MULTIPLICITY RESULTS FOR A CLASS OF SUPERLINEAR ELLIPTIC PROBLEMS

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ABSTRACT. We study a class of superlinear elliptic problems $-\Delta u = \lambda f(u)$ under the Dirichlet boundary condition on a bounded smooth domain in $\mathbb{R}^N$. Assuming that the nonlinearity $f(u)$ is *superlinear* in a neighborhood of $u = 0$, we study the dependence of the number of signed and sign-changing solutions on the parameter λ .

INTRODUCTION

In this paper we consider the question of multiplicity of both signed and sign-changing solutions for the one-parameter family of elliptic problems (P_λ)

$$\begin{cases} -\Delta u &= \lambda f(u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter, Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 3$), and the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function of class C^1 satisfying the following conditions:

- (f_1) there exists $\gamma \in (2, 2^*)$ such that $\limsup_{|u| \rightarrow 0} \frac{f(u)u}{|u|^\gamma} < +\infty$,
- (f_2) there exists $\beta \in (2, 2^*)$ such that $\liminf_{|u| \rightarrow 0} \frac{F(u)}{|u|^\beta} > 0$,
- (f_3) there exists $\mu \in (2, 2^*)$ such that $uf(u) \geq \mu F(u) > 0$ for $0 \neq |u|$ small,
- (f_4) $f(-u) = -f(u) \quad \forall |u| \leq \delta$ (for some $\delta > 0$).

Here $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $F(u) = \int_0^u f(t)dt$. As is well known, if f were assumed to be *superlinear at infinity* (in the sense that (f_1) and (f_2) hold as $|u| \rightarrow \infty$ and (f_3) holds for $|u|$ large), then the associated *energy functional*

$$(0.1) \quad I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} F(u) dx$$

would be of class C^1 on $H_0^1(\Omega)$ and satisfy the Palais-Smale condition, with its critical points being precisely the solutions of problem (P_λ) . Here, the assumptions (f_1) – (f_3) we make on the nonlinearity $f(u)$ refer solely to its behavior in a neighborhood of $u = 0$, and we will show that they suffice for the existence of three solutions of problem (P_λ) when λ is large. In addition, if $f(u)$ is odd near $u = 0$, the number of sign-changing solutions of (P_λ) gets arbitrarily large together with

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λ . With global superlinear conditions these results were known in [1, 3, 4, 5]. More precisely, we will prove the following results.

Theorem 1. *Assume conditions $(f_1) - (f_3)$ are satisfied. Then there exists $\Gamma > 0$ depending only on γ such that, if $(\beta - \gamma)\Gamma < 1$ is satisfied, problem (P_λ) has at least one positive solution, one negative solution, and a sign-changing solution for all λ sufficiently large.*

Theorem 2. *Assume conditions $(f_1) - (f_4)$ are satisfied. Then there exists $\Gamma > 0$ depending only on γ such that, if $(\beta - \gamma)\Gamma < 1$ is satisfied then, for any given $k \geq 1$, problem (P_λ) has k pairs of solutions $\pm v_i$, $i = 1, \dots, k$, with $|v_i|_\infty \leq \delta$, provided λ is sufficiently large. Moreover, $\pm v_i$ for $i = 2, \dots, k$, are sign-changing solutions.*

Our approach is inspired by the results of Costa-Tehrani [2] and is based on the fact that we can show an a priori bound of the form

$$|u|_\infty \leq C\lambda^{-\epsilon}, \quad \epsilon > 0,$$

for a class of solutions of (P_λ) with energy estimates given by minimax methods. The energy estimates for sign-changing solutions rely upon the minimax procedure of Li-Wang [4] for constructing nodal critical points.

The organization of this paper is as follows. Section 1 is reserved for setting the framework and establishing some preliminary results. Theorems 1 and 2 are proved in Section 2.

1. PRELIMINARY RESULTS

Throughout the paper we denote the H_0^1 -norm by $\|\cdot\|$ and the L^r -norm by $|\cdot|_r$, $1 \leq r \leq \infty$. Also, sometimes we denote various positive constants by the same letter C_* .

