SPECTRALLY BOUNDED ϕ -DERIVATIONS ON BANACH ALGEBRAS

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ABSTRACT. Applying the density theorem on algebras with ϕ -derivations, we show that if a ϕ -derivation δ of a unital Banach algebra A is spectrally bounded, then $[\delta(A), A] \subseteq \text{rad}(A)$. Also, $\delta(A) \subseteq \text{rad}(A)$ if and only if $\sup\{r(z^{-1}\delta(z)) \mid z \in A \text{ is invertible}\} < \infty$, where $r(a)$ denotes the spectral radius of $a \in A$.

1. INTRODUCTION

Throughout, unless specially stated, A always denotes a *unital* Banach algebra over the complex field $\mathbf{C}$. We denote by $\text{rad}(A)$ the Jacobson radical of A and by $r(x)$ the spectral radius of $x \in A$. Also, let $Q(A)$ be the set of all quasinilpotent elements of A . A *derivation* $d: A \rightarrow A$ is a linear mapping satisfying $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$; an *inner derivation* is a derivation of the form $d(x) = ax - xa$ for some $a \in A$, and a *generalized inner derivation* is one of the form $d(x) = ax - xb$ for some $a, b \in A$. All derivations or automorphisms of A are always assumed to be linear. A linear mapping $T: A \rightarrow A$ is called *spectrally bounded* if there exists $M \geq 0$ such that $r(T(x)) \leq M r(x)$ for all $x \in A$. In addition, if $M = 0$ (i.e., $T(A) \subseteq Q(A)$), T is called *spectrally infinitesimal*. Clearly, we have the following implications:

$$T(A) \subseteq \text{rad}(A) \Rightarrow T \text{ spectrally infinitesimal} \Rightarrow T \text{ spectrally bounded.}$$

However, none of the above reverse implications is true in general. The Singer–Wermer Theorem [14] states that $d(A) \subseteq \text{rad}(A)$ if d is a bounded derivation of a commutative Banach algebra A . In 1988, Thomas [15] proved the same conclusion without the bounded assumption on the derivation d . The noncommutative Singer–Wermer conjecture states that each derivation d on a Banach algebra A leaves each primitive ideal invariant, which is equivalent to $d(A) \subseteq \text{rad}(A)$ if $[d(a), a] \in \text{rad}(A)$ for all $a \in A$ [12, Theorem 1.8]. Also, d spectrally infinitesimal always implies that $d(A) \subseteq \text{rad}(A)$ for d bounded [11] or arbitrary [16].

In [1] Brešar proved that every spectrally bounded inner derivation maps into the radical. Later, Curto and Mathieu [7] characterized spectrally bounded generalized inner derivations. Finally, Brešar and Mathieu [4] proved that a derivation d of A is spectrally bounded if and only if $d(A) \subseteq \text{rad}(A)$. In [3] Brešar proved an extension

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of the Jacobson density theorem for Banach algebras with automorphisms. As its applications, automorphisms ϕ of Banach algebras such that $\phi - 1$ is spectrally bounded are characterized (see [3, Theorems 3.1 and 3.3]).

Let ϕ be an automorphism of A . By a ϕ -derivation of A we mean an additive map $\delta: A \rightarrow A$ such that $\delta(xy) = \phi(x)\delta(y) + \delta(x)y$ for all $x, y \in A$. Clearly, derivations and the maps $\phi - 1$ are ϕ -derivations. In the present paper we study ϕ -derivations to unify both derivations and maps of type $\phi - 1$ on Banach algebras. Applying the density theorem on algebras with ϕ -derivations we will prove the following two main results:

Theorem 1.1. *Let δ be a spectrally bounded ϕ -derivation on a unital Banach algebra A . Then $[\delta(A), A] \subseteq \text{rad}(A)$.*

Theorem 1.2. *Let δ be a ϕ -derivation of a unital Banach algebra A . Then $\delta(A) \subseteq \text{rad}(A)$ if and only if $\sup\{r(z^{-1}\delta(z)) \mid z \in A \text{ is invertible}\} < \infty$.*

Remarks. (1) Theorem 1.1 was proved by Brešar for $\delta = \phi - 1$ [3, Theorem 3.1]. Moreover, in this case, $[\delta(A), A] \subseteq \text{rad}(A)$ implies that δ is spectrally bounded. However, the general case is unknown even when δ is a derivation. Indeed, in view of [9, Theorem 1.2] the condition that $[\delta(a), a] \in \text{rad}(A)$ for all $a \in A$ implies that $[\delta(A), A] \subseteq \text{rad}(A)$. Hence, if the reverse implication of Theorem 1.1 is true for the derivation case, then δ is spectrally bounded and so, by Brešar and Mathieu's theorem [4], $\delta(A) \subseteq \text{rad}(A)$ follows. Thus the noncommutative Singer–Wermer conjecture is true.

