WEAK BOUNDEDNESS THEOREMS FOR CANONICALLY FIBERED GORENSTEIN MINIMAL 3-FOLDS

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ABSTRACT. Let X be a Gorenstein minimal projective 3-fold with at worst locally factorial terminal singularities. Suppose the canonical map is of fiber type. Denote by F a smooth model of a generic irreducible element in fibers of ϕ_1 , and so F is a curve or a smooth surface. The main result is that there is a computable constant K independent of X such that $g(F) \leq 647$ or $p_g(F) \leq 38$ whenever $p_g(X) \geq K$.

INTRODUCTION

In this paper, we study the boundedness problem of a canonical family of curves or surfaces on certain algebraic 3-folds. Let X be a Gorenstein minimal projective 3-fold with locally factorial terminal singularities. Suppose $p_g(X) \geq 2$. One may define the canonical map ϕ_1 . Denote $\alpha := \dim \phi_1(X)$. We say that ϕ_1 is of *fiber type* if $\alpha < 3$. Denote by F a smooth model of a generic irreducible element in fibers of ϕ_1 . It is interesting to know whether the birational invariants of F are bounded. For technical reasons, we have to assume on X an effective Miyaoka-Yau inequality. This is the reason that we are only able to treat a Gorenstein minimal 3-fold by virtue of ([14]). Apparently we have $\kappa(X) \geq 1$. If $\kappa(X) \leq 2$, by the Base Point Free Theorem ([11, 13]), one can see that F is special. So, throughout, we may suppose $\kappa(X) = 3$, i.e. X is of general type. What we can do is as follows (see Theorem 1.5, Theorem 2.3 and Theorem 2.8).

Theorem 0.1. *There is a computable constant K . For all X , a Gorenstein minimal projective 3-fold with locally factorial terminal singularities, denote by ϕ_1 the canonical map of X and by F a smooth model of a generic irreducible element in fibers of ϕ_1 . The following hold:*

- (1) *If $\dim \phi_1(X) = 2$, then F is a curve with geometric genus $g(F) \leq 647$ provided $p_g(X) \geq K$.*
- (2) *If $\dim \phi_1(X) = 1$, then F is a surface of general type with geometric genus $p_g(F) \leq 38$ provided $p_g(X) \geq K$.*

Remark 0.2. In the above theorem, we do not know the boundedness when $p_g(X) < K$. The relations among $p_g(X)$, $q(X)$ and $h^2(\mathcal{O}_X)$ are expected to be known to

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solve the problem. We also believe that those bounds in 0.1 might be far from sharp.

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1. CANONICAL FAMILY OF SURFACES

1.1. Setting up. Let X be a minimal projective 3-fold with $\mathbb{Q}$ -factorial terminal singularities. Suppose $p_g(X) \geq 2$. We can define the canonical map ϕ_1 , which is usually a rational map. Take the birational modification $\pi : X' \rightarrow X$, according to Hironaka, such that

- (i) X' is smooth;
- (ii) the movable part of $|K_{X'}|$ is base point free;
- (iii) certain $\pi^*(K_X)$ have supports with only normal crossings.

Denote by g the composition $\phi_1 \circ \pi$. So $g : X' \rightarrow W' \subseteq \mathbb{P}^{p_g(X)-1}$ is a morphism. Let $g : X' \xrightarrow{f} W \xrightarrow{s} W'$ be the Stein factorization of g . We can write

$$K_{X'} = \pi^*(K_X) + E = S_1 + Z_1,$$

where S_1 is the movable part of $|K_{X'}|$, Z_1 the fixed part and E is an effective $\mathbb{Q}$ -divisor that is a sum of distinct exceptional divisors.

If $\dim \phi_1(X) < 3$, f is called a *derived fibration* of ϕ_1 . We mean a *generic irreducible element in fibers of ϕ_1* by a general fiber of f .

If $\dim \phi_1(X) = 2$, a general fiber C of f is a smooth curve of genus $g(F)$ which should not be confused with the morphism g .

If $\dim \phi_1(X) = 1$, a general fiber F of f is a smooth projective surface of general type. Denote by F_0 the minimal model of F . Denote by b the genus of the smooth curve W .

