A GENERAL REARRANGEMENT INEQUALITY

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ABSTRACT. We prove a general rearrangement inequality for multiple integrals, using polarization. We introduce a special class of kernels for which the product inequality holds, and then we prove that it also holds when the product is replaced by a so-called function AL_m .

1. INTRODUCTION

Let $f : \Omega \rightarrow \mathbb{R}_+$ be a nonnegative measurable function defined on a measure space (Ω, μ) , which satisfies the condition

$$(1.1) \quad \mu(\{f > t\}) < \infty, \quad \forall t > 0$$

where we write

$$\{f > t\} := \{x \in \Omega : f(x) > t\}.$$

Its *distribution function* λ_f is defined to be

$$\lambda_f(t) = \mu(\{f > t\}), \quad t \in [0, \infty).$$

Two functions f and g are said to be *equimeasurable* if they have the same distribution function and we write $f \sim g$. Functions which are equimeasurable are also said to be *rearrangements* of each other.

Let X denote either $\mathbb{R}^n$, $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, or the n -dimensional hyperbolic space $\mathbb{H}^n$. We equip X with its geodesic distance d and canonical measure μ , and we fix an origin $\mathbf{o}$ in X . For $\mathbb{R}^n$, d is the Euclidean distance, μ is the Lebesgue measure, and $\mathbf{o}$ is $(0, \dots, 0)$. For $\mathbb{S}^n$, d is the great circle distance on the sphere, μ is the surface area measure and $\mathbf{o}$ is $(1, 0, \dots, 0)$. We take as model for $\mathbb{H}^n$ the open unit ball $\{x \in \mathbb{R}^n : |x| < 1\}$ endowed with the distance element $ds = \frac{2}{1-|x|^2}|dx|$; μ is the volume measure associated to ds , and the origin $\mathbf{o}$ is $(0, \dots, 0)$.

Given $f : X \rightarrow \mathbb{R}_+$ satisfying (1.1) we define its *symmetric decreasing rearrangement* $f^\# : X \rightarrow \mathbb{R}_+$ as follows :

$$f^\#(x) = \inf\{t : \lambda_f(t) \leq \mu(B(d(x)))\},$$

where $B(d(x))$ is the ball centered at $\mathbf{o}$ and with radius $d(x) = d(x, \mathbf{o})$. It follows easily that $f^\#$ is constant on each sphere centered at $\mathbf{o}$ and decreases as $d(x)$ increases. Moreover, $f^\# \sim f$. See, e.g., [3].

We define now a very simple rearrangement, called *polarization* (see [2, 3, 6, 7]).

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Let $\mathcal{H}(\mathbb{R}^n)$ be the collection of all $(n-1)$ -dimensional affine subspaces H of $\mathbb{R}^n$ such that $\mathbf{o} \notin H$, $\mathcal{H}(\mathbb{S}^n)$ the collection of all intersections of $\mathbb{S}^n$ with hyperplanes through the origin in $\mathbb{R}^{n+1}$ which do not contain $\mathbf{o}$ and $\mathcal{H}(\mathbb{H}^n)$ the collection of all images under the group of hyperbolic motions of the hyperbolic $n-1$ plane $\{x \in \mathbb{R}^n : |x| < 1, x_n = 0\} \subset \mathbb{H}^n$ which do not contain the origin $\mathbf{o}$. For $H \in \mathcal{H}(X)$, let $\bar{\cdot}$ denote the reflection in H (i.e. $\bar{x}$ is the reflection of x in H), and let H^+ and H^- be the two components of $X \setminus H$, such that $\mathbf{o} \in H^+$.

For $f : X \rightarrow \mathbb{R}$, we define a function $f^H : X \rightarrow \mathbb{R}$ by $f^H(x) = f(x)$ if $x \in H$, and

$$(1.2) \quad f^H(x) = \begin{cases} \max(f(x), f(\bar{x})) & \text{if } x \in H^+, \\ \min(f(x), f(\bar{x})) & \text{if } x \in H^-. \end{cases}$$

We explore, in what follows, some of the rich history of rearrangement inequalities. The most basic rearrangement inequality is due to Hardy and Littlewood [13]:

$$\int_X f(x)g(x) d\mu(x) \leq \int_X f^\sharp(x)g^\sharp(x) d\mu(x),$$

with $f, g : X \rightarrow \mathbb{R}_+$ satisfying (1.1). One can show that the inequality also holds for polarizations.