(2) Theorem 1.2 has been obtained by Brešar and Mathieu [4, Theorem 2.6] for derivations and by Brešar [3, Theorem 3.3] for the ϕ -derivation $\phi - 1$.

2. PRELIMINARY RESULTS

In this section we will quote some results given in [10], which will be used in the sequel. For simplicity, we only state its special form required here. Throughout this section, unless specially stated, we make the following assumptions:

Let A be an algebra over a field k with ${}_A E$ an irreducible left A -module, and let $D = \text{End}({}_A E)$ be the associated division algebra of ${}_A E$. Let ϕ be a k -linear automorphism of A . By the Jacobson density theorem, A acts densely on E_D . We define the map $\pi: A \rightarrow \text{End}(E_D)$ by the left multiplications:

$$a \in A \mapsto \pi(a) \in \text{End}(E_D), \text{ where } \pi(a)m = am \text{ for all } m \in E.$$

By a ϕ -derivation $\delta: A \rightarrow \text{End}(E_D)$ we mean that δ is a k -linear map satisfying $\delta(xy) = \pi\phi(x)\delta(y) + \delta(x)\pi(y)$ for all $x, y \in A$. We remark that if $\delta: A \rightarrow A$ is a k -linear ϕ -derivation, then δ canonically induces a ϕ -derivation $\bar{\delta}: A \rightarrow \text{End}(E_D)$ by the rule: $a \in A \mapsto \bar{\delta}(a) = \pi\delta(a)$.

Definition. (1) A k -linear automorphism ϕ of A is called E -inner if there exists an invertible element $S \in \text{End}_k(E)$ such that $\pi\phi(a) = S^{-1}\pi(a)S$ for all $a \in A$. Otherwise, it is called E -outer.

(2) A ϕ -derivation $\delta: A \rightarrow \text{End}(E_D)$ is called E -inner if there exists $T \in \text{End}_k(E)$ such that $\delta(a) = \pi\phi(a)T - T\pi(a)$ for all $a \in A$. Otherwise, it is called E -outer. Moreover, if a ϕ -derivation $\delta: A \rightarrow A$ is called E -inner (E -outer) if its induced ϕ -derivation $\bar{\delta}: A \rightarrow \text{End}(E_D)$ is E -inner (resp. E -outer).

The following two theorems are special forms of [10, Theorems 2.2 and 2.3] (by considering the case of left modules).

Theorem 2.1. *A ϕ -derivation $\delta: A \rightarrow \text{End}(E_D)$, where ϕ is E -inner, is E -outer if and only if given finitely many D -independent $x_i \in E$ and arbitrary $y_i, z_i \in E$, where $1 \leq i \leq n$, there exists $a \in A$ such that $\pi(a)x_i = y_i$, $\delta(a)x_i = z_i$ for $1 \leq i \leq n$.*

Theorem 2.2. *A ϕ -derivation $\delta: A \rightarrow \text{End}(E_D)$, where ϕ is E -outer, is E -outer if and only if given finitely many D -independent $x_i \in E$ and arbitrary $y_i, z_i, w_i \in E$, where $1 \leq i \leq n$, there exists $a \in A$ such that $\pi(a)x_i = y_i$, $\pi\phi(a)x_i = z_i$, and $\delta(a)x_i = w_i$ for $1 \leq i \leq n$.*

3. PROOFS

We are now ready to proceed with our proofs of the theorems stated in the first section. We begin with the following:

Lemma 3.1. *Let δ be a spectrally bounded ϕ -derivation on A . Then $\text{rad}(A)$ is invariant under δ .*