We start by observing that (f_2) and (f_1) imply the existence of constants $C_0, C_1 > 0$ such that

$$(1.1) \quad F(u) \geq C_0|u|^\beta,$$

$$(1.2) \quad F(u) \leq C_1|u|^\gamma,$$

for $|u|$ small. Now, let $\rho(t)$ be an even cut-off function satisfying $\rho(t) \equiv 1$ if $|t| \leq \delta$, $\rho(t) \equiv 0$ if $|t| \geq 2\delta$, $t\rho'(t) \leq 0$, and $|t\rho'(t)| \leq \frac{2}{\delta}$, where $0 < \delta < \frac{1}{2}$ is chosen such that (1.1), (1.2) and (f_3) hold for $|u| \leq 2\delta$.

Lemma 1.1. *Define $\tilde{F}(u) = \rho(u)F(u) + (1 - \rho(u))F_\infty(u)$, where $F_\infty(u) := C_1|u|^\gamma$. Then*

$$u\tilde{F}'(u) \geq \theta\tilde{F}(u) > 0$$

for all $u \neq 0$, where $\theta = \min\{\mu, \gamma\}$.

Proof. We have

$$(1.3) \quad \tilde{F}'(u) = \rho(u)f(u) + (1 - \rho(u))F'_\infty(u) + \rho'(u)F(u) - \rho'(u)F_\infty(u)$$

so that

$$\begin{aligned}
 \theta \tilde{F}(u) &= \theta \rho(u) F(u) + \theta(1 - \rho(u)) F_\infty(u) \\
 &\leq \rho(u) \frac{\theta}{\mu} f(u) u + (1 - \rho(u)) \frac{\theta}{\gamma} F'_\infty(u) u \\
 &+ \rho'(u) u F(u) - \rho'(u) u F_\infty(u) - \rho'(u) u F(u) + \rho'(u) u F_\infty(u) \\
 &\leq u \tilde{F}'(u) + \rho'(u) u (F_\infty(u) - F(u)) \leq u \tilde{F}'(u) .
 \end{aligned}$$

□

Now let us consider the modified equation of (P_λ) given by $(\tilde{P}_\lambda)$

$$\begin{cases} -\Delta u &= \lambda \tilde{F}'(u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega . \end{cases}$$

The corresponding functional

$$\tilde{I}_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega \tilde{F}(u) dx , \quad u \in E := H_0^1(\Omega) ,$$

is of class C^2 , and its critical points are the solutions of $(\tilde{P}_\lambda)$. We note that critical points of $(\tilde{I}_\lambda)$ with L^∞ -norm less than or equal to δ are also solutions of the original problem (P_λ) .

Lemma 1.2. *The functional $\tilde{I}_\lambda$ satisfies (PS) on E .*

Proof. This is standard in view of Lemma 1.1 and the fact that $\gamma < 2^*$. □

Lemma 1.3. *Let $u \in E$ be a critical point of $\tilde{I}_\lambda$. Then*

$$(1.4) \quad \|u\|^2 \leq \frac{2\theta}{\theta-2} \tilde{I}_\lambda(u) .$$

Proof. This estimate readily follows from Lemma 1.1 and

$$\frac{1}{2} \|u\|^2 - \lambda \int_\Omega \tilde{F}(u) dx = \tilde{I}_\lambda(u) , \quad \|u\|^2 - \lambda \int_\Omega \tilde{F}'(u) u dx = 0 .$$

□

Now, let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ and $\phi_1, \phi_2, \phi_3, \dots$ denote the eigenvalues and corresponding eigenfunctions of $-\Delta$ on $H_0^1(\Omega)$. Also, for $k = 1, 2, \dots$, let

$$Y_k := \text{span} \{ \phi_1, \dots, \phi_k \} , \quad Z_k := \overline{\text{span}} \{ \phi_k, \phi_{k+1}, \dots \} .$$

Lemma 1.4. *Let $b_{k,\lambda} = \sup_{u \in Y_k} \tilde{I}_\lambda(u)$. Then*

$$(1.5) \quad b_{k,\lambda} \leq C_* C_k \lambda^{-\frac{2}{\beta-2}}$$

where $C_* > 0$ depends only on γ, β and Ω and $C_k := \lambda_k^{\frac{\beta}{\beta-2}} + \lambda_k^{\frac{\gamma}{\gamma-2}}$.

