THE FEFFERMAN-STEIN TYPE INEQUALITY FOR THE KAKEYA MAXIMAL OPERATOR IN WOLFF'S RANGE

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ABSTRACT. Let K_δ , $0 < \delta \ll 1$, be the Kakeya (Nikodým) maximal operator defined as the supremum of averages over tubes of eccentricity δ . The (so-called) Fefferman-Stein type inequality:

$$\|K_\delta f\|_{L^p(\mathbf{R}^d, w)} \leq C(1/\delta)^{d/p-1} (\log(1/\delta))^\alpha \|f\|_{L^p(\mathbf{R}^d, K_\delta w)}$$

is shown in the range $1 < p \leq (d+2)/2$, where C and α are some constants depending only on p and the dimension d and w is a weight. The result is a sharp bound up to $\log(1/\delta)$ -factors.

1. INTRODUCTION

In this paper we will discuss the weighted version of the Kakeya problem in Wolff's range.¹ Throughout this paper δ , $0 < \delta \ll 1$, will be a small parameter. For a locally integrable function f on $\mathbf{R}^d$, $d > 1$, the Kakeya (Nikodým) maximal operator K_δ is defined by

$$K_\delta f(x) = \sup_{T \ni x} \frac{1}{|T|} \int_T |f| dy,$$

where the supremum is taken over all tubes T containing $x \in \mathbf{R}^d$ with the property that the eccentricity (the ratio of the radius of the cross section and the length of the axis) is equal to δ . A weight w is a non-negative locally integrable function on $\mathbf{R}^d$. We will represent the norm of the function space $L^p(\mathbf{R}^d, w)$ as

$$\|f\|_{L^p(\mathbf{R}^d, w)} = \left(\int_{\mathbf{R}^d} |f(x)|^p w(x) dx \right)^{1/p}, \quad p > 1.$$

$w(A)$ will denote the $w(x) dx$ measure of a set A .

It is conjectured that the (so-called) Fefferman-Stein type inequality

$$(1) \quad \|K_\delta f\|_{L^d(\mathbf{R}^d, w)} \leq C(\log(1/\delta))^\alpha \|f\|_{L^d(\mathbf{R}^d, K_\delta w)}$$

holds with some constants C and α depending only on the dimension d . This conjecture is known to be true for $d = 2$ (D. Müller and F. Soria, [2]) and for $d > 2$

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¹The developments on the Kakeya problem in recent years are summarized in [4] and [12].

and radial functions (H. Tanaka [6]), but is still open for other cases. This is a weighted version of the long-standing conjecture of A. Córdoba [1].

By using the Fefferman-Stein inequality for the Hardy-Littlewood maximal operator M and the simple majorizations $C_1(Mf) \leq K_\delta f \leq C_2(1/\delta)^{d-1}(Mf)$, one sees that

$$(2) \quad \lambda w(\{x : K_\delta f(x) > \lambda\}) \leq C(1/\delta)^{d-1} \|f\|_{L^1(\mathbf{R}^d, K_\delta w)}, \quad \forall \lambda > 0.$$

Interpolating the conjectured bound (1) with the bound (2), one obtains the equivalent conjecture

$$(3) \quad \|K_\delta f\|_{L^p(\mathbf{R}^d, w)} \leq C(1/\delta)^{d/p-1} (\log(1/\delta))^\alpha \|f\|_{L^p(\mathbf{R}^d, K_\delta w)}, \quad 1 < p \leq d.$$

The main difficulty of this problem lies in making the exponent p as high as possible.

The first achievement in this problem was given by A. Vargas [10], who showed that (3) is true in the range $1 < p \leq (d+1)/2$. Her proof was based on the weighted estimate for a “bush”. This result was improved by the author [7], who showed that (3) is true in the range $1 < p \leq (d^2 - 2)/(2d - 3)$. The proof was based on the weighted estimate for a “hairbrush”. In this paper we shall give a further improvement.

Theorem 1. *Equation (3) is true in the range $1 < p \leq (d+2)/2$.*

Table 2. *Known results for the upper bound of the range*

$d = 3$	$d = 4$	$d = d$	
2	5/2	$(d+1)/2$	<i>A. Vargas, 1994, [10],</i>
7/3	14/5	$(d^2 - 2)/(2d - 3)$	<i>H. Tanaka, 2001, [7],</i>
5/2	3	$(d+2)/2$	<i>our result,</i>
3	4	d	<i>conjecture.</i>

An unweighted case of Theorem 1 (putting $w \equiv 1$) corresponds to Wolff’s result (T. Wolff, [11]). However, our result is a sharp bound up to $\log(1/\delta)$ -factors.