Proof. Let $x \in \text{rad}(A)$ and $a \in A$. Then $\phi^{-1}(a)x \in \text{rad}(A)$, and so $r(\phi^{-1}(a)x) = 0$. Since δ is spectrally bounded, applying the fact that $\delta(\phi^{-1}(a))x \in \text{rad}(A)$ we have

$$0 = r(\delta(\phi^{-1}(a))x) = r(a\delta(x) + \delta(\phi^{-1}(a))x) = r(a\delta(x)),$$

implying that $a\delta(x)$ is quasi-nilpotent and hence is also quasi-invertible in A . Thus $\delta(x) \in \text{rad}(A)$. This proves that $\delta(\text{rad}(A)) \subseteq \text{rad}(A)$, as asserted. $\square$

Let A be a Banach algebra with ϕ a linear automorphism, and let $\delta: A \rightarrow A$ be a ϕ -derivation. Suppose that (π, E) is a continuous irreducible representation of A . It is well known that $\text{End}({}_A E) = \mathbf{C}$. Applying the notion given in §2 we have: (1) An automorphism ϕ of A is E -inner if there exists an invertible $S \in \text{End}_{\mathbf{C}}(E)$ such that $\pi\phi(a) = S^{-1}\pi(a)S$ for all $a \in A$. (2) A ϕ -derivation δ on A is called E -inner if there exists $T \in \text{End}_{\mathbf{C}}(E)$ such that $\pi\delta(a) = \pi\phi(a)T - T\pi(a)$ for all $a \in A$. We need the following result in the proof of Theorem 1.1.

Lemma 3.2. *Let δ be a ϕ -derivation of A , and let (π, E) be a continuous irreducible representation of A with $\dim_{\mathbf{C}} E \geq 2$. Suppose that δ is E -outer. Then there exist $\mathbf{C}$ -independent vectors $u, v \in E$ and $x, y \in A$ such that*

$$(3.1) \quad \begin{aligned} &\pi\delta(x)v = 0, \pi\phi(x)u = v, \pi(x)v = 0 \text{ and} \\ &\pi\delta(y)v = u, \pi\phi(y)v = 0, \pi\phi(y)u = 0, \pi(y)v = 0. \end{aligned}$$

Proof. Suppose first that ϕ is E -outer. Choose $\mathbf{C}$ -independent $u, v \in E$. In view of Theorem 2.2, we can choose $x, y \in A$ satisfying (3.1). Suppose next that ϕ is E -inner. Thus, by definition, there exists an invertible element $S \in \text{End}_{\mathbf{C}}(E)$ such that $\pi\phi(a) = S^{-1}\pi(a)S$ for all $a \in A$. We can choose $\mathbf{C}$ -independent $u, v \in E$ such that Su and v are $\mathbf{C}$ -independent. Indeed, we first choose a nonzero $u \in E$. If $Su = \beta u$ for some $\beta \in \mathbf{C}$, we are done by choosing an arbitrary $v \notin \mathbf{C}u$. Otherwise, $Su \notin \mathbf{C}u$, and let $v = u + Su$ in this case. Applying Theorem 2.1 we can choose $x \in A$ such that

$$\pi\delta(x)v = 0, \pi(x)Su = Sv, \text{ and } \pi(x)v = 0.$$

Thus $\pi\phi(x)u = S^{-1}\pi(x)Su = v$. Since u and v are $\mathbf{C}$ -independent, by Theorem 2.1 there exists $y \in A$ such that $\pi\delta(y)v = u$ and $\pi(y)$ maps the $\mathbf{C}$ -subspace $\mathbf{C}v + \mathbf{C}Su + \mathbf{C}Sv$ into zero. But $S\pi\phi(y) = \pi(y)S$; this means that $\pi\phi(y)v = 0$, $\pi\phi(y)u = 0$, and $\pi(y)v = 0$. This proves the lemma. $\square$

Let $R_i \in \text{Hom}_F(U, V)$, $1 \leq i \leq n$, where U and V are two vector spaces over a field F . We say that the R_i 's are *locally dependent* if, for each $u \in U$, the set $\{R_i u \mid 1 \leq i \leq n\}$ is F -dependent. The following theorem is given in [5, Theorem 2.4]. We state its form needed here.