Proof. We recall that $\delta > 0$ was chosen so that

$$(1.6) \quad \tilde{F}(u) \geq F(u) \geq C_0 |u|^\beta \quad \text{for } |u| \leq 2\delta ,$$

$$(1.7) \quad \tilde{F}(u) = F_\infty(u) = C_1 |u|^\gamma \quad \text{for } |u| \geq 2\delta .$$

For $u \in E$, denote $\Omega_1 = \{x \mid |u| \geq 2\delta\}$, $\Omega_2 = \{x \mid |u| < 2\delta\}$, and let $u_1 = u|_{\Omega_1}$, $u_2 = u|_{\Omega_2}$. In view of Hölder's inequality we obtain

$$\begin{aligned} \int_{\Omega} \tilde{F}(u_2) \, dx &\geq C_* |u_2|_2^\beta, \\ \int_{\Omega} \tilde{F}(u_1) \, dx &\geq C_* |u_1|_2^\gamma, \end{aligned}$$

so that, for $u \in Y_k$, it follows that

$$\begin{aligned} \tilde{I}_\lambda(u) &\leq \frac{\lambda_k}{2} |u|_2^2 - \lambda C_* |u_1|_2^\gamma - \lambda C_* |u_2|_2^\beta \\ &= \frac{\lambda_k}{2} |u_1|_2^2 - \lambda C_* |u_1|_2^\gamma + \frac{\lambda_k}{2} |u_2|_2^2 - \lambda C_* |u_2|_2^\beta \\ &\leq C_* \lambda_k^{\frac{\gamma}{\gamma-2}} \lambda^{-\frac{2}{\gamma-2}} + C_* \lambda_k^{\frac{\beta}{\beta-2}} \lambda^{-\frac{2}{\beta-2}}. \end{aligned}$$

Since $2 < \gamma \leq \beta < 2^*$, we obtain the following estimate:

$$(1.8) \quad b_{k,\lambda} \leq C_* C_k \lambda^{-\frac{2}{\beta-2}},$$

where $C_k := \lambda_k^{\frac{\beta}{\beta-2}} + \lambda_k^{\frac{\gamma}{\gamma-2}}$. □

2. PROOFS OF THE THEOREMS

As is well known, $(\tilde{P}_\lambda)$ has a positive solution $u_{1,\lambda}$ and a negative solution $u_{2,\lambda}$ obtained through an application of the Mountain-Pass Theorem [1] to the functionals

$$\tilde{I}_{1,\lambda}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \lambda \int_{\Omega} \tilde{F}_1(u) \, dx, \quad u \in E := H_0^1(\Omega)$$

and

$$\tilde{I}_{2,\lambda}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \lambda \int_{\Omega} \tilde{F}_2(u) \, dx, \quad u \in E := H_0^1(\Omega),$$

respectively, where $\tilde{F}_1(u) = \tilde{F}(u)$ if $u > 0$, $\tilde{F}_1(u) \equiv 0$ if $u \leq 0$, and $\tilde{F}_2(u) = \tilde{F}(u)$ if $u < 0$, $\tilde{F}_2(u) \equiv 0$ if $u \geq 0$. The corresponding critical values are given by

$$\tilde{d}_{j,\lambda} = \inf_{h \in \Gamma_j} \sup_{0 \leq t \leq 1} \tilde{I}_{j,\lambda}(h(t)), \quad j = 1, 2,$$

where $\Gamma_j = \{h \in C([0, 1], E) \mid h \neq 0, \tilde{I}_{j,\lambda}(h(1)) \leq 0\}$. It is clear that

$$(2.1) \quad \tilde{d}_{j,\lambda} \leq d_{j,\lambda}$$

where

$$d_{1,\lambda} := \inf_{u > 0} \sup_{0 \leq t < \infty} \tilde{I}_\lambda(tu), \quad d_{2,\lambda} := \inf_{u < 0} \sup_{0 \leq t < \infty} \tilde{I}_\lambda(tu).$$

Lemma 2.1. $\tilde{d}_{j,\lambda} \leq C_* \lambda^{-\frac{2}{\beta-2}}$ for $j = 1, 2$ and $\lambda > 0$ large.