In [8], the author showed a weaker result than that of Theorem 1. The reason why he could not reach Wolff’s range is that the inductive argument in [11], which relies on a simple scaling argument, seems difficult to apply to the weighted setting. Instead of this inductive argument we will use a weighted inequality for the Sogge auxiliary maximal function, which was used in the problem of the estimates for the Nikodým maximal function on three-dimensional curved spaces (C. Sogge, [3]).

Reduction of the problem. We make some reduction of the problem. We write $X \lesssim Y$, $Y \gtrsim X$ if there is a constant C such that $X \leq CY$. The constant C must depend only on the dimension d but may vary from line to line. We write further $X \sim Y$ if $X \lesssim Y$ and $X \gtrsim Y$.

If $\chi_{\mathcal{A}}$ is the indicator function of a compact set $\mathcal{A} \subset \mathbf{R}^d$, define the maximal function $(\chi_{\mathcal{A}})_\delta(x)$ by

$$(\chi_{\mathcal{A}})_\delta(x) = \sup_{T \ni x} \frac{|\mathcal{A} \cap T|}{|T|},$$

where the supremum is taken over all δ -tubes T that contain $x \in \mathbf{R}^d$ and whose axes make an angle less than $\pi/8$ with the d -th coordinate direction e_d . (See [11].) Hereafter, a δ -tube is the tube with length 1 and radius of cross section δ . Let $Q^d = (-1, 1)^d$ and $\tilde{Q}^d = (-1/2, 1/2)^d$. We divide Q^d into δ -cubes Q_i centered at $i \in \mathbf{I}$, where $\mathbf{I}$ is the set of lattice points with the δ -separation. Our result can then

be reduced to the following (discrete) proposition by using (sieve) arguments and the three-points interpolation lemma. (See [2], [10] and also [6].)

Proposition 3. *Let $d > 2$ and $\mathcal{A} \subset \tilde{Q}^d$. Then for any λ , $0 < \lambda < 1$, we have*

$$\lambda^{(d+2)/2} \delta^{(d-2)/2} \sum_{i \in I} w(Q_i) \lesssim (\log(1/\delta))^{(d+5)/2} K_\delta w(\mathcal{A}),$$

where $I = \{i \in \mathbf{I} : (\chi_{\mathcal{A}})_\delta(i) > \lambda\}$.

Lemmas. We will use the following lemmas. Lemma 4 corresponds to L^2 -bounds for the Sogge auxiliary maximal function.

Given any line L in $\mathbf{R}^d$ and a set $\mathcal{E} \subset \tilde{Q}^d$, define the maximal function $(\chi_{\mathcal{E}})_\delta^L(x)$ by

$$(\chi_{\mathcal{E}})_\delta^L(x) = \sup_{T \ni x} \frac{|\mathcal{E} \cap T|}{|T|},$$

where the supremum is taken over all δ -tubes T that contain x and whose axes intersect L .

Lemma 4. *Given σ , $2\delta < \sigma < 1$, and ρ , $\delta < \rho < \sigma/2$, if $\mathcal{E} \subset \{y : \rho < \text{dist}(y, L) < 2\rho\} \cap \tilde{Q}^d$, then for any λ , $0 < \lambda < 1$, we have*

$$\lambda^2 \sum_{i \in I} w(Q_i) \lesssim (\log(1/\delta))^2 (\sigma/\rho)^{d-2} K_\delta w(\mathcal{E}),$$

where $I = \{i \in \mathbf{I} \cap \{y : \text{dist}(y, L) < \sigma\} : (\chi_{\mathcal{E}})_\delta^L(i) > \lambda\}$.

This lemma is just a minor modification of Theorem 2 in [6]. We postpone the proof until the end of the next section.

Let $\mathcal{B}_{\geq \delta}$ be the class of all rectangles in $\mathbf{R}^d$ with the property that the ratio of the shortest and the longest sidelength is bigger than or equal to δ . The corresponding maximal function associated to this base $\mathcal{B}_{\geq \delta}$ will be denoted by $K_{\geq \delta}$.