Theorem 3.3 (Brešar and Šemrl [5]). *Let $R_i \in \text{Hom}_F(U, V)$, $i = 1, 2, 3$, where U and V are two vector spaces over an infinite field F , $\text{char } F \neq 2$. Suppose that R_1, R_2 , and R_3 are locally dependent but not F -dependent. Then either $\dim_F R_i U \leq 3$ for all i or there exists $P \in \text{End}_F(V)$ an idempotent of rank 1 such that*

$$\dim \text{span}\{(I_V - P)R_1, (I_V - P)R_2, (I_V - P)R_3\} = 1.$$

We are now in a position to give the

Proof of Theorem 1.1. We may assume that A is semisimple. Indeed, note that $\phi(\text{rad}(A)) = \text{rad}(A)$. Thus ϕ canonically induces an automorphism $\bar{\phi}$ on $A/\text{rad}(A)$. By Lemma 3.1, $\delta(\text{rad}(A)) \subseteq \text{rad}(A)$. We set $\bar{x} = x + \text{rad}(A) \in A/\text{rad}(A)$ for $x \in A$. Then δ also induces a $\bar{\phi}$ -derivation $\bar{\delta}$ on $A/\text{rad}(A)$ by the rule: $\bar{\delta}(\bar{x}) = \overline{\delta(x)}$ for all $x \in A$. Moreover,

$$r(\bar{\delta}(\bar{x})) = r(\delta(x)) \leq Mr(x) = Mr(\bar{x})$$

for all $x \in A$. That is, $\bar{\delta}$ is still a spectrally bounded $\bar{\phi}$ -derivation on $A/\text{rad}(A)$, a semisimple Banach algebra. Thus $[\bar{\delta}(A/\text{rad}(A)), A/\text{rad}(A)] = 0$, and so $[\delta(A), A] \subseteq \text{rad}(A)$, as asserted. So we can assume from the start that A is semisimple.

By [8] and [6, Corollary 4.3], both ϕ and δ are continuous. Let (π, E) be a continuous irreducible representation of A . Clearly, it suffices to prove that $[\delta(A), A] \subseteq \ker(\pi)$ for each (π, E) . If $\dim_{\mathbf{C}} E = 1$, then $[\pi(A), \pi(A)] = 0$ and so $[A, A] \subseteq \ker(\pi)$, as asserted. We now consider the case that $\dim_{\mathbf{C}} E \geq 2$. The aim is to prove that $\pi\delta(A)E = 0$.

Case 1. Suppose that δ is E -outer. Let $u, v \in E$ and $x, y \in A$ be given as in Lemma 3.2. That is,

$$\pi\delta(y)v = u, \quad \pi\phi(y)v = 0, \quad \pi\phi(y)u = 0, \quad \text{and} \quad \pi(y)v = 0.$$

We set $z_n = e^{ny}$ for $n \geq 0$. Applying the continuity of both π and δ we have

$$\pi(z_n)v = v, \quad \pi\phi(z_n)v = v, \quad \text{and}$$

$$\pi\delta(z_n)v = (\delta(1) + \pi(n\delta(y)) + \frac{1}{2!}(\pi\delta(n^2y^2) + \cdots))v = nu,$$

since

$$\pi\delta(n^k y^k)v = n^k((\pi\phi(y))^{k-1}\pi\delta(y) + \pi\delta(y^{k-1})\pi(y))v = 0$$

for $k \geq 2$. On the other hand, by (3.1) we have $\pi\delta(x)v = 0$, $\pi\phi(x)u = v$ and $\pi(x)v = 0$. We compute

$$\begin{aligned} & \pi\delta(z_n^{-1}xz_n)v \\ &= \pi(\phi(z_n^{-1})\phi(x)\delta(z_n))v + \pi(\phi(z_n^{-1})\delta(x)z_n)v + \pi(\delta(z_n^{-1})xz_n)v = nv. \end{aligned}$$

Then $Mr(x) = Mr(z_n^{-1}xz_n) \geq r(\delta(z_n^{-1}xz_n)) \geq r(\pi\delta(z_n^{-1}xz_n)) \geq n$ for all $n \geq 1$, a contradiction.

