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δ-FUNCTION OF AN OPERATOR: A WHITE NOISE APPROACH

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ABSTRACT. Let $(E) \subset (L^2) \subset (E)^*$ be the canonical framework of white noise analysis over the Gel'fand triple $S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S^*(\mathbb{R})$ and $\mathcal{L} \equiv \mathcal{L}[(E),(E)^*]$ be the space of continuous linear operators from (E) to $(E)^*$. Let Q be a self-adjoint operator in (L^2) with spectral representation $Q = \int_{\mathbb{R}} \lambda\, P_Q(d\lambda)$. In this paper, it is proved that under appropriate conditions upon Q, there exists a unique linear mapping $Z : S^*(\mathbb{R}) \longmapsto \mathcal{L}$ such that $Z(f) = \int_{\mathbb{R}} f(\lambda)\, P_Q(d\lambda)$ for each $f \in S(\mathbb{R})$. The mapping is then naturally used to define $\delta(Q)$ as $Z(\delta)$, where δ is the Dirac δ-function. Finally, properties of the mapping Z are investigated and several results are obtained.

1. Introduction

Let δ be the Dirac δ-function, which is a Schwartz generalized function, and Q an observable, i.e., a self-adjoint operator in a Hilbert space. Then $\delta(Q)$, called the δ-function of Q, is of physical significance (cf. [1]). However, from the mathematical point of view, it is a very singular object. What is the mathematical meaning of $\delta(Q)$ which is both reasonable and rigorous? In [1], the authors gave an interpretation in the context of Hilbert space theory.

On the other hand, white noise analysis initiated by Hida [2], which is essentially an infinite-dimensional analogue of Schwartz generalized function theory, has been considerably developed and successfully applied to many fields including stochastic analysis and quantum physics (see, e.g., [2, 3, 4, 5], [7, 8], [12] and references cited therein). The mathematical framework of the theory is the Gel'fand triple

$$(E) \subset (L^2) \subset (E)^*$$

over $S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S^*(\mathbb{R})$, where (E) (resp. $(E)^*$) is known as Hida testing (resp. generalized) functional space. Let $\mathcal{L} \equiv \mathcal{L}[(E),(E)^*]$ be the space of continuous linear operators from (E) to $(E)^*$. Elements of $\mathcal{L}$ are usually called generalized operators, which are significant generalizations of bounded operators on the Hilbert space (L^2).

The main purpose of the present paper is to define $\delta(Q)$ reasonably and rigorously in the context of white noise analysis. The paper is organized as follows. In Section 2

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we recall some necessary notions, notation and facts in white noise analysis. In Section 3, we first prove that for a self-adjoint operator Q in (L^2) with spectral representation $Q = \int_{\mathbb{R}} \lambda \, P_Q(d\lambda)$, under appropriate conditions upon Q there exists a unique linear mapping $Z : S^*(\mathbb{R}) \longmapsto \mathcal{L}$ such that $Z(f) = \int_{\mathbb{R}} f(\lambda) \, P_Q(d\lambda)$ for each $f \in S(\mathbb{R})$. We then naturally use the mapping Z to define $\delta(Q)$ as $Z(\delta)$. Finally, we show that the mapping $Z : S^*(\mathbb{R}) \longmapsto \mathcal{L}$ is continuous and positivity-preserving.

2. Framework of white noise analysis

In this section we briefly recall some notions, notation and facts in white noise analysis. For details see [2], [5], [7] and [9].

We first fix some general notation. Throughout the paper, $\mathbb{R}$ and $\mathbb{C}$ stand for the real line and complex plane, respectively. For any real locally convex space V, we denote by $V_{\mathbb{C}}$ the complexification of V. Let $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on $V^* \times V$; then the canonical bilinear forms on $V_{\mathbb{C}}^* \times V_{\mathbb{C}}$ and $(V_{\mathbb{C}}^{\otimes n})^* \times V_{\mathbb{C}}^{\otimes n}$ are still denoted by $\langle \cdot, \cdot \rangle$. Similarly, if V is a real Hilbert space with norm $|\cdot|$, then the norms of $V_{\mathbb{C}}$ and $V_{\mathbb{C}}^{\otimes n}$ are also denoted by the same symbol $|\cdot|$.

