt{M+M\sqrt{2}}} \right)^2$, and let $d := \sqrt{K}$. Then since $\|z_{n+1} - x^*\| > \|z_n - x^*\|$ by our assumption, from (3.4) we get that $2\lambda_n\theta_n\langle z_n - w, z_n - x^* \rangle \leq \lambda_n^2M$, which gives $\langle z_n - w, z_n - x^* \rangle \leq \frac{\lambda_n M}{2\theta_n} \leq \frac{M}{2}K$, since $\frac{\lambda_n}{\theta_n} \leq K = d^2, \forall n \geq 1$ by our assumption. Now adding $\langle w - x^*, z_n - x^* \rangle$ to both sides of this inequality we get that

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \frac{M}{2}K + \langle w - x^*, z_n - x^* \rangle \leq \frac{M}{2}K + \|w - x^*\|\|z_n - x^*\| \\
&\leq \frac{M}{2}K + \frac{r}{2}\|z_n - x^*\|.
\end{aligned}$$

Solving this quadratic inequality for $\|z_n - x^*\|$ and using the estimate $\sqrt{\frac{r^2}{16} + \frac{M}{2}K} \leq \frac{r}{4} + \sqrt{\frac{M}{2}K}$, we obtain that $\|z_n - x^*\| \leq \frac{r}{2} + \sqrt{\frac{M}{2}K}$. But in any case, $\|z_{n+1} - x^*\| \leq \|z_n - x^*\| + \lambda_n\|Tz_n + \theta_n(z_n - w)\|$ so that $\|z_{n+1} - x^*\| \leq \frac{r}{2} + \sqrt{\frac{M}{2}K} + \lambda_n M \leq r$, by the original choices of K and λ_n , and this contradicts the assumption that z_{n+1} is not in B . Therefore $z_n \in B$ for all positive integers n . Thus $\{z_n\}$ is bounded. Now we show that $z_n \rightarrow y^*$, for some $y^* \in N(T)$. Since T satisfies the range condition, so does $\theta^{-1}T$ for $\theta > 0$. Thus for each $n \in \mathbb{N}$ there exists a unique $y_n \in D(T)$ such that

$$(3.5) \quad y_n = (I + \frac{1}{\theta_n}T)^{-1}(w).$$

From this and (3.1) we get that

$$\begin{aligned}
 \|z_{n+1} - y_n\|^2 &\leq \|z_n - y_n - \lambda_n(Tz_n + \theta_n(z_n - w))\|^2 \\
 &\leq \|z_n - y_n\|^2 - 2\lambda_n \langle Tz_n + \theta_n(z_n - w), z_n - y_n \rangle \\
 &\quad + \lambda_n^2 \|Tz_n + \theta_n(z_n - w)\|^2 \\
 &\leq (1 - 2\lambda_n \theta_n) \|z_n - y_n\|^2 - 2\lambda_n \langle Tz_n + \theta_n(y_n - w), z_n - y_n \rangle \\
 &\quad + \lambda_n^2 \|Tz_n + \theta_n(z_n - w)\|^2.
 \end{aligned}
 \tag{3.6}$$

Moreover, since from (3.5) we have that $\theta_n(w - y_n) = Ty_n$, we obtain that $\langle Tz_n + \theta_n(y_n - w), z_n - y_n \rangle \geq 0$. Furthermore, $\{z_n\}$ and hence $\{Tz_n\}$ are bounded. The sequence $\{y_n\}$ is also bounded, because it is convergent by Theorem SR. Thus, there exists $M_1 > 0$ such that $\|Tz_n + \theta_n(z_n - w)\| \leq M_1$. Therefore, inequality (3.6) gives

$$(3.7) \quad \|z_{n+1} - y_n\|^2 \leq (1 - 2\lambda_n \theta_n) \|z_n - y_n\|^2 + M_1 \lambda_n^2.$$