We will prove inequalities from my thesis [10], of the form

$$\begin{aligned} & \int_X \Psi(f_1(x_1), \dots, f_m(x_m)) K(x_1, \dots, x_m) dx \\ & \leq \int_X \Psi(f_1^\sharp(x_1), \dots, f_m^\sharp(x_m)) K(x_1, \dots, x_m) dx, \end{aligned}$$

where $dx = d\mu(x_1) \dots d\mu(x_m)$, which generalize the Hardy-Littlewood inequality. See also [11].

The theory of rearrangement inequalities is well-developed in $\mathbb{R}^n$. The following inequality, known as Riesz-Sobolev, was proved for $n = 1$ by F. Riesz in 1930 and for $n \geq 1$ by S. L. Sobolev in 1938 :

$$\int_{\mathbb{R}^{2n}} f(x)g(y)h(x-y) dx dy \leq \int_{\mathbb{R}^{2n}} f^\sharp(x)g^\sharp(y)h^\sharp(x-y) dx dy.$$

The functions f , g , and h are nonnegative.

A general rearrangement inequality involving nonnegative functions is due to H. J. Brascamp, E. H. Lieb, and J. M. Luttinger [5]:

$$(1.3) \quad \int_{(\mathbb{R}^n)^p} \prod_{j=1}^k f_j \left(\sum_{m=1}^p a_{jm} x_m \right) dx \leq \int_{(\mathbb{R}^n)^p} \prod_{j=1}^k f_j^\sharp \left(\sum_{m=1}^p a_{jm} x_m \right) dx$$

where a_{jm} are constants, $dx = dx_1 \dots dx_p$, and each $x_i \in \mathbb{R}^n$. When $p = 2$, $k = 3$ and a_{jm} are suitably chosen, (1.3) implies the Riesz-Sobolev inequality.

The proof of (1.3) uses the Brunn-Minkowski inequality [15], which states that if K and L are convex bodies (compact convex sets with nonempty interiors) in $\mathbb{R}^k$ and $0 < \lambda < 1$, then

$$V((1-\lambda)K + \lambda L)^{1/k} \geq (1-\lambda)V(K)^{1/k} + \lambda V(L)^{1/k}.$$

V denotes the k -dimensional volume and $+$ is the vector sum of K and L . Further generalizations of (1.3) were given by M. Christ [9] and R. E. Pfeifer [16].

The proofs of the rearrangement inequalities on $\mathbb{R}^n$ require the product structure $\mathbb{R}^n = \mathbb{R} \times \dots \times \mathbb{R}$, as well as the Brunn-Minkowski inequality. These tools are not available on $\mathbb{S}^n$ or $\mathbb{H}^n$, and for this reason complete analogues of (1.3) for $\mathbb{S}^n$ and $\mathbb{H}^n$ are not known. To get rearrangement inequalities on X , polarization proved to be an effective tool. A. Baernstein and B. A. Taylor [2] used it to derive inequalities for $\mathbb{S}^n$, and W. Beckner [4] noticed that it can also be used for $\mathbb{H}^n$.

Baernstein and Taylor showed that the symmetric decreasing rearrangement of a function can be approximated through a sequence of polarizations [2]. The polarization is easier to handle and if an inequality holds for polarizations in all hyperplanes in $\mathcal{H}(X)$, then in most cases one can apply a similar approximation argument to deduce the inequality for the symmetric decreasing rearrangement. We state here the Baernstein-Taylor inequality:

$$\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f(x)g(y)K(d(x,y)) d\mu(x) d\mu(y) \leq \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f^\sharp(x)g^\sharp(y)K(d(x,y)) d\mu(x) d\mu(y),$$

for $K : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a decreasing function. This inequality was proved for f^H and g^H in place of $f^\sharp$ and $g^\sharp$, for every $H \in \mathcal{H}(X)$, and then they used the approximation argument to deduce it for $f^\sharp$ and $g^\sharp$.