Now let $H \equiv L^2(\mathbb{R}, dt; \mathbb{R})$ be the Hilbert space of real-valued square integrable functions on $\mathbb{R}$ with norm $|\cdot|_0$ and inner product $\langle \cdot, \cdot \rangle$. Let $A = 1 + t^2 - d^2/dt^2$ be the harmonic oscillator. Then A has a self-adjoint extension in H, which is still denoted by A.

For each integer p, let E_p be the completion of $\operatorname{Dom} A^p$ with respect to the Hilbertian norm $|\cdot|_p = |A^p \cdot|_0$. Then E_p and E_{-p} can be regarded as each other's dual if we identify H with its dual. Let E be the projective limit of $\{ E_p \mid p \geq 0 \}$ and E^* the topological dual of E. Then E is a nuclear space and E^* is the inductive limit of $\{ E_{-p} \mid p \geq 0 \}$. Hence we have a real Gel'fand triple $E \subset H \subset E^*$. It is known (cf. [2]) that E and E^* coincide with Schwartz rapidly decreasing function space $S(\mathbb{R})$ and generalized function space $S^*(\mathbb{R})$, respectively. We denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear form on $E^* \times E$, which is consistent with the inner product of H.

Let μ be the standard Gaussian measure on E^*, i.e., its characteristic function is

$$\int_{E^*} e^{i\langle x, f \rangle} \, \mu(dx) = e^{-\frac{1}{2}|f|_0^2}, \quad f \in E. \tag{2.1}$$

The measure space (E^*, μ) is known as *white noise*. Let $(L^2) \equiv L^2(E^*, \mu; \mathbb{C})$ be the Hilbert space of complex-valued μ-square integrable functionals on E^* with the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and norm $\|\cdot\|_0$. Then, by the well-known Wiener-Itô-Segal isomorphism theorem, for each $\varphi \in (L^2)$ there exists a unique sequence $(f_n)_{n=0}^{\infty}$ with $f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}$ such that $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$ in norm $\|\cdot\|_0$ and

$$\|\varphi\|_0^2 = \sum_{n=0}^{\infty} n! |f_n|_0^2 \tag{2.2}$$

where $I_n(f_n)$ denotes the multiple Wiener integral of order n with kernel f_n.

Note that the harmonic oscillator A also has a self-adjoint extension in $H_{\mathbb{C}}$, which is still denoted by A. Let $\Gamma(A)$ be the second quantization operator of A defined by

$$\Gamma(A)\varphi = \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n) \tag{2.3}$$

where $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$. Then $\Gamma(A)$ is a positive self-adjoint operator with Hilbert-Schmidt inverse in (L^2).

Similarly, for each integer p, let (E_p) be the completion of $\mathrm{Dom}\,\Gamma(A)^p$ with respect to the Hilbertian norm $\|\cdot\|_p = \|\Gamma(A)^p \cdot\|_0$. Then (E_p) becomes a complex Hilbert space. In particular, $(E_0) = (L^2)$. Let (E) be the projective limit of $\{\,(E_p) \mid p \geq 0\,\}$ and $(E)^*$ the inductive limit of $\{\,(E_{-p}) \mid p \geq 0\,\}$. Then (E) and $(E)^*$ can be regarded as each other's dual. Moreover, (E) is a nuclear space and we come to a complex Gel'fand triple

$$(E) \subset (L^2) \subset (E)^*,$$

which is known as the *canonical framework of white noise analysis.* Elements of (E) (resp. $(E)^*$) are called *Hida testing* (resp. *generalized*) *functionals.* In the following, we denote by $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ the canonical bilinear form on $(E)^* \times (E)$.