Case 2. Assume that δ is E -inner, but ϕ is E -outer. In this case, there exists $T \in \text{End}_{\mathbf{C}}(E)$ such that $\pi\delta(x) = \pi\phi(x)T - T\pi(x)$ for all $x \in A$. Suppose first that T is not a scalar. Choose $v \in E$ such that Tv and v are $\mathbf{C}$ -independent. In view of

[3, Theorems 2.1 and 1.2], there exist invertible $z_n \in A$ for each $n \geq 1$ and $x \in A$ such that

$$\begin{aligned}\pi\phi(z_n)Tv &= nv, \quad \pi\phi(z_n)v = Tv, \quad \pi(z_n)v = v, \\ \pi\phi(x)v &= Tv, \quad \text{and} \quad \pi(x)v = 0.\end{aligned}$$

Then $\pi\phi(z_n^{-1})Tv = v$, and so

$$\pi\delta(z_n^{-1}xz_n)v = \pi(\phi(z_n^{-1})\phi(x)\phi(z_n))Tv - T\pi(z_n^{-1}xz_n)v = nv.$$

This derives a contradiction as in Case 1. Hence, we assume that T is a nonzero scalar. Without loss of generality, we may assume that $T = 1$. Choose $\mathbf{C}$ -independent $u, v \in E$. By [3, Theorems 2.1 and 1.2] again, there exist invertible $z_n \in A$ for each $n \geq 1$ and $x \in A$ such that

$$\begin{aligned}\pi(z_n)v &= v, \quad \pi(z_n)u = u, \quad \pi\phi(z_n)v = u, \quad \pi\phi(z_n)u = v - nu, \\ \pi(x)v &= u, \quad \pi\phi(x)v = 0, \quad \text{and} \quad \pi\phi(x)u = v.\end{aligned}$$

Then $\pi\delta(z_n^{-1}xz_n)v = \pi(\phi(z_n^{-1})\phi(x)\phi(z_n) - z_n^{-1}xz_n)v = nv$, a contradiction.

Case 3. Assume that both δ and ϕ are E -inner. Thus there exist $S, T \in \text{End}_{\mathbf{C}}(E)$ with S invertible such that

$$\pi\delta(x) = \pi\phi(x)T - T\pi(x) \quad \text{and} \quad \pi\phi(x) = S^{-1}\pi(x)S \quad \text{for all } x \in A.$$

By the above we see that

$$\pi\delta(x) = S^{-1}[\pi(x), U] \quad \text{for all } x \in A,$$

where $U = ST$. Note that if U is a scalar, then $\pi\delta(A)E = 0$ and so $\delta(A) \subseteq \ker(\pi)$. We are done in this case. Assume that U is not a scalar.

Suppose there exists a vector $v \in E$ such that Sv, Uv and v are $\mathbf{C}$ -independent. In view of [13, p. 36], for $n \geq 0$ we choose an invertible element $z_n \in A$ satisfying

$$\pi(z_n)Sv = Sv, \quad \pi(z_n)Uv = v + nSv, \quad \text{and} \quad \pi(z_n)v = Uv.$$

Applying the Jacobson density theorem, we can choose $x \in A$ such that $\pi(x)v = 0$, $\pi(x)Uv = 0$ and $\pi(x)Sv = Sv$. We now compute

$$\begin{aligned}\pi\delta(z_n^{-1}xz_n)v &= (S^{-1}\pi(z_n^{-1})\pi(x)\pi(z_n)U - S^{-1}U\pi(z_n^{-1})\pi(x)\pi(z_n))v \\ &= S^{-1}\pi(z_n^{-1})\pi(x)\pi(z_n)Uv = nv.\end{aligned}$$

This derives a contradiction. Thus we have proved that, for each $v \in E$, the three vectors Sv, Uv and v are $\mathbf{C}$ -dependent. That is, I_E, S, U are locally dependent.

In view of Theorem 3.3, either $\dim_{\mathbf{C}} E \leq 3$, or there exists $P \in \text{End}_{\mathbf{C}}(E)$ an idempotent of rank 1 such that $\dim \text{span}\{I_E - P, (I_E - P)S, (I_E - P)U\} = 1$, or I_E, U and S are $\mathbf{C}$ -dependent. For brevity, we denote $I = I_E$.

Consider first that $\dim_{\mathbf{C}} E \leq 3$.