Lemma 5 (Theorem 3 in [5]). *Given a weight w , there exist constants C_1 and C_2 depending only on the dimension d such that for any $x \in \mathbf{R}^d$*

$$C_1 K_\delta w(x) \leq K_{\geq \delta} w(x) \leq C_2 K_\delta w(x).$$

2. PROOF OF PROPOSITION 3

Write $W = K_\delta w$ and $l(\delta) = \log(1/\delta)$. Fix $\mathcal{A} \subset \tilde{Q}^d$ and λ , $0 < \lambda < 1$. Recall $I = \{i \in \mathbf{I} : (\chi_{\mathcal{A}})_\delta(i) > \lambda\}$. Then for every $i \in I$ we can select a δ -tube T_i , which contains i and whose axis makes an angle less than $\pi/8$ with e_d , such that

$$(4) \quad |\mathcal{A} \cap T_i| > |T_i| \lambda.$$

It follows from (4) and the Schwarz inequality that

$$\begin{aligned} (5) \quad & (\delta^{d-1} \lambda)^2 \left(\sum_{i \in I} w(Q_i) \right)^2 \\ & \lesssim \left(\sum_{i \in I} w(Q_i) |\mathcal{A} \cap T_i| \right)^2 \sim \left(\int_{\mathcal{A}} \sum_{i \in I} w(Q_i) \chi_{T_i} \right)^2 \\ & \lesssim \left(\int \left(\sum_{i \in I} w(Q_i) \chi_{T_i} \right)^2 W^{-1} \right) \cdot W(\mathcal{A}). \end{aligned}$$

We shall estimate the quantity

$$(6) \quad \int \left(\sum_{i \in I} w(Q_i) \chi_{T_i} \right)^2 W^{-1}$$

by using the bilinearity introduced in [9].

Let $L(T_i)$ be the axis of T_i . We represent the point $\omega_1(T_i)$ by $\omega_1(T_i) = \Pi \cap L(T_i)$, where $\Pi = \{x \in \mathbf{R}^d : x_d = 0\}$. We represent further the point $\omega_2(T_i)$ by $\omega_2(T_i) = (\xi_1, \dots, \xi_{d-1}, 0)$, where $(\xi_1, \dots, \xi_d) \in S^{d-1}$ is the direction of $L(T_i)$. We will identify $\omega_1(T_i)$ and $\omega_2(T_i)$ with the points of $\mathbf{R}^{d-1}$ in the usual manner.

Let $I_1 = \{\omega_1(T_i) : i \in I\}$ and $I_2 = \{\omega_2(T_i) : i \in I\}$. Then without the loss of generality we may assume that I_1 and I_2 have the δ -separations. We will need the following geometric observation:

Lemma 6. *We have*

$$|T_i \cap T_{i'}| \lesssim \frac{\delta^d}{|\omega_2(T_i) - \omega_2(T_{i'})| + \delta}.$$

Moreover,

$$|\omega_2(T_i) - \omega_2(T_{i'})| + \delta \gtrsim |\omega_1(T_i) - \omega_1(T_{i'})|$$

if $T_i \cap T_{i'} \neq \emptyset$.

This lemma is proved by using the fact that the axis of T_i makes an angle less than $\pi/8$ with e_d .

Let $Q^{2d-2} = (-1, 1)^{2d-2}$. We introduce a Whitney decomposition (see [9]). For each integer $j > 0$ such that $\delta \lesssim 2^{-j}$ we divide Q^{2d-2} into $2^{(2d-2)j}$ dyadic subcubes τ_k^j of sidelength 2^{1-j} . If τ_k^j and τ_l^j are two cubes with the same sidelength that are not adjacent but have adjacent parents, we say that these cubes are close and write $\tau_k^j \sim \tau_l^j$. We partition (6) as

$$(6) \sim \sum_j \sum_{k,l: \tau_k^j \sim \tau_l^j} \int \left(\sum_{i \in I: \omega_1(T_i) \times \omega_2(T_i) \in \tau_k^j} w(Q_i) \chi_{T_i} \right) \cdot \left(\sum_{i' \in I: \omega_1(T_{i'}) \times \omega_2(T_{i'}) \in \tau_l^j} w(Q_{i'}) \chi_{T_{i'}} \right) W^{-1}.$$