For $\xi \in E_{\mathbb{C}}$, the exponential functional ϕ_ξ associated with ξ is defined as

$$\phi_\xi(x) = e^{\langle x,\xi\rangle - \langle \xi,\xi\rangle/2} = \sum_{n=0}^{\infty} \Big\langle :x^{\otimes n}:, \frac{1}{n!}\xi^{\otimes n}\Big\rangle, \quad x \in E^*. \tag{2.4}$$

It is known that the set $\{\,\phi_\xi \mid \xi \in E_{\mathbb{C}}\,\}$ is total in the Hilbert space (E_p) for each integer p. Hence $\mathrm{Span}\{\,\phi_\xi \mid \xi \in E_{\mathbb{C}}\,\}$ is a dense subspace of (E).

Continuous linear operators from (E) to $(E)^*$ are usually called *generalized operators.* The space of all generalized operators is denoted by $\mathcal{L} \equiv \mathcal{L}[(E),(E)^*]$. For $X \in \mathcal{L}$, its *symbol* $\widehat{X}$ is defined as

$$\widehat{X}(\xi,\eta) = \langle\!\langle X\phi_\xi, \phi_\eta \rangle\!\rangle, \quad \xi,\eta \in E_{\mathbb{C}}. \tag{2.5}$$

The next lemma (cf. [8] and [9]) will be used later.

Lemma 2.1. *Let $\{\,X_n\,\}_{n\geq 1} \subset \mathcal{L}$ be such that*

(1) *$\forall\, \xi,\eta \in E_{\mathbb{C}}$, the sequence $\{\,\widehat{X_n}(\xi,\eta)\,\}_{n\geq 1}$ is convergent in $\mathbb{C}$,*
(2) *there exist constants $a,k,p \geq 0$ such that*

$$\sup_{n\geq 1} \big|\widehat{X_n}(\xi,\eta)\big| \leq a\exp\{k(|\xi|_p^2 + |\eta|_p^2)\}, \quad \xi,\eta \in E_{\mathbb{C}}. \tag{2.6}$$

Then there exists a unique $X \in \mathcal{L}$ such that $X_n \longrightarrow X$ in $\mathcal{L}$.

3. δ-FUNCTION OF AN OPERATOR

We first make some necessary assumptions. Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-field of the real line $\mathbb{R}$ and $\mathcal{P}[(L^2)]$ the set of projections in (L^2).

Let Q be a given self-adjoint operator in (L^2) with spectral representation

$$Q = \int_{\mathbb{R}} \lambda\, P_Q(d\lambda), \tag{3.1}$$

where $P_Q : \mathcal{B}(\mathbb{R}) \longmapsto \mathcal{P}[(L^2)]$ is the spectral measure of Q (cf. [10]). It is well known that for a Borel measurable function f on $\mathbb{R}$, $f(Q) = \int_{\mathbb{R}} f(\lambda)\, P_Q(d\lambda)$ makes sense as a densely defined operator in (L^2). Moreover, $f(Q)$ is a bounded operator in (L^2) if f is a bounded Borel measurable function (see [10] for details).

For each $\xi,\eta \in E_{\mathbb{C}}$, define $\nu^Q_{\xi,\eta} : \mathcal{B}(\mathbb{R}) \longmapsto \mathbb{C}$ as

$$\nu^Q_{\xi,\eta}(S) = \langle\!\langle P_Q(S)\phi_\xi, \phi_\eta \rangle\!\rangle, \quad S \in \mathcal{B}(\mathbb{R}). \tag{3.2}$$

Obviously $\nu^Q_{\xi,\eta}$ is a complex-valued measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Throughout the section, we make the following hypothesis.