(I) Suppose that $\dim_{\mathbf{C}} E = 3$. In this case, $\pi(A) \cong M_3(\mathbf{C})$. For brevity, we write $\delta(x) = S^{-1}[x, U]$ for all $x \in A$. Writing U in its Jordan form modulo a scalar, we can choose U to be one of the following three types:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \quad \text{where } \alpha, \gamma \in \mathbf{C} \text{ with } \alpha \neq 0.$$

Since the first two cases have the same argument, we choose $U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ as

an example. Let $S = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Applying the fact that, for each $v \in E$,

Sv , Uv and v are $\mathbf{C}$ -dependent to $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ we see that $a_{32} = 0 =$

a_{31} . Thus $S^{-1} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & g \end{pmatrix}$. Let $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $z_n = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for

$n \geq 1$. Then $[z_n^{-1}xz_n, U] = \begin{pmatrix} n & 1 & 0 \\ 0 & -n & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and so $\delta(z_n^{-1}xz_n) = S^{-1}[z_n^{-1}xz_n, U] =$

$\begin{pmatrix} na & a - nb & 0 \\ nc & c - nd & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Let $\mu_n, \nu_n, 0$ be the eigenvalues of $\delta(z_n^{-1}xz_n)$. Then $\mu_n\nu_n =$

$na(c - nd) - nc(a - nb) = n^2(bc - ad) \neq 0$. This means $\lim_{n \rightarrow \infty} r(\delta(z_n^{-1}xz_n)) = \infty$, a contradiction.

We next consider the case $U = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}$. Let $S = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Apply-

ing the fact that, for each $v \in E$, Sv , Uv and v are $\mathbf{C}$ -dependent to $v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and

$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ we see that $a_{31} = 0 = a_{32}$. Thus $S^{-1} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & g \end{pmatrix}$. Let $x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and $z_n = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $[z_n^{-1}xz_n, U] = \begin{pmatrix} 0 & -\alpha & 0 \\ -n^2\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and so

$$\delta(z_n^{-1}xz_n) = S^{-1}[z_n^{-1}xz_n, U] = \begin{pmatrix} -n^2b\alpha & -a\alpha & 0 \\ -n^2d\alpha & -c\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let μ_n, ν_n and 0 be the eigenvalues of $\delta(z_n^{-1}xz_n)$. Then $\mu_n\nu_n = n^2\alpha^2(bc - ad) \neq 0$, implying that $\lim_{n \rightarrow \infty} r(\delta(z_n^{-1}xz_n)) = \infty$, a contradiction.

(II) Suppose that $\dim_{\mathbf{C}} E = 2$. In this case we may choose $U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or

$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$, where $0 \neq \alpha \in \mathbf{C}$. The two cases can be proved by the same arguments above.

Consider the next case: there exists $P \in \text{End}_{\mathbf{C}}(E)$ an idempotent of rank 1 such that $\dim \text{span}\{I - P, (I - P)S, (I - P)U\} = 1$. We also assume that I, U and S are $\mathbf{C}$ -independent. By the case above we may suppose that $\dim_{\mathbf{C}} E > 3$. By assumption, there exist $\alpha, \beta \in \mathbf{C}$ such that $(I - P)S = \alpha(I - P)$ and $(I - P)U = \beta(I - P)$. Then $S = \alpha I + P(S - \alpha I)$ and $U = \beta I + P(U - \beta I)$. Set

$Q_1 = P(S - \alpha I)$ and $Q_2 = P(U - \beta I)$. Then $\text{rank}(Q_i) \leq 1$ for $i = 1, 2$. According to the $\mathbf{C}$ -independence of I, S and U we have $Q_1 \notin \mathbf{C}$. Thus there exists $v \in E$ such that v and $Q_1 v$ are $\mathbf{C}$ -independent. Also, since $Q_2 \notin \mathbf{C}$ and $\text{rank}(Q_2) = 1$, there exists $w \in E$ such that $Q_2 w \neq 0$. Let W be the $\mathbf{C}$ -subspace of E generated by $v, Q_1 v, Q_2 w$. Then $\dim_{\mathbf{C}} W \leq 3$. Note that $Q_1 E = \mathbf{C}Q_1 v$ and $Q_2 E = \mathbf{C}Q_2 w$. Thus W is invariant under S and U . We consider the subalgebra $A' = \{x \in A \mid \pi(x)W \subseteq W\}$. Then, for $\pi(x) \in \pi(A')$, we have $\pi\delta(x)W = S[\pi(x), U]W \subseteq W$. Moreover, $M_3(\mathbf{C})$ is a homomorphic image of $\pi(A')$. To reduce the present case to the case that $\dim_{\mathbf{C}} E \leq 3$, we must only make a further remark: Let $a \in \text{End}_{\mathbf{C}}(W)$ be invertible. Then, by the density theorem [13, Theorem 6.7], there exists an invertible $z \in A$ such that $z|_W = a$ and hence $z^{-1}|_W = a^{-1}$.