Observing the terms of j and l with the maximum values in the above expression, we can select j_0 and k' so that

$$(7) \quad (6) \lesssim l(\delta) \sum_k \int \left(\sum_{i \in I: \omega_1(T_i) \times \omega_2(T_i) \in \tau_k^{j_0}} w(Q_i) \chi_{T_i} \right) \cdot \left(\sum_{i' \in I: \omega_1(T_{i'}) \times \omega_2(T_{i'}) \in \tau_{k'}^{j_0}} w(Q_{i'}) \chi_{T_{i'}} \right) W^{-1}.$$

We notice that the separation of $\tau_k^{j_0}$ and $\tau_{k'}^{j_0}$ is $O(2^{-j_0})$. Observing the terms with the maximum value again, we can select T_{i_k} so that

$$(8) \quad (7) \lesssim l(\delta) \sum_k \sum_{i \in I: \omega_1(T_i) \times \omega_2(T_i) \in \tau_k^{j_0}} w(Q_i) \cdot \int_{T_{i_k}} \left(\sum_{i' \in I: \omega_1(T_{i'}) \times \omega_2(T_{i'}) \in \tau_{k'}^{j_0}} w(Q_{i'}) \chi_{T_{i'}} \right) W^{-1}.$$

Set

$$\begin{cases} I_k = \{i \in I : \omega_1(T_i) \times \omega_2(T_i) \in \tau_k^{j_0}\}, \\ I'_k = \{i' \in I : \omega_1(T_{i'}) \times \omega_2(T_{i'}) \in \tau_{k'}^{j_0} \text{ and } T_{i_k} \cap T_{i'} \neq \emptyset\}, \\ s(T_{i_k}) = \inf_{y \in T_{i_k}} W(y), \\ \sigma = 2^{-j_0} \gtrsim \delta. \end{cases}$$

Then it follows from Lemma 6 that

$$(9) \quad (8) \lesssim l(\delta) \frac{\delta^d}{\sigma} \sum_k \sum_{i \in I_k} w(Q_i) \left(s(T_{i_k})^{-1} \sum_{i' \in I'_k} w(Q_{i'}) \right).$$

The following observation is due to Sogge [3]. This will allow us to avoid the inductive argument in [11].

By the pigeon-hole principle we can select ρ , $\delta < \rho < \sigma/2$, and $I''_k \subset I'_k$, so that

$$(10) \quad (9) \lesssim l(\delta)^2 \frac{\delta^d}{\sigma} \sum_k \sum_{i \in I_k} w(Q_i) \left(s(T_{i_k})^{-1} \sum_{i' \in I''_k} w(Q_{i'}) \right)$$

and

$$(11) \quad |\mathcal{A} \cap T_{i'} \cap \{y : \rho < \text{dist}(y, L(T_{i_k})) < 2\rho\}| > c_1 l(\delta)^{-1} |T_{i'}| \lambda, \quad \forall i' \in I''_k.$$

Let $N_k = s(T_{i_k})^{-1} \sum_{i' \in I''_k} w(Q_{i'})$. If

$$(6) \lesssim l(\delta)^2 \frac{\delta^d}{\sigma} \sum_{k: N_k \lesssim \delta^{d/2}} N_k \sum_{i \in I_k} w(Q_i),$$

then we have

$$(6) \lesssim l(\delta)^2 \delta^{d-1} \delta^{d/2} \sum_{i \in I} w(Q_i).$$

Inserting this expression into (5), we have

$$\lambda^2 \delta^{(d-2)/2} \sum_{i \in I} w(Q_i) \lesssim l(\delta)^2 W(\mathcal{A}).$$

Thus, we may assume that $N_k \gtrsim \delta^{d/2}$. We see also that $N_k \lesssim 1$ by Lemma 5. From these facts and the pigeon-hole principle again, there exists a number N such that

$$(12) \quad (6) \lesssim l(\delta)^3 \frac{\delta^d}{\sigma} \sum_{k: N \leq N_k \leq 2N} N_k \sum_{i \in I_k} w(Q_i).$$

Put $K = \{k : N \leq N_k \leq 2N\}$. Then we now have the following claim.