Hypothesis. For each $\xi, \eta \in E_{\mathbb{C}}$, there exists a function $\rho^Q_{\xi,\eta} \in E_{\mathbb{C}}$ such that

$$\nu^Q_{\xi,\eta}(S) = \int_S \rho^Q_{\xi,\eta}(\lambda) d\lambda, \quad S \in \mathcal{B}(\mathbb{R}). \tag{3.3}$$

We call the function $\rho^Q_{\xi,\eta}$ the spectral density of the operator Q associated with ξ, η.

Proposition 3.1. *The spectral density $\rho^Q_{\xi,\eta}$ is positive definite in the sense that for each $n \geq 1$ and any $z_i \in \mathbb{C},\ \xi_i \in E_{\mathbb{C}},\ i = 1, 2, \cdots, n$,*

$$\sum_{i,j=1}^n z_i \overline{z_j}\, \rho^Q_{\xi_i,\bar{\xi}_j} \geq 0 \quad \textit{as a function on } \mathbb{R}. \tag{3.4}$$

Proof. Let $\varphi = \sum_{i=1}^n z_i\, \phi_{\xi_i}$. Then for each $S \in \mathcal{B}(\mathbb{R})$, we have

$$\begin{aligned} \int_S \sum_{i,j=1}^n z_i \overline{z_j}\, \rho^Q_{\xi_i,\bar{\xi}_j}(\lambda)\, d\lambda &= \sum_{i,j=1}^n z_i \overline{z_j}\, \nu^Q_{\xi_i,\bar{\xi}_j}(S) \\ &= \sum_{i,j=1}^n z_i \overline{z_j}\, \langle\!\langle P_Q(S)\, \phi_{\xi_i}, \phi_{\bar{\xi}_j} \rangle\!\rangle \\ &= \langle\!\langle P_Q(S)\, \varphi, \overline{\varphi} \rangle\!\rangle \\ &= \|P_Q(S)\, \varphi\|_0^2 \\ &\geq 0 \end{aligned}$$

where $\|\cdot\|_0$ denotes the norm of (L^2). Hence $\sum_{i,j=1}^n z_i \overline{z_j}\, \rho^Q_{\xi_i,\bar{\xi}_j} \geq 0$ as a function on $\mathbb{R}$. □

Proposition 3.2. *Let $Dom\, Q^n$ be the domain of Q^n, where $n \geq 0$. Then $\{\, \phi_\xi \mid \xi \in E_{\mathbb{C}} \,\} \subset Dom\, Q^n$.*

Proof. Let $\xi \in E_{\mathbb{C}}$. By Proposition 3.1 and the Hypothesis, we have

$$0 \leq \int_{\mathbb{R}} \lambda^{2n}\, \rho^Q_{\xi,\bar{\xi}}(\lambda)\, d\lambda < +\infty.$$

Hence

$$\begin{aligned} \int_{\mathbb{R}} |\lambda^n|^2 \langle\!\langle P_Q(d\lambda)\phi_\xi, \phi_\xi \rangle\!\rangle &= \int_{\mathbb{R}} \lambda^{2n}\, \langle\!\langle P_Q(d\lambda)\phi_\xi, \overline{\phi_\xi} \rangle\!\rangle \\ &= \int_{\mathbb{R}} \lambda^{2n}\, \nu^Q_{\xi,\bar{\xi}}(d\lambda) \\ &= \int_{\mathbb{R}} \lambda^{2n}\, \rho^Q_{\xi,\bar{\xi}}(\lambda)\, d\lambda \\ &< +\infty, \end{aligned}$$

which implies that $\phi_\xi \in \mathrm{Dom}\, Q^n$. □

The above propositions show useful properties of the operator Q. Now we use them to define Schwartz generalized functions of Q.