We introduce now a special class of functions of particular interest for our result. For $\underline{x}^1 = (x_1^1, \dots, x_m^1)$, $\underline{x}^2 = (x_1^2, \dots, x_m^2) \in \mathbb{R}_+^m$ we write $\underline{x}^1 \leq \underline{x}^2$ if $x_i^1 \leq x_i^2$, for every $i = 1, \dots, m$.

Definition 1.1. Let $\Psi : \mathbb{R}_+^m \rightarrow \mathbb{R}_+$ be a continuous function such that

$$\Psi_{|\{x_i=0\}} = 0, \text{ for every } i = 1, \dots, m.$$

We say that $\Psi \in AL_m(\mathbb{R}_+^m)$ if, for every $\underline{x}^1, \underline{x}^2 \in \mathbb{R}_+^m$, such that $\underline{x}^1 \leq \underline{x}^2$, the following condition holds:

$$(1.4) \quad \sum_{i_1, \dots, i_m \in \{1,2\}} (-1)^{i_1 + \dots + i_m} \Psi(x_1^{i_1}, \dots, x_m^{i_m}) \geq 0.$$

If $\Psi \in C^m(\mathbb{R}_+^m)$, then (1.4) is equivalent to $\frac{\partial^m \Psi}{\partial x_1 \dots \partial x_m} \geq 0$.

Consider now a family of m positive random variables $X_1, \dots, X_m$ on some probability space (Ω, P) , and define

$$\Psi(x_1, \dots, x_m) = P(X_1 \leq x_1, \dots, X_m \leq x_m),$$

their joint distribution function. Then Ψ satisfies (1.4).

One can show that $\Psi \in AL_m(\mathbb{R}_+^m)$ implies Ψ is increasing in each argument, when the rest are fixed. This can be seen, for example, by taking $\underline{x}^1 = (x_1, 0, \dots, 0)$ and $\underline{x}^2 = (y_1, x_2, \dots, x_m)$ with $\underline{x}^1 \leq \underline{x}^2$, and using (1.4) and the fact that Ψ vanishes on the boundary.

We say that a continuous function $\Psi : \mathbb{R}_+^m \rightarrow \mathbb{R}_+$ is in $AL_2(\mathbb{R}_+^m)$ if for every fixed $m-2$ variables Ψ is in $AL_2(\mathbb{R}_+^2)$ in the remaining two arguments [6]. This class of functions was studied by Almgren and Lieb [1].

Brock [6] proved an extension of the Hardy-Littlewood inequality: If $\Psi \in AL_2(\mathbb{R}_+^m)$, then

$$(1.5) \quad \int_X \Psi(f_1(x), \dots, f_m(x)) d\mu(x) \leq \int_X \Psi(f_1^\sharp(x), \dots, f_m^\sharp(x)) d\mu(x),$$

for every $f_1, \dots, f_m : X \rightarrow \mathbb{R}_+$ satisfying (1.1). As before, X is any of the spaces $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$.

In the same spirit, Morpurgo [14] proved the following inequality: For $K_{ij} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $1 \leq i < j \leq m$, decreasing functions and $\Psi_0 : \mathbb{R} \rightarrow \mathbb{R}$ a continuous convex function:

$$(1.6) \quad \int_{X^m} \prod_{i < j} K_{ij}(d(x_i, x_j)) \Psi_0\left(\sum_{l=1}^m f_l(x_l)\right) dx \\ \leq \int_{X^m} \prod_{i < j} K_{ij}(d(x_i, x_j)) \Psi_0\left(\sum_{l=1}^m f_l^\sharp(x_l)\right) dx,$$

where $dx = dx_1 \dots dx_m$. A similar result was proved by Burchard and Schmuckenschläger, [8]. Both Morpurgo's result (1.6) and Brock's result (1.5) were first proved for polarization, then obtained for the symmetric decreasing rearrangement by the approximation argument. Also, Morpurgo's inequality contains the Baernstein-Taylor inequality as a special case.