Proof. From Lemma 1.3 and (2.1), we have the estimates

$$(2.2) \quad \|u_{j,\lambda}\| \leq C_* \sqrt{\tilde{d}_{j,\lambda}}, \quad j = 1, 2.$$

Let us consider $u_{1,\lambda}$ since the same argument applies to $u_{2,\lambda}$. Recalling from (1.6) that $\tilde{F}(u) \geq C_0 |u|^\beta$ for $|u| \leq 2\delta$ and defining

$$J_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \lambda \int_{\Omega} C_0 |u|^\beta \, dx,$$

we have

$$(2.3) \quad \begin{aligned} d_{1,\lambda} &= \inf_{u>0} \sup_{0 \leq t < \infty} \tilde{I}_\lambda(tu) \leq \inf_{0 < u \leq 2\delta} \sup_{0 \leq t < \infty} \tilde{I}_\lambda(tu) \\ &\leq \inf_{0 < u \leq 2\delta} \sup_{0 \leq t < \infty} J_\lambda(tu) = \inf_{u \neq 0} \sup_{0 \leq t < \infty} J_\lambda(tu), \end{aligned}$$

where the last equality holds provided $\lambda > 0$ is sufficiently large so that the *ground states* of $J_\lambda(u)$ have L^∞ -norm less than 2δ . In fact, we obtain through straightforward calculations as in [2, Thm. 1.1] that

$$\inf_{u \neq 0} \sup_{0 \leq t < \infty} J_\lambda(tu) \leq C_* \lambda^{-\frac{2}{\beta-2}}.$$

Therefore, keeping in mind (2.3) and (2.2) above, we conclude that

$$(2.4) \quad d_{1,\lambda} \leq C_* \lambda^{-\frac{2}{\beta-2}}$$

and

$$(2.5) \quad \|u_{1,\lambda}\| \leq C_* \lambda^{-\frac{1}{\beta-2}}$$

for $\lambda > 0$ large. It follows by Sobolev's inequality that

$$(2.6) \quad |u_{1,\lambda}|_{2^*} \leq C_* \lambda^{-\frac{1}{\beta-2}}$$

for $\lambda > 0$ large. The proof is complete in view of (2.4) and (2.1). $\square$

Lemma 2.2. *Let u_λ be solutions of (P_λ) satisfying $\tilde{I}_\lambda(u_\lambda) \leq C_* \lambda^{-\frac{2}{\beta-2}}$. Then there is an integer m depending only on γ such that for all $\lambda \geq 1$,*

$$|u_\lambda|_\infty \leq C_* \lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2} \sum_{i=0}^{m-1} (\gamma-1)^i}.$$

Proof. Since the cut-off function $\rho(t)$ satisfies $|\rho'(t)t| \leq 2/\delta$, we obtain from (1.3) that

$$\left| \tilde{F}'(u_\lambda) \right| \leq C_* |u_\lambda|^{\gamma-1};$$

hence

$$\tilde{F}'(u_\lambda) \in L^{\frac{2^*}{\gamma-1}}(\Omega).$$

Now, if $r_1 = \frac{2^*}{\gamma-1}$, then L^{r_1} -estimates give

$$\begin{aligned} \|u_\lambda\|_{W^{2,r_1}} &\leq C_* \left(|u_\lambda|_{r_1} + \lambda \left| \tilde{F}'(u_\lambda) \right|_{r_1} \right) \\ &\leq C_* \left(C_* |u_\lambda|_{2^*} + \lambda C_* |u_\lambda|^{\gamma-1}_{2^*} \right) \\ &\leq C_* \left(\lambda^{-\frac{1}{\beta-2}} + \lambda \lambda^{-\frac{\gamma-1}{\beta-2}} \right) \\ &\leq C_* \left(\lambda^{-\frac{1}{\beta-2}} + \lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2}} \right); \end{aligned}$$

hence

$$|u_\lambda|_{s_1} \leq C_* \left(\lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2}} \right),$$

where $\frac{1}{s_1} = \frac{1}{r_1} - \frac{2}{N}$. Note that $s_1 > 2^*$ since $\gamma < 2^*$. Next, letting $r_2 = \frac{s_1}{\gamma-1}$ and using L^{r_2} -estimates, we obtain