Proposition 1.2. *Keep the above notation. Suppose X is Gorenstein minimal of general type with locally factorial terminal singularities, $\dim \phi_1(X) = 1$ and $b = 0$. If $p_g(X) \geq 2k + 2$ for certain $k \geq 4$, then there is an effective $\mathbb{Q}$ -divisor E'_k on X' such that*

$$\pi^*(K_X)|_F =_{\mathbb{Q}} \frac{k}{k+1} \sigma^*(K_{F_0}) + E'_k$$

where $\sigma : F \rightarrow F_0$ is contraction onto the minimal model.

Proof. Let M_{k+1} be the movable part of $|(k+1)K_{X'}|$. Then we can write

$$(k+1)\pi^*(K_X) \sim M_{k+1} + E'_{k+1}$$

where E'_{k+1} is an effective divisor with respect to k . Therefore we see that $(k+1)\pi^*(K_X)|_F \geq M_{k+1}|_F$. Let N_k be the movable part of $|kK_F|$. According to [3], $|kK_{F_0}|$ is base point free. Thus $N_k = \sigma^*(kK_{F_0})$ where $\sigma : F \rightarrow F_0$ is contraction onto the minimal model.

We use the approach in [12] (Corollary 4.8) to prove $M_{k+1}|_F \geq N_k$. In fact, we have a fibration $f : X' \rightarrow \mathbb{P}^1$. Because $p_g(X) \geq 2k + 2$, we can see that $\mathcal{O}(2k+1) \hookrightarrow f_*\omega_{X'}$. Thus we have

$$\mathcal{E} := \mathcal{O}(1) \otimes f_*\omega_{X'/\mathbb{P}^1}^k = \mathcal{O}(2k+1) \otimes f_*\omega_{X'}^k \hookrightarrow f_*\omega_{X'}^{k+1}.$$

Note that $H^0(\mathbb{P}^1, f_*\omega_{X'}^{k+1}) \cong H^0(X', \omega_{X'}^{k+1})$. It is well known that $\mathcal{E}$ is generated by global sections and that $f_*\omega_{X'/\mathbb{P}^1}^k$ is a sum of line bundles with non-negative degree ([8, 16, 17]). Thus the global sections of $\mathcal{E}$ can distinguish different fibers of f . On the other hand, the local sections of $f_*\omega_{X'}^k$ give the k -canonical map of F , and these local sections can be extended to global sections of $\mathcal{E}$. This means $M_{k+1}|_F \geq N_k$. The proposition is proved. $\square$

Theorem 1.3 ([4], Theorem 1). *Let X be a minimal projective 3-fold of general type with at worst $\mathbb{Q}$ -factorial terminal singularities. Suppose $\dim \phi_1(X) = 1$. If $b \geq 2$, then either*

$$\begin{aligned} p_g(F) = 1 \text{ and } p_g(X) &\geq b - 1 \text{ or} \\ b = p_g(F) = p_g(X) &= 2. \end{aligned}$$

1.4. Reduction to the case $b = 0$. We mainly study a Gorenstein minimal 3-fold. By virtue of 1.3, one only has to consider the case $b \leq 1$. If $b = 1$, then $p_g(F) \leq 38$ according to [4] (Theorem 2(1)). From now on within this section, we may suppose $b = 0$. So $W = \mathbb{P}^1$.

Theorem 1.5. *There exists a constant $K_1 \leq 800$ such that, for all X , a Gorenstein minimal projective 3-fold of general type with at worst locally factorial terminal singularities, satisfying $\dim \phi_1(X) = 1$, the following hold:*

- (1) $p_g(F) \leq 38$ provided $p_g(X) \geq K_1$.
- (2) $p_g(F) \leq 5002$ provided $p_g(X) \geq 23$.