Claim 7. *We have*

$$(13) \quad (6) \lesssim l(\delta)^3 \frac{\delta^d}{\sigma} N \sum_{i \in I} w(Q_i).$$

To proceed, we shall estimate the quantity

$$(14) \quad \sum_{k \in K} N_k \sum_{i \in I_k} w(Q_i).$$

Let $T_{i_k}^\sigma$ be the tubes concentric with T_{i_k} and with length ~ 1 , the radius of the cross section $\sim \sigma$. We now observe the geometric fact that

$$\bigcup_{i \in I_k} Q_i \subset T_{i_k}^\sigma.$$

From the definition of N_k , this observation and Lemma 5,

$$\begin{aligned}
 (15) \quad N_k \sum_{i \in I_k} w(Q_i) & \lesssim \sigma^{d-1} s(T_{i_k})^{-1} \frac{w(T_{i_k}^\sigma)}{|T_{i_k}^\sigma|} \sum_{i' \in I_k''} w(Q_{i'}) \\
 & \lesssim \sigma^{d-1} \sum_{i' \in I_k''} w(Q_{i'}).
 \end{aligned}$$

Define the cylindric sets

$$D_k^\rho = T_{i_k}^\sigma \cap \{y : \rho < \text{dist}(y, L(T_{i_k})) < 2\rho\}.$$

Applying Lemma 4 with $\mathcal{E} = \mathcal{A} \cap D_k^\rho$ and $L = L(T_{i_k})$, we obtain that

$$(16) \quad \lambda^2 \sum_{i' \in I_k''} w(Q_{i'}) \lesssim l(\delta)^4 (\sigma/\rho)^{d-2} W(\mathcal{A} \cap D_k^\rho)$$

by (11). From (15) and (16),

$$(17) \quad (14) \lesssim (1/\lambda)^2 l(\delta)^4 \sigma^{d-1} (\sigma/\rho)^{d-2} \sum_{k \in K} W(\mathcal{A} \cap D_k^\rho).$$

The inequalities (12), (14) and (17) imply

$$(18) \quad (6) \lesssim (1/\lambda)^2 \delta^d l(\delta)^7 \sigma^{d-2} (\sigma/\rho)^{d-2} \sum_{k \in K} W(\mathcal{A} \cap D_k^\rho).$$

The final step is a bush-type analysis.

Rewrite

$$\sum_{k \in K} W(\mathcal{A} \cap D_k^\rho) = \int_{\mathcal{A}} \left(\sum_{k \in K} \chi_{D_k^\rho} \right) W.$$

Observing the points with the maximum value, we can select a point $a \in \mathcal{A}$ so that

$$(19) \quad W(\mathcal{A})^{-1} \sum_{k \in K} W(\mathcal{A} \cap D_k^\rho) \lesssim \text{card}(\{k \in K : a \in D_k^\rho\}).$$

Putting $s = \min_{k \in K} s(T_{i_k})$, we have

$$(20) \quad Ns \lesssim \sum_{i' \in I_k''} w(Q_{i'}), \quad k \in K.$$

Applying Lemma 4 with $\mathcal{E} = \mathcal{A} \cap D_k^\rho \setminus B(a, c_1 l(\delta)^{-1} \lambda/3)$ and $L = L(T_{i_k})$, we obtain that

$$(21) \quad \lambda^2 \sum_{i' \in I_k''} w(Q_{i'}) \lesssim l(\delta)^4 (\sigma/\rho)^{d-2} W(\mathcal{A} \cap D_k^\rho \setminus B(a, c_1 l(\delta)^{-1} \lambda/3))$$

by (11). The inequalities (20) and (21) imply the following claim.

Claim 8. *We have*

$$(22) \quad Ns \lesssim (1/\lambda)^2 l(\delta)^4 (\sigma/\rho)^{d-2} W(\mathcal{A} \cap D_k^\rho \setminus B(a, c_1 l(\delta)^{-1} \lambda/3)), \quad k \in K.$$

From (19), (22) and the fact that the set $\{D_k^\rho : a \in D_k^\rho, k \in K\}$ has the σ -separation in the direction sides

$$\begin{aligned} & W(\mathcal{A})^{-1} \sum_{k \in K} W(\mathcal{A} \cap D_k^\rho) \\ & \lesssim (Ns)^{-1} (1/\lambda)^2 l(\delta)^4 (\sigma/\rho)^{d-2} \sum_{k \in K: a \in D_k^\rho} W(\mathcal{A} \cap D_k^\rho \setminus B(a, c_1 l(\delta)^{-1} \lambda/3)) \\ & \lesssim (Ns)^{-1} (1/\lambda)^{d+1} l(\delta)^{d+3} (\rho/\sigma) W(\mathcal{A}). \end{aligned}$$

Hence we reach that

$$(23) \quad \sum_{k \in K} W(\mathcal{A} \cap D_k^\rho) \lesssim (Ns)^{-1} (1/\lambda)^{d+1} l(\delta)^{d+3} (\rho/\sigma) W(\mathcal{A})^2,$$

and we have the following claim by (18) and (23).