$$\begin{aligned} \|u_\lambda\|_{W^{2,r_2}} &\leq C_* \left(|u_\lambda|_{r_2} + \lambda \left| \tilde{F}'(u_\lambda) \right|_{r_2} \right) \\ &\leq C_* \left(C_* |u_\lambda|_{s_1} + \lambda C_* |u_\lambda|_{s_1}^{\gamma-1} \right) \\ &\leq C_* \left(\lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2}} + \lambda \lambda^{-\frac{\gamma-1}{\beta-2}} \lambda^{\frac{(\gamma-1)(\beta-\gamma)}{\beta-2}} \right) \\ &\leq C_* \left(\lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2}} + \lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\gamma(\beta-\gamma)}{\beta-2}} \right). \end{aligned}$$

Iterating $m \geq 1$ times yields

$$(2.7) \quad \|u_\lambda\|_{W^{2,r_m}} \leq C_* \lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2} \sum_{i=0}^{m-1} (\gamma-1)^i},$$

and the result follows by taking m so that $r_m > \frac{N}{2}$. $\square$

Remark 2.3. The number m of iterations needed to have $W^{2,r_m}(\Omega) \subset L^\infty(\Omega)$ in Lemma 2.2 depends only on N and γ . In other words, since the space dimension N is given, the positive number

$$\Gamma := \sum_{i=0}^{m-1} (\gamma-1)^i$$

above depends solely on γ . Therefore, if $\beta \geq \gamma$ is such that

$$(\beta - \gamma)\Gamma < 1,$$

we get a negative exponent for λ in (2.7).

Proof of Theorem 1. Let $2 < \gamma \leq \beta < 2^*$ be such that $(\beta - \gamma)\Gamma < 1$, where $\Gamma > 0$ was defined in Remark 2.3. Then

$$|u_{j,\lambda}|_\infty \leq C_* \lambda^{-\frac{1-(\beta-\gamma)\Gamma}{\beta-2}},$$

where the exponent of λ is negative, so that there exists $\Lambda_0 > 0$ such that

$$C_* \lambda^{-\frac{1-(\beta-\gamma)\Gamma}{\beta-2}} \leq \delta$$

for all $\lambda \geq \Lambda_0$. It follows that $u_{1,\lambda} > 0$ and $u_{2,\lambda} < 0$ are solutions of our original problem (P_λ) for all $\lambda \geq \Lambda_0$.

Since f is assumed to be of class C^1 and $\inf_{s \in \mathbb{R}} \tilde{F}'(s) > -\infty$ by construction, we also prove that there is a sign-changing solution for all λ large. We employ the method in [4]. On E , let us define

$$P_E = \{u \in E \mid u(x) \geq 0, \text{ a.e. in } \Omega\},$$

which is a closed convex cone. Then, the Banach space $X = C_0^1(\Omega)$ is densely embedded in E , and

$$P = P_E \cap X$$

is a closed convex cone in X . Furthermore, $P = \overset{\circ}{P} \cup \partial P$ under the topology of X , i.e., there exist interior points in P . So, as in [4, Section 3], we may define a partial order relation in X : $u, v \in X$, $u > v \iff u - v \in P \setminus \{0\}$; $u \gg v \iff u - v \in \overset{\circ}{P}$. We also define $W = P \cup (-P)$.

We follow the arguments of Example 3.2 and Corollary 3.2 in [4]. On Y_2 consider $Q = \{u = s\phi_1 + t\phi_2 \mid |s| \leq R, 0 \leq t \leq R, \|u\| \leq R\}$ and for $0 < r < R$, $T = \{u \in Z_2 \mid \|u\| = r\}$. Note that the boundary of Q contains two parts: Q_1 is

the part contained in Y_1 and Q_2 is the part such that $\|u\| = R$. From the conditions on $\tilde{F}$ we may assume $R > 0$ is large enough so that $\tilde{I}_\lambda(u) \leq 0$ for all $u \in Y_2$ with $\|u\| = R$ and for all $\lambda \geq 1$. Also we may assume $r > 0$ is small enough so that $\tilde{I}_\lambda(u) > 0$ for all $u \in T$ and for all $\lambda \geq 1$. Define

$$\Gamma = \{h \in C(Q, X) \mid h(Q_1) \in W, h(u) = u, \text{ for } u \in Q_2\},$$

$$c_\lambda = \inf_{h \in \Gamma} \sup_{h(Q) \setminus W} \tilde{I}_\lambda.$$

Then it follows from [4] that $c_\lambda > 0$ is a critical value of $\tilde{I}_\lambda$ having a sign-changing critical point u_λ at this critical value. From the construction of c_λ and Lemma 1.4, we have $c_\lambda \leq b_{2,\lambda} \leq C_* \lambda^{-\frac{2}{\beta-2}}$. From Lemma 2.2 and for λ large, the u_λ 's are solutions of the original problem (P_λ) . $\square$