Our result gives a larger class of kernels K for which the polarization inequality holds in Morpurgo's theorem, when $\Psi_0(\sum_{l=1}^m f_l(x_l))$ is replaced by $f_1(x_1) \dots f_m(x_m)$ and hence by $\Psi(f_1(x_1), \dots, f_m(x_m))$, where $\Psi \in AL_m(\mathbb{R}_+^m)$. We begin with a discrete result.

2. A DISCRETE REARRANGEMENT INEQUALITY

Let $Q = \{1, 2\}$ be a set with two points, and let $K_0 : Q^m \rightarrow \mathbb{R}_+$ be a nonnegative function. We think of Q^m as the vertices of a cube in $\mathbb{R}^m$. Take $k \in \{1, \dots, m\}$ and $S \subset \{1, \dots, m\}$, with $|S| = k$. We write $S = \{i_1, \dots, i_k\}$, with $i_1 < \dots < i_k$. For every S and $y = (y_1, \dots, y_k) \in Q^k$, let

$$A(y, S) = \{x \in Q^m : x_{i_1} = y_1, \dots, x_{i_k} = y_k\}.$$

We can think of $A(y, S)$ as an $(m - k)$ -dimensional subcube of Q^m .

Let $1_k = (1, \dots, 1) \in Q^k$. Then $A(1_k, S)$ is the set of points x in Q^m for which $x_{i_1} = \dots = x_{i_k} = 1$.

We define

$$(2.1) \quad K_0(y, S) = \sum_{x \in A(y, S)} K_0(x), \quad y \in Q^k.$$

Definition 2.1. A function $K_0 : Q^m \rightarrow \mathbb{R}_+$ is called a "good" cube kernel if

$$(2.2) \quad K_0(y, S) \leq K_0(1_k, S),$$

for every $y \in Q^k$, every $S \subset \{1, \dots, m\}$ with $|S| = k$, and every $k \in \{1, \dots, m\}$.

Let us consider some particular cases. For $k = m$, we have $A(y, S) = \{y\}$ and $S = \{1, \dots, m\}$. Then (2.2) says that

$$K_0(y) \leq K_0(1_m), \quad \forall y \in Q^m.$$

For $k = m - 1$ and $S = \{1, \dots, m\} \setminus \{j\}$, (2.2) asserts that

$$(2.3) \quad K_0(y_1, \dots, y_{j-1}, 1, y_{j+1}, \dots, y_m) + K_0(y_1, \dots, y_{j-1}, 2, \dots, y_m) \\ \leq K_0(1, 1, \dots, 1) + K_0(1, 1, \dots, 2, \dots, 1), \quad \forall y \in Q^{m-1}.$$

One may picture the left-hand side of (2.3) as the sum of K_0 over an edge of the m -cube Q^m , and then (2.3) says that among all parallel edges, the maximal edge sum is achieved when the edge contains $(1, \dots, 1)$.

Likewise, for every S , with $|S| = k$, $A(y, S)$ represents the sum of K_0 over the vertices of an $m - k$ subcube of Q^m and (2.2) requires that among all parallel $m - k$ subcubes, the maximal sum is attained when the subcube contains $(1, \dots, 1)$.

For $g : Q \rightarrow \mathbb{R}_+$ we define g^* , its decreasing rearrangement, as follows:

$$g^*(1) = \max(g(1), g(2)) \quad \text{and} \quad g^*(2) = \min(g(1), g(2)).$$

The following lemma is immediate from the definition of K_0 .

Lemma 2.2. *Let $K_0 : Q^m \rightarrow \mathbb{R}_+$ be a nonnegative function. Then, the following inequality is true:*

$$(2.4) \quad \int_{Q^m} g_1(\varepsilon_1) \cdots g_m(\varepsilon_m) K_0(\varepsilon_1, \dots, \varepsilon_m) d\varepsilon_1 \dots d\varepsilon_m \\ \leq \int_{Q^m} g_1^*(\varepsilon_1) \cdots g_m^*(\varepsilon_m) K_0(\varepsilon_1, \dots, \varepsilon_m) d\varepsilon_1 \dots d\varepsilon_m$$

for every $g_1, \dots, g_m : Q \rightarrow \mathbb{R}_+$, if and only if K_0 is a good cube kernel.