Claim 9. *We have*

$$(24) \quad (6) \lesssim (Ns)^{-1} (1/\lambda)^{d+3} \delta^d l(\delta)^{d+10} \sigma^{d-2} (\sigma/\rho)^{d-3} W(\mathcal{A})^2.$$

Completion of the proof. First, we notice the fact that $\sigma/\rho \lesssim l(\delta)(1/\lambda)$ by (11) and a simple geometric observation.

$$\begin{aligned} \underline{\text{CASE 1:}} \text{ If } \sigma \geq s^{-1/(d-1)} \left(\sum_{i \in I} w(Q_i) \right)^{1/(d-1)}, \text{ then (5), (13) and (22) yield} \\ \sigma s \lambda^{d+2} \delta^{d-2} \sum_{i \in I} w(Q_i) \lesssim l(\delta)^{d+5} W(\mathcal{A})^2, \end{aligned}$$

and hence

$$(25) \quad \lambda^{(d+2)/2} \delta^{(d-2)/2} s^{(d-2)/(2d-2)} \left(\sum_{i \in I} w(Q_i) \right)^{d/(2d-2)} \lesssim l(\delta)^{(d+5)/2} W(\mathcal{A}).$$

$$\underline{\text{CASE 2:}} \text{ If } \sigma \leq s^{-1/(d-1)} \left(\sum_{i \in I} w(Q_i) \right)^{1/(d-1)}, \text{ then (13) and (24) yield}$$

$$(6) \lesssim s^{-1/2} \left(\sum_{i \in I} w(Q_i) \right)^{1/2} (1/\lambda)^d \delta^d l(\delta)^{d+5} \sigma^{(d-3)/2} W(\mathcal{A}),$$

and hence we have the same inequality as (25) by (5).

Lastly, we notice the fact that $\sum_{i \in I} w(Q_i) \lesssim s$ by Lemma 5. Thus, we obtain the desired inequality

$$\lambda^{(d+2)/2} \delta^{(d-2)/2} \sum_{i \in I} w(Q_i) \lesssim l(\delta)^{(d+5)/2} W(\mathcal{A}). \quad \square$$

Remark 10. If we put $w \equiv 1$ in (25), then we recover the $L^p \rightarrow L^q$ bound of Wolff. But, our result is a sharp bound up to $\log(1/\delta)$ -factors.

Proof of Lemma 4. Let $F = \{y : \rho < \text{dist}(y, L) < 2\rho\} \cap \tilde{Q}^d$ and for $\mu, \delta < \mu < \sigma/2$, let $J = \{i \in \mathbf{I} \cap \{y : \mu < \text{dist}(y, L) < 2\mu\} : (\chi_{\mathcal{E}})_\delta^L(i) > \lambda\}$. We first notice that the lemma clearly follows from the dyadic estimate

$$(26) \quad \lambda^2 \sum_{i \in J} w(Q_i) \lesssim l(\delta)(\sigma/\rho)^{d-2} K_\delta w(\mathcal{E}).$$

Just the same as at the beginning of this section, for every $i \in J$ we can select a δ -tube T_i , which contains i and whose axis intersects L , such that $|\mathcal{E} \cap T_i| > |T_i| \lambda$ and we have

$$(27) \quad (\delta^{d-1} \lambda)^2 \left(\sum_{i \in J} w(Q_i) \right)^2 \lesssim \left(\int_F \left(\sum_{i \in J} w(Q_i) \chi_{T_i} \right)^2 (K_\delta w)^{-1} \right) \cdot K_\delta w(\mathcal{E}).$$

We shall estimate the quantity

$$(28) \quad \int_F \left(\sum_{i \in J} w(Q_i) \chi_{T_i} \right)^2 (K_\delta w)^{-1}.$$

Observing the terms with the maximum value, we can select T_{i_0} so that

$$(29) \quad (28) \lesssim \left(\sum_{i \in J} w(Q_i) \right) \int_{F \cap T_{i_0}} \left(\sum_{i \in J} w(Q_i) \chi_{T_i} \right) (K_\delta w)^{-1}.$$