Proof. We only have to prove the theorem for the case $b = 0$. In this case, $W = \mathbb{P}^1$. We have a derived fibration $f : X' \rightarrow \mathbb{P}^1$. We have

$$q(X) = h^1(\mathbb{P}^1, R^1 f_* \omega_{X'}) \leq q(F).$$

Because F is of general type, we have $p_g(F) \geq 2q(F) - 4$ according to [2]. Thus the Miyaoka-Yau inequality ([14]) becomes

$$\begin{aligned} K_X^3 &\leq -72[1 - q(X) + h^2(\mathcal{O}_X) - p_g(X)] \\ &\leq 72[(p_g(X) - 1) + q(F)] \\ &\leq 72[p_g(X) - 1] + 36p_g(F) + 144. \end{aligned}$$

When $p_g(X) \geq 2k + 2$ for $k \geq 4$, Proposition 1.2 gives

$$K_X^3 \geq \frac{k^2}{(k+1)^2} \cdot K_{F_0}^2 \cdot [p_g(X) - 1].$$

Denote $A := p_g(X) - 1$ and $B := \frac{k^2}{(k+1)^2}$. Also using the inequality $K_{F_0}^2 \geq 2p_g(F) - 4$, we obtain the following inequality:

$$p_g(F) \leq \frac{36A + 2AB + 72}{AB - 18} = \frac{(36 + 2B) + \frac{72}{A}}{B - \frac{18}{A}}.$$

Noting that $A \rightarrow +\infty$ and $B \rightarrow 1$ if $k \rightarrow +\infty$, we can easily find a constant K_1 such that $p_g(F) \leq 38$ whenever $p_g(X) \geq K_1$. The proof is completed. $\square$

Remark 1.6. Among known examples, the largest possible $p_g(F)$ is 5 ([7], Example 2(e)) where the total space is a smooth minimal 3-fold and the canonical system is composed of a pencil of surfaces over a rational curve. Can one construct more examples with larger $p_g(F)$?

2. CANONICAL FAMILY OF CURVES

2.1. Setting up. Within this section, we keep the same notation as in 1.1 and suppose X is a Gorenstein minimal 3-fold of general type with at worst locally factorial terminal singularities and $\dim \phi_1(X) = 2$. In this case, we may even suppose W is a smooth surface, and so a general fiber C of f is a smooth projective curve.

2.2. An inequality. We need a nontrivial inequality in terms of $g(C)$, namely,

$$K_X^3 \geq \frac{2}{3}(g(C) - 1)(p_g(X) - 2)$$

according to [6] (Theorem 4.1(ii)). We omit the proof here simply because, on the one hand, this type of inequality is expected and, on the other hand, the proof there is quite clear and short.

Theorem 2.3. *There is a computable constant K_2 such that for all X , a Gorenstein minimal projective 3-fold of general type with at worst locally factorial terminal singularities, satisfying $\dim \phi_1(X) = 2$ and $\kappa(W) \geq 0$, the following hold:*

- (1) $g(C) \leq 109$ provided $p_g(X) \geq K_2$.
- (2) $g(C) \leq 11989$ provided $p_g(X) > 109$.

Proof. We have a canonically derived fibration $f : X' \rightarrow W$. According to [12] and [16, 17], we know that $R^1 f_* \omega_{X'} \cong \omega_W$ and $f_* \omega_{X'}$ is torsion free. We have

$$\begin{aligned} q(X) &= h^1(W, R^1 f_* \omega_{X'}) + h^2(W, f_* \omega_{X'}) \\ &= q(W) + h^2(W, f_* \omega_{X'}) \leq q(W) + g(C), \\ h^2(\mathcal{O}_X) &= h^0(W, R^1 f_* \omega_{X'}) + h^1(W, f_* \omega_{X'}) \\ &= p_g(W) + h^1(W, f_* \omega_{X'}). \end{aligned}$$

On the surface W , we have $q(W) - p_g(W) = 1 - \chi(\mathcal{O}_W)$. By virtue of the birational classification theory on surfaces, we can see that

$$\chi(\mathcal{O}_W) \geq 0 \quad \text{whenever} \quad \kappa(W) \geq 0.$$

Thus we definitely have $q(W) - p_g(W) \leq 1$ under the assumption of the theorem. So we get

$$\begin{aligned} q(X) - h^2(\mathcal{O}_X) &\leq q(W) - p_g(W) + g(C) \leq g(C) + 1, \\ \chi(\omega_X) &= p_g(X) + q(X) - h^2(\mathcal{O}_X) - 1 \leq p_g(X) + g(C). \end{aligned}$$