The proof is achieved by considering each g_i to be the characteristic function of a point in Q . This result implies the following:

Proposition 2.3. *Let $f_i : Q \rightarrow \mathbb{R}_+$ be m nonnegative functions, let $\Psi \in AL_m(\mathbb{R}_+^m)$ and let $K_0 : Q^m \rightarrow \mathbb{R}_+$ be a good cube kernel. We define*

$$I[g_1, \dots, g_m] = \int_{Q^m} \Psi(g_1(\varepsilon_1), \dots, g_m(\varepsilon_m)) K_0(\varepsilon_1, \dots, \varepsilon_m) d\varepsilon_1 \dots d\varepsilon_m.$$

Sometimes we write $I(g_1, \dots, g_m, \Psi, K)$ if we want to emphasize Ψ and K . Then

$$I[g_1, \dots, g_m] \leq I[g_1^*, \dots, g_m^*].$$

Conversely, if for some $K_0 : Q^m \rightarrow \mathbb{R}_+$ we assume that the inequality holds for every $\Psi \in AL_m(\mathbb{R}_+^m)$ and all choices of g_i , then K_0 is a good cube kernel.

Proof. For every $a > 0$ we define the set

$$C(a) = \mathbb{R}_+^m \setminus \{(x_1, \dots, x_m) : x_i > a, \forall i = 1, \dots, m\}.$$

We assume first that $\Psi \in C^m$ (Ψ is m times continuously differentiable) and that it vanishes on $C(a)$, for some a . We already have the inequality for the case when Ψ is the product function, $\Psi(y_1, \dots, y_m) = y_1 \cdots y_m$. So, we need to express Ψ in terms of a product. To condense notation, let

$$\Phi(s) = \frac{\partial^m \Psi}{\partial s_1 \dots \partial s_m}(s),$$

$d\varepsilon = d\varepsilon_1 \dots d\varepsilon_m$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$, $ds = ds_1 \dots ds_m$, $s = (s_1, \dots, s_m)$. Then $\Phi \geq 0$ since $\Psi \in AL_m$.

Since all the boundary terms are zero, we can write

$$\Psi(y_1, \dots, y_m) = \int_0^{y_1} \int_0^{y_2} \dots \int_0^{y_m} \Phi(s) ds_1 \dots ds_m \\ = \int_0^\infty \dots \int_0^\infty \Phi(s) \chi_{\{s_1 < y_1\}}(s_1) \cdots \chi_{\{s_m < y_m\}}(s_m) ds.$$

Setting $y_i = g_i(\varepsilon_i)$, multiplying by $K_0(\varepsilon)$ and integrating, we obtain

$$I[g_1, \dots, g_m] = \int_{Q^m} \int_{(0, \infty)^m} \Phi(s) \chi_{\{g_1 > s_1\}}(\varepsilon_1) \cdots \chi_{\{g_m > s_m\}}(\varepsilon_m) K_0(\varepsilon) ds d\varepsilon.$$

Now reverse the order of integration. For fixed $s_1, \dots, s_m$, positive,

$$\chi_{\{g_i > s_i\}}^* = \chi_{\{g_i^* > s_i\}}, \quad \text{as functions on } Q,$$

and we can apply Lemma 2.2 to show that

$$\begin{aligned} & \int_{Q^m} \chi_{\{g_1 > s_1\}}(\varepsilon_1) \cdots \chi_{\{g_m > s_m\}}(\varepsilon_m) K_0(\varepsilon) d\varepsilon \\ & \leq \int_{Q^m} \chi_{\{g_1^* > s_1\}}(\varepsilon_1) \cdots \chi_{\{g_m^* > s_m\}}(\varepsilon_m) K_0(\varepsilon) d\varepsilon. \end{aligned}$$

It follows that $I(g_1, \dots, g_m, \Psi, K_0) \leq I(g_1^*, \dots, g_m^*, \Psi, K_0)$ when $\Psi \in C^m$ and Ψ vanishes on $C(a)$.