Theorem 3.3. *Assume that the spectral density $\rho^Q_{\xi,\eta}$ satisfies that for each $q \geq 0$ there exist constants a, k, $p \geq 0$ such that*

$$|\rho^Q_{\xi,\eta}|_q \leq k\exp\{a(|\xi|_p^2 + |\eta|_p^2)\}, \quad \xi,\, \eta \in E_{\mathbb{C}}. \tag{3.5}$$

Then for each Schwartz generalized function $\omega \in E^* = S^*(\mathbb{R})$, there exists a unique generalized operator $X^Q_\omega \in \mathcal{L}$ such that

$$\widehat{X^Q_\omega}(\xi,\eta) = \langle \omega, \rho^Q_{\xi,\eta}\rangle, \quad \xi,\, \eta \in E_{\mathbb{C}}. \tag{3.6}$$

Proof. Obviously (3.6) implies the uniqueness of X^Q_ω. Now we prove the existence. Let $\omega \in E^*$. Then there is $q \geq 0$ such that $\omega \in E_{-q}$. Since E is dense in E_{-q}, we can take a sequence $\{ f_n \}_{n\geq 1} \subset E$ such that $f_n \longrightarrow \omega$ in the norm $|\cdot|_{-q}$. For each $n \geq 1$, $f_n(Q) = \int_{\mathbb{R}} f_n(\lambda) P_Q(d\lambda)$ is a bounded linear operator on (L^2) since f_n is a bounded function. Hence $f_n(Q) \in \mathcal{L}$ for all $n \geq 1$.

We assert that the sequence $\{ f_n(Q) \}$ satisfies the two conditions of Lemma 2.1.

In fact, for each ξ, $\eta \in E_{\mathbb{C}}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \widehat{f_n(Q)}(\xi,\eta) &= \lim_{n\to\infty} \int_{\mathbb{R}} f_n(\lambda)\, \nu^Q_{\xi,\eta}(d\lambda) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} f_n(\lambda)\, \rho^Q_{\xi,\eta}(\lambda)\, d\lambda \\
&= \lim_{n\to\infty} \langle f_n, \rho^Q_{\xi,\eta}\rangle \\
&= \langle \omega, \rho^Q_{\xi,\eta}\rangle.
\end{aligned}$$

On the other hand, by the assumption we have

$$\begin{aligned}
|\widehat{f_n(Q)}(\xi,\eta)| &= |\langle f_n, \rho^Q_{\xi,\eta}\rangle| \\
&\leq |f_n|_q |\rho^Q_{\xi,\eta}|_q \\
&\leq \alpha k \exp\{a(|\xi|_p^2 + |\eta|_p^2)\}
\end{aligned}$$

$\forall\, \xi,\, \eta \in E_{\mathbb{C}}$, where $\alpha = \sup_{n\geq 1} |f_n|_{-q} < \infty$ since $\{ f_n \}_{n\geq 1}$ is convergent in the norm $|\cdot|_{-q}$.

By Lemma 2.1, there exists a generalized operator, denoted by X^Q_ω, such that $f_n(Q) \longrightarrow X^Q_\omega$ in $\mathcal{L}$, which implies

$$\lim_{n\to\infty} \widehat{f_n(Q)}(\xi,\eta) = \widehat{X^Q_\omega}(\xi,\eta), \quad \forall\, \xi,\, \eta \in E_{\mathbb{C}},$$

which implies (3.6). □

Proposition 3.4. *Let $\rho^Q_{\xi,\eta}$ be as in Theorem 3.3 and $f \in E = S(\mathbb{R})$. Then*

$$X^Q_f = f(Q) \qquad \textit{(as generalized operators)} \tag{3.7}$$

where $f(Q) = \int_{\mathbb{R}} f(\lambda)\, P_Q(d\lambda)$, which is well known as the f-function of Q.

Proof. $f(Q)$ is a bounded operator on (L^2), which means $f(Q) \in \mathcal{L}$. For each ξ, $\eta \in E_{\mathbb{C}}$, by a straightforward computation, we find that

$$\widehat{f(Q)}(\xi,\eta) = \widehat{X^Q_f}(\xi,\eta),$$

which implies (3.7). □

Motivated by Proposition 3.4, we now give the definition of Schwartz generalized functions of the operator Q as follows.