By 2.2 and Miyaoka's inequality, we have

$$\frac{2}{3}(g(C) - 1)[p_g(X) - 2] \leq 72[p_g(X) + g(C)].$$

Denote $A := p_g(X) - 1$. The above inequality becomes

$$g(C) \leq \frac{109A + 108}{A - 108} = \frac{109 + \frac{108}{A}}{1 - \frac{108}{A}}.$$

It is easy to find a constant K_2 such that $g(C) \leq 109$ provided $p_g(X) \geq K_2$. $\square$

2.4. The case $\kappa(W) = -\infty$. We know that W is birationally equivalent to a ruled surface or $\mathbb{P}^2$. Denote $b_0 := q(W)$. This situation is more complicated. We have to do some preparation in order to prove the boundedness.

Corollary 2.5. *There exists a computable constant K_3 such that for all X , a Gorenstein minimal projective 3-fold of general type with at worst locally factorial terminal singularities, satisfying $\dim \phi_1(X) = 2$ and $\kappa(W) = -\infty$, we have $g(C) \leq b_0 + 109$ provided $p_g(X) \geq K_3$.*

Proof. According to the proof of 2.3, we only have to estimate $\chi(\omega_X)$. Because $q(X) \leq b_0 + g(C)$, we have

$$\chi(\omega_X) = p_g(X) + q(X) - h^2(\mathcal{O}_X) - 1 \leq p_g(X) + b_0 + g(C) - 1.$$

By 2.2 and Miyaoka's inequality, we have

$$\frac{2}{3}(g(C) - 1)[p_g(X) - 2] \leq 72[p_g(X) + b_0 + g(C) - 1].$$

Denote $A_1 = p_g(X) - 2$. The inequality is

$$g(C) \leq 109 \cdot \frac{A_1}{A_1 - 108} + b_0 \cdot \frac{108}{A_1 - 108} + \frac{108}{A_1 - 108}.$$

Noting that

$$\frac{A_1}{A_1 - 108} \mapsto 1, \quad \frac{108}{A_1 - 108} \mapsto 0 \quad \text{when } p_g(X) \mapsto +\infty,$$

we can easily find a constant K_3 such that $g(C) \leq b_0 + 109$ provided $p_g(X) \geq K_3$. $\square$

Lemma 2.6. *Let X be a Gorenstein minimal projective 3-fold of general type with at worst locally factorial terminal singularities. Suppose $\dim \phi_1(X) = 2$, $\kappa(W) = -\infty$ and $b_0 \geq 3$. Then $p_g(S_1) \leq \frac{9}{2}K_X^3 - 2p_g(X) + 6$ where S_1 is by definition the movable part of $|K_{X'}|$.*

Proof. Because W is birationally ruled, we have a projection map $p : W \rightarrow B$ onto a smooth curve B with genus b_0 . Because $b_0 > 0$, p is a morphism and a fibration. Because ϕ_1 factors through f and $\dim \phi_1(X) = 2$, $\dim f(S_1) = 1$. This means f maps a general S_1 onto a curve H in W and H is exactly the pull-back of a hyperplane section H' in W' under the map $s : W \rightarrow W'$. We see that H is a nef and big divisor on W and H is a smooth irreducible curve. Therefore there is a finite map $\theta : H \rightarrow B$, because p doesn't contract H . So $b_1 := g(H) \geq g(B) = b_0$. We have the following commutative diagram:

$$\begin{array}{ccccc} S_1 & \xrightarrow{\alpha=f|_{S_1}} & H & \xrightarrow{\theta} & B \\ j_1 \downarrow & & \downarrow j_2 & & \downarrow \parallel \\ X' & \xrightarrow{f} & W & \xrightarrow{p} & B \\ \pi \downarrow & & \downarrow s & & \\ X & \xrightarrow{\text{the rational map } \phi_1} & W' \subset \mathbb{P}^N & & \end{array}$$

where j_1 and j_2 are both inclusion. We can see that $\alpha : S_1 \rightarrow H$ is a fibration where a general fiber C' lies in the same numerical class as a fiber of f . Thus $g(C') = g(C)$. Because $b_0 \geq 3$, $q(S_1) \geq b_1 \geq b_0 \geq 3$. So $|K_{S_1}|$ defines a generically finite map onto its image according to [18]. Denote by N_1 the movable part of $|K_{S_1}|$. Then, according to Noether's inequality, $N_1^2 \geq 2p_g(S_1) - 4$. We can bound