On the other hand, by the monotonicity of T we have that

$$\begin{aligned}
 \|y_{n-1} - y_n\| &\leq \|y_{n-1} - y_n + \frac{1}{\theta_n}(Ty_{n-1} - Ty_n)\| \\
 &\leq \frac{\theta_{n-1} - \theta_n}{\theta_n} (\|y_{n-1}\| + \|w\|) = \left(\frac{\theta_{n-1}}{\theta_n} - 1\right) (\|y_{n-1}\| + \|w\|).
 \end{aligned}$$

Therefore, this estimate with inequality (3.7) gives that

$$\|z_{n+1} - y_n\|^2 \leq (1 - 2\lambda_n \theta_n) \|z_n - y_{n-1}\|^2 + M_2 \left(\frac{\theta_{n-1}}{\theta_n} - 1\right) + M_2 \lambda_n^2,$$

for some constant $M_2 > 0$. Thus by Lemma 2.1, $z_{n+1} - y_n \rightarrow 0$. Hence, since by Theorem SR, $y_n \rightarrow y^* := [u^*, v^*] \in N(T)$ we have $z_n \rightarrow y^*$. But since $z_n = [u_n, v_n]$, this implies that $u_n \rightarrow u^*$ and $v_n \rightarrow v^*$. This completes the proof. $\square$

Corollary 3.5. *Let H be a real Hilbert space. Let $F, K : H \rightarrow H$ be bounded monotone mappings satisfying the range condition. Suppose the equation $0 = u + KF u$ has a solution in H . Let $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in $(0, 1]$ satisfying the following conditions:*

- (i) $\lim_{n \rightarrow \infty} \theta_n = 0$;
- (ii) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$, $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\theta_n} = 0$;
- (iii) $\lim_{n \rightarrow \infty} \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n \theta_n} = 0$.

Let $u_0, v_0 \in H$ be arbitrary, and let a sequence $\{u_n\}$ and $\{v_n\}$ be generated from u_0 and v_0 , respectively, by

$$\begin{aligned}
 u_{n+1} &= u_n - \lambda_n(Fu_n - v_n + \theta_n(u_n - w)), \\
 v_{n+1} &= v_n - \lambda_n(Kv_n + u_n + \theta_n(v_n - w)),
 \end{aligned}$$

where $w \in H$ is arbitrary but fixed. Then there exists $d > 0$ such that if $\lambda_n \leq d$ and $\frac{\lambda_n}{\theta_n} \leq d^2$ for all $n \geq 0$, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^ and v^* , respectively, in H , where u^* is a solution of the equation $0 = u + KF u$ and $v^* = Fu^*$.*

Theorem 3.6. *Let H be a real Hilbert space. Let $F, K : H \rightarrow H$ be bounded continuous monotone mappings. Suppose there exist $u^0, v^0 \in H$ such that*

$$\|F(u^0) - v^0\|^2 + \|Kv^0 + u^0\|^2 < r \leq \liminf_{u,v \in H, \|u\|, \|v\| \rightarrow \infty} (\|Fu - v\|^2 + \|Kv + u\|^2),$$

for some $r > 0$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in $(0, 1]$ satisfying the following conditions:

- (i) $\lim_{n \rightarrow \infty} \theta_n = 0$;
- (ii) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$, $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\theta_n} = 0$;
- (iii) $\lim_{n \rightarrow \infty} \frac{(\frac{\theta_{n-1}}{\theta_n} - 1)}{\lambda_n \theta_n} = 0$.