Proof of Theorem 2. To prove Theorem 2 we shall use the arguments in [3] and [4] to get solutions for $(\tilde{P}_\lambda)$ first. In order to get estimates on the critical values we shall use the proofs in [3] for the existence and the proofs in [4] for the nodal property of the solutions. Fix an integer k . We shall show (P_λ) has k pairs of solutions for large λ , including $k-1$ pairs of nodal solutions.

Choose $R > 0$ such that $\tilde{I}_\lambda(u) \leq 0$ for all $u \in Y_k$ with $\|u\| \geq R$, and for all $\lambda \geq 1$. Let $D = B_R \cap Y_k$. Define $G = \{h \in C(D, E) \mid h \text{ is odd and } h(u) = u, \text{ for } \|u\| = R\}$. We denote the genus of a symmetric subset A by $i(A)$ and, for $j = 1, \dots, k$, we let

$$\Gamma_j = \{h(\overline{D \setminus B}) \mid h \in G, k \geq j, i(B) \leq k-j\},$$

$$c_{j,\lambda} = \inf_{A \in \Gamma_j} \sup_{u \in A} \tilde{I}_\lambda(u).$$

Then, by [3] (Proposition 9.30, p. 58), and under our conditions on f , we have that $0 < c_{1,\lambda} \leq c_{2,\lambda} \leq \dots \leq c_{k,\lambda}$ are all critical values of $\tilde{I}_\lambda$ and there are at least k pairs of critical points at these critical values. Since $Id \in G$ we have

$$c_{k,\lambda} \leq b_{k,\lambda} \leq C_k \lambda^{-\frac{2}{\beta-2}},$$

where $b_{k,\lambda}$ is as in Lemma 1.4. Then, by Lemma 2.2 and for λ large, these k pairs of critical points are also solutions of the original problem.

Finally, we make use of the fact that $\inf_{s \in \mathbb{R}} \tilde{F}'(s) > -\infty$ (by construction) and follow the idea in [4] by restricting the above minimax procedure to X and taking the maximum outside W . Choose $R > 0$ such that $\tilde{I}_\lambda(u) \leq 0$ for all $u \in Y_k$ with $\|u\| \geq R$, and for all $\lambda \geq 1$. Let $D = B_R \cap Y_k$ and define $G = \{h \in C(D, X) \mid h \text{ is odd and } h(u) = u, \text{ for } \|u\| = R\}$. We also denote the genus of a symmetric subset A in X by again $i(A)$ and, for $j = 2, \dots, k$, let

$$\Gamma_j = \{h(\overline{D \setminus B}) \mid h \in G, k \geq j, i(B) \leq k-j\},$$

$$c_{j,\lambda} = \inf_{A \in \Gamma_j} \sup_{A \setminus W} \tilde{I}_\lambda(u).$$

Then, by [4] (Theorem 2.3, p. 3214) and, again, by the properties of $\tilde{F}'$, we have that $0 < c_{2,\lambda} \leq c_{3,\lambda} \leq \dots \leq c_{k,\lambda}$ are all critical values of $\tilde{I}_\lambda$ and there are at least $(k-1)$ pairs of sign-changing critical points at these critical values. Also, since $Id \in G$, we have

$$c_{k,\lambda} \leq b_{k,\lambda} \leq C_k \lambda^{-\frac{2}{\beta-2}},$$

with $b_{k,\lambda}$ as in Lemma 1.4. Therefore, it follows from Lemma 2.2 and for λ large that these $(k - 1)$ pairs of sign-changing critical points are also solutions of the original problem. $\square$

Remark. We point out that all the above results are still true when $N = 1, 2$.

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