We now consider the last case: I, U and S are $\mathbf{C}$ -dependent. If S is a scalar, then $\pi\delta(x) = [\pi(x), S^{-1}U] = [\pi(x), T]$ for all $x \in A$. Suppose that $\pi\delta(A) \neq 0$. Then T is not central in $\text{End}_{\mathbf{C}}(E)$. Choose $\zeta \in E$ such that ζ and $T\zeta$ are $\mathbf{C}$ -independent. By the density of $\pi(A)$ on $\text{End}_{\mathbf{C}}(E)$, there exists $x \in A$ such that $\pi(x)\zeta = 0$ and $\pi(x)T\zeta = \zeta$. On the other hand, by [13, Theorem 6.7] there exist invertible $z_n \in A$ for $n \geq 1$ such that $\pi(z_n)\zeta = \zeta$ and $\pi(z_n)T\zeta = nT\zeta$. We then compute

$$\pi\delta(z_n^{-1}xz_n)\zeta = (\pi(z_n^{-1})\pi(x)\pi(z_n)T - T\pi(z_n^{-1})\pi(x)\pi(z_n))\zeta = n\zeta,$$

a contradiction. Suppose next that S is not a scalar. Then $U = \alpha I + \beta S$ for some $\alpha, \beta \in \mathbf{C}$. Thus

$$\pi\delta(x) = S^{-1}[\pi(x), U] = S^{-1}[\pi(x), \alpha I + \beta S] = \beta(S^{-1}\pi(x)S - \pi(x))$$

for all $x \in A$. We follow the argument of [3, Theorem 3.1]. Choose $v \in E$ such that v and Sv are $\mathbf{C}$ -independent. Suppose first that S^2v, Sv and v are $\mathbf{C}$ -independent. Choose $x \in A$ and invertible $z_n \in A$ for $n \geq 1$ [13, Theorem 6.7] such that

$$\pi(x)Sv = S^2v, \quad \pi(x)S^2v = v,$$

$$\pi(z_n)v = Sv, \quad \pi(z_n)Sv = -(nSv + S^2v), \quad \text{and} \quad \pi(z_n)S^2v = v + nS^2v.$$

We then compute

$$\pi\delta(z_n^{-1}xz_n)v = \beta(S^{-1}\pi(z_n^{-1})\pi(x)\pi(z_n)S - \pi(z_n^{-1})\pi(x)\pi(z_n))v = n\beta v,$$

a contradiction. Suppose next that $S^2v \in \mathbf{C}Sv + \mathbf{C}v$. Set $W = \mathbf{C}Sv + \mathbf{C}v$. Then $\dim_{\mathbf{C}} W = 2$ and W is invariant under S . This case is then reduced to the first case. We now finish the proof. $\square$

Proof of Theorem 1.2. “ $\Rightarrow$ ”: Let $z \in A$ be invertible. By assumption, $\delta(z) \in \text{rad}(A)$ and so $z^{-1}\delta(z) \in \text{rad}(A)$. Thus $r(z^{-1}\delta(z)) = 0$.

“ $\Leftarrow$ ”: Suppose that $\sup\{r(z^{-1}\delta(z)) \mid z \in A \text{ is invertible}\} < M$, where M is a positive number. Let $a \in \text{rad}(A)$; then $(1+a)^{-1} = 1 - a(1+a)^{-1}$. Then

$$\begin{aligned} M &> r((1+a)^{-1}\delta(1+a)) = r((1+a)^{-1}\delta(a)) \\ &= r((1-a(1+a)^{-1})\delta(a)) = r(\delta(a)), \end{aligned}$$

since $a(1+a)^{-1}\delta(a) \in \text{rad}(A)$. For $x \in A$ and a positive integer n we have

$$M > r(\delta(nax)) = r(\phi(a)\delta(nx) + \delta(a)nx) = nr(\delta(a)x),$$

implying that $r(\delta(a)x) = 0$ and so $\delta(a) \in \text{rad}(A)$. Thus $\delta(\text{rad}(A)) \subseteq \text{rad}(A)$, and so δ canonically induces a $\overline{\phi}$ -derivation $\overline{\delta}$ on $\overline{A} = A/\text{rad}(A)$. Note that $z \in A$ is invertible if and only if $\overline{z} = z + \text{rad}(A)$ is invertible in $\overline{A}$. Also, $r(z^{-1}\delta(z)) = r(\overline{z}^{-1}\overline{\delta}(\overline{z}))$. Thus we may reduce to the case that A is semisimple. We just claim

that $\delta = 0$. By [6, Corollary 4.3], both ϕ and δ are continuous. Let (π, E) be a continuous irreducible representation of A .