Definition 3.1. Let $\rho^Q_{\xi,\eta}$ be as in Theorem 3.3. For a Schwartz generalized function $\omega \in E^*$, we define

$$\omega(Q) = X^Q_\omega \tag{3.8}$$

and call it the ω-function of Q.

Remark 3.1. Let δ be the Dirac δ-function. Then $\delta \in E^*$. Hence, under the above conditions upon Q and $\rho^Q_{\xi,\eta}$, $\delta(Q)$ makes sense as a generalized operator.

In the following, we investigate properties of the Schwartz generalized functions of Q defined above.

Theorem 3.5. *Let $\rho^Q_{\xi,\eta}$ be as in Theorem 3.3 and $n \geq 0$. Let $\omega_n \in E^*$ be defined by*

$$\langle \omega_n, f\rangle = \int_{\mathbb{R}} \lambda^n f(\lambda)\, d\lambda, \quad f \in E. \tag{3.9}$$

Then

$$\omega_n(Q)\varphi = Q^n \varphi, \quad \varphi \in \mathcal{D} \tag{3.10}$$

where $\mathcal{D} \equiv Span\{\, \phi_\xi \mid \xi \in E_{\mathbb{C}}\,\}$ is the linear subspace of (E) spanned by $\{\, \phi_\xi \mid \xi \in E_{\mathbb{C}}\,\}$.

Proof. Firstly, by Proposition 3.2, we see that $\mathcal{D} \subset \operatorname{Dom} Q^n$. On the other hand, for each ξ, $\eta \in E_{\mathbb{C}}$, we have

$$\begin{aligned}
\langle\!\langle Q^n\phi_\xi, \phi_\eta\rangle\!\rangle &= \left(\!\!\left(\int_{\mathbb{R}} \lambda^n\, P_Q(d\lambda)\phi_\xi, \overline{\phi_\eta}\right)\!\!\right) \\
&= \int_{\mathbb{R}} \lambda^n (\!(P_Q(d\lambda)\phi_\xi, \phi_{\bar\eta})\!) \\
&= \int_{\mathbb{R}} \lambda^n \langle\!\langle P_Q(d\lambda)\phi_\xi, \phi_\eta\rangle\!\rangle \\
&= \int_{\mathbb{R}} \lambda^n\, \rho^Q_{\xi,\eta}(\lambda)\, d\lambda \\
&= \langle \omega_n, \rho^Q_{\xi,\eta}\rangle \\
&= \langle\!\langle \omega_n(Q)\phi_\xi, \phi_\eta\rangle\!\rangle,
\end{aligned}$$

where $(\!(\cdot,\cdot)\!)$ means the inner product of (L^2). Hence (3.10) follows. □

Remark 3.2. Let $\rho^Q_{\xi,\eta}$ be as in Theorem 3.3. Then from Theorem 3.5 we see that

$$Q\varphi = \omega_1(Q)\,\varphi, \quad \varphi \in \mathcal{D}. \tag{3.11}$$

Note that $\mathcal{D}$ is not only a dense subspace of (E) but also a dense subspace of (L^2). Hence Q itself can be viewed as a generalized operator.

Theorem 3.6. *Let $\rho^Q_{\xi,\eta}$ be as in Theorem 3.3. Define a mapping $Z : E^* \longmapsto \mathcal{L}$ as follows:*

$$Z(\omega) = \omega(Q), \quad \omega \in E^*. \tag{3.12}$$

Then $Z : E^* \longmapsto \mathcal{L}$ is a continuous linear mapping.