If Ψ vanishes on $C(a)$ and is not smooth, then we can approximate it by smooth functions by convolving it with C_0^∞ functions. First, we extend Ψ to $\mathbb{R}^m$ by defining it to be zero outside $\mathbb{R}_+^m$, its domain. The new function Ψ is continuous on $\mathbb{R}^m$.

Let $G : \mathbb{R}^m \rightarrow \mathbb{R}_+$ be a C_0^∞ function such that $\int_{\mathbb{R}^m} G = 1$. Define $G_\delta(x) = \delta^{-m} G(x/\delta)$ and

$$\Psi_\delta(x) = (\Psi * G_\delta)(x) = \int_{\mathbb{R}^m} \Psi(y) G_\delta(x - y) dy.$$

Then Ψ_δ is smooth and in $AL_m(\mathbb{R}_+^m)$, and since $\Psi = 0$ on $C(a)$, Ψ_δ is zero on $C(a_\delta)$, when δ is small enough, for some $a_\delta > 0$. Thus we can apply our inequality for Ψ_δ to get

$$I(g_1, \dots, g_m, \Psi_\delta, K_0) \leq I(g_1^*, \dots, g_m^*, \Psi_\delta, K_0).$$

Since the Ψ_δ are uniformly close to Ψ on the discrete set $f_1(Q) \times \dots \times f_m(Q)$, we let $\delta \rightarrow 0$ to get

$$I(g_1, \dots, g_m, \Psi, K_0) \leq I(g_1^*, \dots, g_m^*, \Psi, K_0),$$

for $\Psi \in AL_m(\mathbb{R}_+^m)$ and $\Psi = 0$ on $C(a)$.

For general $\Psi \in AL_m(\mathbb{R}_+^m)$, let

$$\Psi_k(x) = \Psi(x - (1/k)1_m), \quad \text{with } 1_m = (1, \dots, 1) \in \mathbb{R}^m, \quad x \in \mathbb{R}_+^m \setminus C(1/k).$$

Then we can define Ψ_k to be zero on $C(1/k)$. The new Ψ_k 's are in $AL_m(\mathbb{R}_+^m)$. Since $\Psi_k \uparrow \Psi$ on R_+^m , and since

$$I(g_1, \dots, g_m, \Psi_k, K_0) \leq I(g_1^*, \dots, g_m^*, \Psi_k, K_0),$$

we can pass to the limit under the integral sign, and we obtain

$$I(g_1, \dots, g_m, \Psi, K_0) \leq I(g_1^*, \dots, g_m^*, \Psi, K_0).$$

The converse statement of Proposition 2.3 follows from Lemma 2.2, by taking $\Psi(y_1, \dots, y_m) = y_1 \cdots y_m$. $\square$

3. MAIN RESULT

We return now to analysis on X . We consider a measurable function $K : X^m \rightarrow \mathbb{R}_+$, and take $H \in \mathcal{H}(X)$. For each $(x_1, \dots, x_m) \in (H^+)^m$, we define $K_0 : Q^m \rightarrow \mathbb{R}_+$ as follows:

$$K_0(\varepsilon_1, \dots, \varepsilon_m) := K(y_1, \dots, y_m), \quad \text{with } y_i = \begin{cases} x_i & \text{if } \varepsilon_i = 1, \\ \bar{x}_i & \text{if } \varepsilon_i = 2. \end{cases}$$

As usual, $\bar{x}_i$ is the reflection of x_i in the hyperplane H . We note that K_0 depends on $(x_1, \dots, x_m)$.

Definition 3.1. We call $K : X^m \rightarrow \mathbb{R}_+$ a good polarization kernel if each K_0 is a good cube kernel for almost every $(x_1, \dots, x_m) \in (H^+)^m$ and every $H \in \mathcal{H}(X)$ with $0 \in H^+$.

