ON A MULTIDIMENSIONAL FORM OF F. RIESZ’S “RISING SUN” LEMMA

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ABSTRACT. A multidimensional version of the Riesz rising sun lemma is proved by means of a generalized dyadic process.

The “rising sun” lemma of F. Riesz [11] is very important in differentiation theory, in the theory of the one-dimensional Hardy-Littlewood maximal function (see [3], [12]), and, as was mentioned by E. M. Stein [14], “...had implicitly played a key role in the earlier treatment of the Hilbert transform.”

Lemma (F. Riesz (cf. [5])). *Let f be a summable function on some interval $I_0 \subset \mathbb{R}$, and suppose $|I_0|^{-1} \int_{I_0} f(x) dx \leq A$. Then there is a finite or countable set of pairwise disjoint subintervals $I_j \subset I_0$ such that $|I_j|^{-1} \int_{I_j} f(x) dx = A$ ($j = 1, 2, \dots$), and $f(x) \leq A$ for almost all $x \in I_0 \setminus \left(\bigcup_{j \geq 1} I_j\right)$.*

All known proofs of this lemma (and its variants) are based on the topological structure of the real line. In particular, the lemma does not rely on Lebesgue measure, and it works for arbitrary absolutely continuous measures. This fact can be used, for example, to show that the one-dimensional Hardy-Littlewood maximal function with respect to any measure is of weak type $(1, 1)$. As is well known, for $n \geq 2$ the situation is quite different [13].

A classical multidimensional substitute of Riesz’s lemma is the Calderón-Zygmund lemma [2], which is probably one of the most important propositions of harmonic analysis.

Lemma (Calderón-Zygmund). *Let $f \geq 0$ be a summable function on some cube $Q_0 \subset \mathbb{R}^n$, and suppose $|Q_0|^{-1} \int_{Q_0} f(x) dx \leq A$. Then there is a finite or countable set of pairwise disjoint subcubes $Q_j \subset Q_0$ such that $A < |Q_j|^{-1} \int_{Q_j} f(x) dx \leq 2^n A$ ($j = 1, 2, \dots$), and $f(x) \leq A$ for almost all $x \in Q_0 \setminus \left(\bigcup_{j \geq 1} Q_j\right)$.*

In the one-dimensional case an “interval” and a “cube” are the same. Thus Riesz’s lemma is a more precise result than the Calderón-Zygmund lemma. This

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makes the use of Riesz's lemma preferable for obtaining some sharp results. For instance, sharp estimates of non-increasing rearrangements of functions from BMO [5], and of functions satisfying Gehring, Muckenhoupt, and Gurov-Reshetnyak type conditions [6], [7], [8] were obtained by means of Riesz's lemma.

A natural question arises as to whether one can improve the Calderón-Zygmund lemma for $n \geq 2$; in other words, whether one can get a multidimensional variant of Riesz's lemma, where by an "interval" we mean a "cube". Unfortunately, simple examples show that this is not the case. Indeed, it suffices to consider $f(x) = 1 - \chi_{[1/2, 1]^2}(x)$, $Q_0 = [0, 1]^2$, and a number close to 1, for instance, $A = 7/8$.

It is well known that the geometry of rectangular intervals $I = \prod_{i=1}^n [a_i, b_i]$ (which we will simply call rectangles) is much richer in comparison with cubes. Therefore, it is either very surprising or very natural that rectangles play a main role in an actual multidimensional analogue of Riesz's lemma, and this is the result which we shall prove.

Lemma. *Let I_0 be a rectangle in $\mathbb{R}^n$, and let μ be an absolutely continuous Borel measure on I_0 . Let $f \in L_\mu(I_0)$ and $\frac{1}{\mu(I_0)} \int_{I_0} f(x) d\mu \leq A$. Then there is a finite or countable set of pairwise disjoint rectangles $I_j \subset I_0$ with $(\mu(I_j))^{-1} \int_{I_j} f(x) d\mu = A$ ($j = 1, 2, \dots$), and $f(x) \leq A$ for μ -almost all points $x \in I_0 \setminus (\bigcup_{j \geq 1} I_j)$.*

Before proving this lemma, several remarks are in order.

Remark 1. Crucial steps in this direction were made by the first author, who suggested a weak version of Riesz's lemma in the multidimensional case [9], [10].

Remark 2. In the two-dimensional case the lemma is contained implicitly in Besicovitch's paper [1, Lemma 1]. However, an attempt to extend the method of that paper to $n \geq 3$ allows us only to get the lemma with an additional restriction on the growth of f ; namely, f must belong to $L(\log^+ L)^{n-2}$.

Remark 3. As we mentioned above, known proofs of Riesz's lemma are essentially based on the order structure of the real line. The proof of the Calderón-Zygmund lemma is different; namely, it relies on the concept of dyadic cubes and on a stopping-time argument. Nevertheless, both proofs are unified by the fact that the union of the relevant intervals is a level set $\{x \in P_0 : \widetilde{M}f(x) > A\}$. In the case of Riesz's lemma, $P_0 = I_0$ and $\widetilde{M}f$ is the one-sided (left or right) Hardy-Littlewood maximal function. In the case of the Calderón-Zygmund lemma, $P_0 = Q_0$ and $\widetilde{M}f$ is the dyadic maximal function. Our proof does not rely on the strong maximal function. More precisely, our main idea is a generalized dyadic process; that is, for any number A , we construct an individual differential basis of rectangles that has nice covering properties similar to the basis of dyadic cubes.

Remark 4. In the case $n = 1$ we get a new simple proof of Riesz's lemma itself.

Proof of the lemma. If $f_{I_0} = A$, then the lemma is obvious. Assume that $f_{I_0} < A$. Divide I_0 into two rectangles by the $(n-1)$ -dimensional hyperplane passing through the middle of the largest side. The mean value of f over at least one of the partial rectangles must be less than A . If the mean value of f over both rectangles is less than A , we continue the division of each of them. Otherwise we obtain that the mean value of f over one of them is less than A , while over the other one is greater than A . In this case we translate the hyperplane until we get two rectangles such

that the mean value of f over one of them is exactly equal to A , while over the other one is less than A (here we use the absolute continuity of μ). We put the larger rectangle (over which the mean value is equal to A) into the family $\{I_j\}$, while the smaller one will be further divided. Thus we have described the division argument.

We will continue the above process applying each time the division argument to the rectangle that should be divided. As a result we will get two families of rectangles: a finite or countable family of pairwise disjoint rectangles $\{I_j\}$ such that $f_{I_j} = A$ ($j \geq 1$), and a countable family of rectangles $\{J_j\}$ for which $f_{J_j} < A$ ($j \geq 1$).

We now observe that the rectangles from $\{J_j\}$ have the following “dyadic” property: if $J_i \cap J_k \neq \emptyset$, then either $J_i \subset J_k$ or $J_k \subset J_i$ for all $J_i, J_k \in \{J_j\}$. In the division argument, each time we divided the longest side, we took the smaller part; hence, for any $x \notin \bigcup_j I_j$ there is a sequence of rectangles $J_m \in \{J_j\}$ containing x and such that their diameters tend to zero. Therefore, the rectangles $\{J_j\}$ form a differential basis (see [4, Ch. 2, §2] for definitions) on the set $E \equiv I_0 \setminus \left(\bigcup_{j \geq 1} I_j\right)$ with nice covering properties. In particular, clearly that “dyadic” property implies the Vitali covering property [4, Ch. 1], and hence the basis $\mathcal{B} = \{J_j\}$ differentiates $L^1(E)$. By the standard differentiation argument we obtain $f(x) \leq A$ for μ -almost all x from E . $\square$

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