Case 1. δ is E -outer. Suppose first that ϕ is E -outer. Let $\xi \in E \setminus \{0\}$. Then, by Theorem 2.2, we can choose $y \in A$ such that $\pi\delta(y)\xi = \xi$, $\pi\phi(y)\xi = 0$ and $\pi(y)\xi = 0$. Suppose next that ϕ is E -inner. Write $\pi\phi(x) = S^{-1}\pi(x)S$ for all $x \in A$, where $S \in \text{End}_{\mathbf{C}}(E)$ is invertible. If S is a scalar, we choose an arbitrary $0 \neq \xi \in E$. Otherwise, there exists $\xi \in E$ such that $S\xi$ and ξ are $\mathbf{C}$ -independent. In view of Theorem 2.1, there exists $y \in A$ such that $\pi\delta(y)\xi = \xi$, $\pi(y)S\xi = 0$ and $\pi(y)\xi = 0$. Thus $\pi\phi(y)\xi = 0$ follows.

In either case, we define $z_n = e^{ny}$ for $n \geq 0$. Then each z_n is invertible and $\pi\delta(z_n)\xi = n\xi$, $\pi(z_n)\xi = \xi$. So $\pi(z_n^{-1}\delta(z_n))\xi = n\xi$, and hence $r(z_n^{-1}\delta(z_n)) \geq n$, a contradiction.

Case 2. δ is E -inner. We write $\pi\delta(x) = \pi\phi(x)T - T\pi(x)$ for all $x \in A$, where $T \in \text{End}_{\mathbf{C}}(E)$.

Suppose that ϕ is E -outer. If T is a scalar, by [3, Theorem 3.1] we choose $0 \neq \xi \in E$ and $z_n \in A$ invertible for $n \geq 1$ such that $\pi\phi(z_n)\xi = n\xi$ and $\pi(z_n)\xi = \xi$. Then

$$\pi(z_n^{-1}\delta(z_n))\xi = \pi(z_n^{-1})(\pi\phi(z_n)T - T\pi(z_n))\xi = \pi(z_n^{-1})T((n-1)\xi) = (n-1)T\xi,$$

a contradiction. Assume that T is not a scalar. Choose $0 \neq \xi \in E$ such that ξ and $T\xi$ are $\mathbf{C}$ -independent. For $n \geq 0$ we choose an invertible $z_n \in A$ such that $\pi(z_n)\xi = \xi$ and $\pi\phi(z_n)T\xi = T\xi + n\xi$. Then $\pi(z_n^{-1}\delta(z_n))\xi = n\xi$, implying that $r(z_n^{-1}\delta(z_n)) \geq n$. This is absurd.

Suppose next that ϕ is E -inner. Then $\pi\phi(x) = S^{-1}\pi(x)S$ for all $x \in A$, where $S \in \text{End}_{\mathbf{C}}(E)$ is invertible. Write $\pi\delta(x) = S^{-1}[\pi(x), U]$, where $U = ST$. Suppose on the contrary that $\delta \neq 0$. Thus U is not a scalar. Choose $\xi \in E$ such that $U\xi$ and ξ are $\mathbf{C}$ -independent. Then, for an integer $n \geq 0$, we can choose $\beta_n = 1$ or 2 such that ξ and $U\xi + n\beta_n S\xi$ are $\mathbf{C}$ -independent. Thus, by [13, Theorem 6.7], there exist $z_n \in A$ invertible for $n \geq 0$ such that $\pi(z_n)\xi = \xi$ and $\pi(z_n)U\xi = U\xi + n\beta_n S\xi$. Then

$$\pi(z_n^{-1}\delta(z_n))\xi = \pi(z_n^{-1})S^{-1}(\pi(z_n)U - U\pi(z_n))\xi = n\beta_n S\xi,$$

a contradiction. Thus $\delta = 0$ follows. This proves the theorem. $\square$

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