Let $u_0, v_0 \in H$ be arbitrary, and let a sequence $\{u_n\}$ and $\{v_n\}$ be generated from u_0 and v_0 , respectively, by

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n(Fu_n - v_n + \theta_n(u_n - w)), \\ v_{n+1} &= v_n - \lambda_n(Kv_n + u_n + \theta_n(v_n - w)), \end{aligned}$$

where $w \in H$. Then there exists $d > 0$ such that if $\lambda_n \leq d$ and $\frac{\lambda_n}{\theta_n} \leq d^2$ for all $n \geq 0$, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^ and v^* , respectively, in H , where u^* is a solution of the equation $0 = u + KFu$ and $v^* = Fu^*$.*

Proof. Since F and K are bounded continuous monotone mappings, we have that T defined as in the proof of Theorem 3.4 is a bounded continuous monotone mapping, and hence by Theorem 2 of [8], T satisfies the range condition. Clearly, T is maximal monotone. Moreover, for $z^0 = [u^0, v^0] \in H \times H$,

$$\begin{aligned} \|T(z^0)\| = \|T[u^0, v^0]\| &= \left(\|F(u^0) - v^0\|^2 + \|Kv^0 + u^0\|^2 \right)^{\frac{1}{2}} < r^{\frac{1}{2}} \\ &\leq \liminf_{u,v \in H, \|u\|, \|v\| \rightarrow \infty} \left(\|Fu - v\|^2 + \|Kv + u\|^2 \right)^{\frac{1}{2}} \\ &= \liminf_{\|z\| \rightarrow \infty} \|Tz\|. \end{aligned}$$

Thus, by Theorem TZ we have that $N(T) \neq \emptyset$, and hence the equation $0 = u + KFu$ has a solution in H . Therefore, by Theorem 3.4 we obtain that $z_n := [u_n, v_n] \rightarrow z^* = [u^*, v^*] \in N(T)$. The conclusion follows. $\square$

Remark 3.7. Real sequences that satisfy the hypotheses of the above theorems and corollaries are $\lambda_n = (n+1)^{-a}$ and $\theta_n = (n+1)^{-b}$ with $0 < b < a$ and $a+b < 1$.

Remark 3.8. If F and K are *uniformly* monotone mappings, we have simpler schemes that converge to the solution of the equation $u + KFu = 0$. In fact, we have the following theorem.

Theorem 3.9. *Let H be a real Hilbert space. Let $F : D(F) \subseteq H \rightarrow H$, $K : D(K) \subseteq H \rightarrow H$ be bounded uniformly monotone mappings with $\mathcal{R}(F) \subseteq D(K)$, where $D(F)$ and $D(K)$ are closed convex subsets of H . Suppose that the equation $0 = u + KFu$ has a solution in $D(F)$. Let $\lambda_n \in (0, 1)$ be such that $\lambda_n \rightarrow 0$ and $\sum \lambda_n = \infty$. Let sequences $\{u_n\} \subseteq D(F)$ and $\{v_n\} \subseteq D(K)$ be generated from*

$u_0 \in D(F)$ and $v_0 \in D(K)$, respectively, by

$$\begin{aligned} u_{n+1} &= P_{D(F)}\left(u_n - \lambda_n(Fu_n - v_n)\right), \\ v_{n+1} &= P_{D(K)}\left(v_n - \lambda_n(Kv_n + u_n)\right). \end{aligned}$$

Then there exists $d > 0$ such that if $\lambda_n \leq d$ for all $n \geq 0$, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^* , respectively, in H , where u^* is a solution of the equation $0 = u + KF u$ and $v^* = Fu^*$.

Proof. Since F and K are bounded uniformly monotone mappings, by Remark 3.2, T defined as in Theorem 3.4 is a bounded uniformly monotone mapping. Moreover, since $0 = u + KF u$ has a solution in $D(F)$, $N(T) \neq \emptyset$. Therefore, by Theorem CZ, $z_n := [u_n, v_n] \rightarrow z^* := [u^*, v^*]$, where u^* is a solution of the equation $0 = u + KF u$ and $v^* = Fu^*$. The conclusion follows. $\square$

Remark 3.10. The boundedness of F and K in our theorems can be dispensed with if one assumes instead that F and K are bounded on $\{u_n\}$ and $\{v_n\}$, respectively.

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