Here is an equivalent definition of a good polarization kernel, in terms of sums. Consider $k \in \{1, \dots, m\}$ and $S \subset \{1, \dots, m\}$, $S = \{i_1, \dots, i_k\}$ with $i_1 < \dots < i_k$, and $H \in \mathcal{H}(X)$ with $(x_1, \dots, x_m) \in (H^+)^m$ being fixed. For every $y = (y_1, \dots, y_k) \in \{x_{i_1}, \bar{x}_{i_1}\} \times \dots \times \{x_{i_k}, \bar{x}_{i_k}\}$, let

$$A_H(y, S) = \{x \in \{x_1, \bar{x}_1\} \times \dots \times \{x_m, \bar{x}_m\} : x_{i_1} = y_1, \dots, x_{i_k} = y_k\}.$$

We define

$$(3.1) \quad K(y, S) = \sum_{x \in A_H(y, S)} K(x).$$

Let $y^* = (x_{i_1}, \dots, x_{i_k})$. Then we see that our definition is equivalent to the condition $K(y, S) \leq K(y^*, S)$, for every $S \subset \{1, \dots, m\}$, $H \in \mathcal{H}(X)$ and a.e. $(x_1, \dots, x_m) \in (H^+)^m$.

Now, consider $\Psi : \mathbb{R}_+^m \rightarrow \mathbb{R}_+$ and $f_i : X \rightarrow \mathbb{R}_+$, $1 \leq i \leq m$. Define

$$(3.2) \quad I(f_1, \dots, f_m, K, \Psi) = \int_{X^m} \Psi(f_1(x_1), \dots, f_m(x_m)) K(x_1, \dots, x_m) dx_1 \dots dx_m.$$

If we split each integral $\int_X = \int_{H^+} + \int_{H^-}$ and use reflection in the second integral and collect terms, we get in (3.2):

$$(3.3) \quad I(f_1, \dots, f_m, K, \Psi) = \int_{(H^+)^m} I(g_1, \dots, g_m, \Psi, K_0) dx_1 \dots dx_m.$$

Here the $g_i : Q \rightarrow \mathbb{R}_+$ depend on x_i and are defined as follows:

$$g_i(1) = f_i(x_i), \quad g_i(2) = f_i(\bar{x}_i), \quad x_i \in X.$$

Theorem 3.2. Let $X = \mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$, let $f_i : X \rightarrow \mathbb{R}_+$ be m nonnegative functions, let $\Psi \in AL_m(\mathbb{R}_+^m)$, and let K be a good polarization kernel. We define

$$I[f_1, \dots, f_m] = \int_{X^m} \Psi(f_1(x_1), \dots, f_m(x_m)) K(x_1, \dots, x_m) dx_1 \dots dx_m.$$

Then, the following inequalities hold:

$$(3.4) \quad I[f_1, \dots, f_m] \leq I[f_1^H, \dots, f_m^H] \quad \forall H \in \mathcal{H}(X)$$

and

$$(3.5) \quad I[f_1, \dots, f_m] \leq I[f_1^\sharp, \dots, f_m^\sharp].$$

If $K : X^m \rightarrow \mathbb{R}_+$ is such that the polarization inequality (3.4) holds for every $H \in \mathcal{H}(X)$ and every $\Psi \in AL_m(\mathbb{R}_+^m)$, then K is a good polarization kernel.

Proof. The polarization inequality in Theorem 3.2 follows from Proposition 2.3 applied to the integrand in (3.3), using the fact that if $g_1, \dots, g_m$ correspond to $f_1, \dots, f_m$ for a certain $(x_1, \dots, x_m)$, then $g_1^*, \dots, g_m^*$ correspond to $f_1^H, \dots, f_m^H$. Also, inequality (3.5) can be deduced using an approximation argument similar to the one presented in [2].

Conversely, to prove that K must be a good polarization kernel if (3.4) always holds, let $\Psi(y_1, \dots, y_m) = y_1 \cdots y_m$. Let us fix $H \in \mathcal{H}(X)$ and consider

$(x_1, \dots, x_m) \in (H^+)^m$. For $S = \{i_1, \dots, i_k\}$ and $y = (y_1, \dots, y_k) \in \{x_{i_1}, \bar{x}_{i_1}\} \times \dots \times \{x_{i_k}, \bar{x}_{i_k}\}$, define

$$f_{i_1} = (1/c_\varepsilon)\chi_{B(y_1, \varepsilon)}, \quad \dots, \quad f_{i_k} = (1/c_\varepsilon)\chi_{B(y_k, \varepsilon)}$$

and $f_j = (1/c_\varepsilon)\chi_{B(x_j, \varepsilon)} + (1/c_\varepsilon)\chi_{B(\bar{x}_j, \varepsilon)}$, for $j \in S^c$. Here c_ε represents the measure of any ball of radius ε in X .