Proof. Z is obviously linear. We now prove its continuity. Let $\{\omega^{(k)}\}_{k\geq 1} \subset E^*$ and $\omega \in E^*$ be such that $\omega^{(k)} \longrightarrow \omega$ in E^*. Then there exists some $q \geq 0$ such that $\omega, \omega^{(k)} \in E_{-q}$, $k \geq 1$ and

$$\omega^{(k)} \longrightarrow \omega \qquad (\text{in the norm } |\cdot|_{-q}).$$

With an argument similar to that in the proof of Theorem 3.3, we can get a generalized operator X such that

$$Z(\omega^{(k)}) = \omega^{(k)}(Q) \longrightarrow X \quad (\text{in } \mathcal{L}).$$

On the other hand, we have

$$\begin{aligned}
\widehat{X}(\xi,\eta) &= \lim_{k\to\infty} \widehat{Z(\omega^{(k)})}(\xi,\eta) \\
&= \lim_{k\to\infty} \langle \omega^{(k)}, \rho^Q_{\xi,\eta}\rangle \\
&= \langle \omega, \rho^Q_{\xi,\eta}\rangle \\
&= \widehat{\omega(Q)}(\xi,\eta) \\
&= \widehat{Z(\omega)}(\xi,\eta),
\end{aligned}$$

$\forall\, \xi,\eta \in E_{\mathbb{C}}$, which implies $X = Z(\omega)$. Hence $Z(\omega^{(k)}) \longrightarrow Z(\omega)$ in $\mathcal{L}$. □

Theorem 3.7. *Let $\rho^Q_{\xi,\eta}$ be as in Theorem 3.3 and $Z : E^* \longmapsto \mathcal{L}$ as in Theorem 3.6. Then Z is positivity-preserving in the sense that*

$$\langle\!\langle Z(\omega)\varphi, \overline{\varphi}\rangle\!\rangle \geq 0, \quad \varphi \in (E) \tag{3.13}$$

whenever $\omega \in E^*$ and $\omega \geq 0$.

Proof. Let $\omega \in E^*$ with $\omega \geq 0$. To prove (3.13), we only need to show that for each $n \geq 1$ and any $z_i \in \mathbb{C}$, $\xi_i \in E_{\mathbb{C}}$, $i = 1, 2, \cdots, n$,

$$\left\langle\!\!\left\langle Z(\omega)\sum_{i=1}^n z_i\phi_{\xi_i}, \overline{\sum_{i=1}^n z_i\phi_{\xi_i}}\right\rangle\!\!\right\rangle \geq 0.$$

In fact, we have

$$\begin{aligned}
\left\langle\!\!\left\langle Z(\omega)\sum_{i=1}^n z_i\phi_{\xi_i}, \overline{\sum_{i=1}^n z_i\phi_{\xi_i}}\right\rangle\!\!\right\rangle &= \sum_{i,j=1}^n z_i\overline{z_j}\langle\!\langle Z(\omega)\phi_{\xi_i}, \phi_{\bar{\xi_j}}\rangle\!\rangle \\
&= \sum_{i,j=1}^n z_i\overline{z_j}\langle \omega, \rho^Q_{\xi_i,\bar{\xi_j}}\rangle \\
&= \left\langle \omega, \sum_{i,j=1}^n z_i\overline{z_j}\rho^Q_{\xi_i,\bar{\xi_j}}\right\rangle \\
&\geq 0,
\end{aligned}$$

where, by Proposition 3.1, $\sum_{i,j=1}^n z_i\overline{z_j}\,\rho^Q_{\xi_i,\bar{\xi_j}} \geq 0$ as a function on $\mathbb{R}$. □

By Theorem 3.7, we immediately come to the following proposition.

Proposition 3.8. *$\delta(Q)$ is positive, i.e., $\langle\!\langle \delta(Q)\varphi, \overline{\varphi}\rangle\!\rangle \geq 0$, $\forall\, \varphi \in (E)$.*

Remark 3.3. The physical meaning of the fact that $\delta(Q)$ is positive can be interpreted as follows. From the physical point of view, the self-adjoint operator Q stands for an observable. Naturally, as a generalized operator, $\delta(Q)$ can be viewed as an observable associated with the observable Q. Hence the positivity property of $\delta(Q)$ implies that it is a positive observable.

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