Putting $s_0 = \inf_{y \in F \cap T_{i_0}} K_\delta w(y)$, we see

$$(30) \quad \int_{F \cap T_{i_0}} \left(\sum_{i \in J} w(Q_i) \chi_{T_i} \right) (K_\delta w)^{-1} \lesssim s_0^{-1} \int_{F \cap T_{i_0}} \left(\sum_{i \in J} w(Q_i) \chi_{T_i} \right).$$

Now we claim that

$$(31) \quad \int_{F \cap T_{i_0}} \left(\sum_{i \in J} w(Q_i) \chi_{T_i} \right) \lesssim \delta^{2d-2} l(\delta)(\sigma/\rho)^{d-2} s_0.$$

Clearly (31) with (27)–(30) imply (26). We shall divide the proof into two cases.

THE CASE $\mu \geq \rho$: Choose two-planes Π_k containing L so that any point $i \in J$ belongs to at most ~ 1 Π_k^δ 's, where Π_k^δ is the δ -neighbourhood of Π_k . We partition the left-hand side of (31) as

$$\sum_k \int_{F \cap T_{i_0}} \left(\sum_{i \in J \cap \Pi_k^\delta} w(Q_i) \chi_{T_i} \right).$$

By a simple geometric observation for every k and $i \in J \cap \Pi_k^\delta$, we can select $\tilde{\Pi}_k^\delta$ of the $(\sim \delta)$ -neighbourhood of Π_k so that $F \cap T_i \subset \tilde{\Pi}_k^\delta$ and $Q_i \subset \tilde{\Pi}_k^\delta$. We can also select a δ -tube $\tilde{T}_{i_0}$ so that $F \cap T_{i_0} \subset \tilde{T}_{i_0} \subset \tilde{\Pi}_k^\delta$. We see further that the multiplicity of $\tilde{\Pi}_k^\delta$'s on $F \cap T_{i_0}$ is at most $\sim (\mu/\rho)^{d-2}$.

For $l, m \in \mathbf{Z} \cap [1, \nu]$, $\nu \sim (1/\delta)$, let $\gamma_l = \{y \in \Pi_k^\delta : (l-1)\delta \leq \text{dist}(y, \tilde{T}_{i_0}) < l\delta\}$ and $\Gamma_m = \{y \in \tilde{\Pi}_k^\delta : \text{dist}(y, \tilde{T}_{i_0}) < m\delta\}$. Then we have

$$\begin{aligned}
& \int_{F \cap T_{i_0}} \left(\sum_{i \in J \cap \Pi_k^\delta} w(Q_i) \chi_{T_i} \right) \\
& \sim \sum_{l=1}^{\nu} \int_{F \cap T_{i_0}} \left(\sum_{i \in J \cap \gamma_l} w(Q_i) \chi_{T_i} \right) \\
& \lesssim \sum_{l=1}^{\nu} (\delta^{d-1}/l) \sum_{i \in J \cap \gamma_l} w(Q_i) \\
& \sim \delta^{d-1} \sum_{l=1}^{\nu} \left(\sum_{m=l}^{\nu} \frac{1}{m(m+1)} + \frac{1}{\nu+1} \right) \sum_{i \in J \cap \gamma_l} w(Q_i) \\
& \lesssim \delta^{2d-2} \left(\sum_{m=1}^{\nu} \frac{1}{m+1} \frac{w(\Gamma_m)}{|\Gamma_m|} + \frac{w(\Gamma_\nu)}{|\Gamma_\nu|} \right) \\
& \lesssim \delta^{2d-2} l(\delta) s_0,
\end{aligned}$$

where we used Lemma 5 and Córdoba's observation: $|\tilde{T}_{i_0} \cap T_i| \lesssim \frac{\delta^d}{\text{dist}(i, \tilde{T}_{i_0}) + \delta}$. Thus, we obtain (31) in this case.

THE CASE $\mu < \rho$: Let $\tilde{\Pi}_\delta$ be the $(\sim \delta)$ -neighbourhood of the two-plane determined by $\tilde{L}$ and the axis of T_{i_0} . By a simple geometric observation we notice that only the set that contributes to the left-hand side of (31) is $\{i : i \in J \cap \tilde{\Pi}_\delta\}$. Thus, we have (31) by using the same inequality as (32).

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