For ε sufficiently small,

$$f_{i_1}^H = (1/c_\varepsilon)\chi_{B(x_{i_1}, \varepsilon)}, \quad \dots, \quad f_{i_k}^H = (1/c_\varepsilon)\chi_{B(x_{i_k}, \varepsilon)}$$

and $f_j = f_j^H$ for $j \in S^c$.

Without loss of generality, we can assume that K is integrable ($K \in L^1(X^m)$).

Since

$$I(f_1, \dots, f_m) \leq I(f_1^H, \dots, f_m^H),$$

and by passing to the limit as $\varepsilon \rightarrow 0$ we get that $\text{LHS} \rightarrow K(y, S)$, $\text{RHS} \rightarrow K(y^*, S)$ for a.e. $(x_1, \dots, x_m) \in (H^+)^m$ (see [12, p. 93]), it follows that

$$K(y, S) \leq K(y^*, S), \quad \text{for a.e. } (x_1, \dots, x_m) \in (H^+)^m.$$

Since $H \in \mathcal{H}(X)$ was arbitrarily chosen, this proves that K is a good polarization kernel. $\square$

We conclude this section with a few remarks. Firstly, when $m = 2$, $K(x, y) = K_1(d(x, y))$, with K_1 a decreasing function, is a good polarization kernel.

Kernels of the form $K(x_1, \dots, x_m) = k(|x_1|^{\alpha_1} + |x_2|^{\alpha_2} + \dots + |x_m|^{\alpha_m})$, where $\alpha_1, \dots, \alpha_m \geq 0$ are positive real numbers and k is a decreasing function of one variable, are good polarization kernels. Here $|x|$ denotes the distance to the origin, $d(x, \mathbf{o})$. To see this, let $H \in \mathcal{H}(X)$ and $(x_1, \dots, x_m) \in (H^+)^m$. With the notation in (3.1), $K(x) \leq K(z)$ for $x \in A_H(y, S)$ and $z \in A_H(y^*, S)$ such that $x_i = z_i$ for every $i \in S^c$, since any component of z has distance to the origin less than or equal to the distance to the origin of the corresponding component of x and since k is decreasing. There is a one-to-one correspondence between elements in $A_H(y, S)$ and $A_H(y^*, S)$, and thus

$$K(y, S) = \sum_{x \in A_H(y, S)} K(x) \leq \sum_{z \in A_H(y^*, S)} K(z) = K(y^*, S),$$

for every $H \in \mathcal{H}(X)$, and $(x_1, \dots, x_m) \in (H^+)^m$ and y . This proves that K is a good polarization kernel.

More generally, $K(x_1, \dots, x_m)$, where K is symmetric decreasing in each argument, is a good polarization kernel.

We show now that Morpurgo's kernels are good polarization kernels. Let $K(x_1, \dots, x_m) = \prod_{i < j} K_{ij}(d(x_i, x_j))$ where $x_1, \dots, x_m \in X$, and $K_{ij} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are decreasing. We have $I(f_1, \dots, f_m, \Psi, K) \leq I(f_1^H, \dots, f_m^H, \Psi, K)$ for

$$\Psi(y_1, \dots, y_m) = y_1 \cdots y_m \in AL_m(\mathbb{R}_+^m)$$

and all nonnegative f_i 's and $H \in \mathcal{H}(X)$. This follows from Morpurgo's result (1.6), which was proved using polarization, and by taking $\Psi_0(y) = e^y$ and using the fact that $e^{f^H} = (e^f)^H$. The proof of the necessity part of Theorem 3.2 required only the case when Ψ is the product function. Thus, we can conclude that K is a good polarization kernel.

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