$p_g(S_1)$ by studying ϕ_3 . It is obvious that $|K_{X'} + \pi^*(K_X) + S_1| \subset |3K_{X'}|$. Denote by M'_3 the movable part of $|K_{X'} + \pi^*(K_X) + S_1|$ and by M_3 the movable part of $|3K_{X'}|$. Then we have $3\pi^*(K_X) \geq M_3 \geq M'_3$. The vanishing theorem gives

$$|K_{X'} + \pi^*(K_X) + S_1|_{|S_1} = |K_{S_1} + L| \supset |N_1 + L_0|,$$

where $L = \pi^*(K_X)|_{S_1}$ and $L_0 = S_1|_{S_1}$. Because $S_1^3 = 0$, we can see that $|L_0|$ is composed of a free pencil of curves on the surface S_1 . We can write $L_0 \equiv a_1 C'$ where $a_1 \geq p_g(X) - 2$. Since $N_1 + L_0$ is movable, we get from [5] (Lemma 2.7) that

$$3L \geq M'_3|_{S_1} \geq N_1 + L_0.$$

Thus $9L^2 \geq (N_1 + L_0)^2$. Noting that $N_1 \cdot C' \geq 2$, we can easily derive the inequality

$$p_g(S_1) \leq \frac{9}{2}K_X^3 - 2p_g(X) + 6.$$

□

Proposition 2.7. *Let X be a Gorenstein minimal projective 3-fold of general type with at worst locally factorial terminal singularities. Suppose $\dim \phi_1(X) = 2$, $\kappa(W) = -\infty$ and $b_0 \geq 3$. If $g(C) \geq 648$, then $b_0 \leq p_g(X) + 653$.*

Proof. According to the proof of Lemma 2.6, we have a surface fibration $\alpha : S_1 \rightarrow H$ where $g(C') = g(C)$ and $b_1 = g(H) \geq b_0$. We shall use a filtration of vector bundles on H which was first studied by G. Xiao ([18]). Considering the natural map

$$H^0(X', S_1) \xrightarrow{\gamma_1} \Lambda_1 \subset H^0(S_1, L_0)$$

where Λ_1 is the image of γ_1 , we can see that α is exactly obtained by taking the Stein factorization of Φ_{Λ_1, S_1} . Note that $L_0 \leq K_{S_1}$ and $H^0(S_1, \omega_{S_1}) \cong H^0(H, \alpha_*\omega_{S_1})$ where $\alpha_*\omega_{S_1}$ is a vector bundle of rank $g(C)$ on the curve H . Denote by $\mathcal{L}_0$ the saturated sub-bundle of $\alpha_*\omega_{S_1}$, which is generated by sections in Λ_1 . Then it is obvious that $\mathcal{O}_{L_0}$ is a line bundle because $|L_0|$ is composed of fibers of α . So we get an extension of $\alpha_*\omega_{S_1}$:

$$0 \rightarrow \mathcal{L}_0 \rightarrow \alpha_*\omega_{S_1} \rightarrow \mathcal{L}_1 \rightarrow 0.$$

Because $\alpha_*\omega_{S_1/H}$ is semi-positive, we have $\deg(\mathcal{L}_1 \otimes \omega_H^{-1}) \geq 0$, i.e., $\deg(\mathcal{L}_1) \geq 2(g(C)-1)(b_1-1)$. The Riemann-Roch theorem gives $h^0(H, \mathcal{L}_1) \geq (g(C)-1)(b_1-1)$. Noting that $\deg(\mathcal{L}_0) > 0$, we get, by applying Clifford's theorem, that $h^1(H, \mathcal{L}_0) \leq b_1 - 1$. From the long exact sequence

$$0 \rightarrow H^0(\mathcal{L}_0) \rightarrow H^0(\alpha_*\omega_{S_1}) \rightarrow H^0(\mathcal{L}_1) \rightarrow H^1(\mathcal{L}_0) \rightarrow \cdots,$$

we have the inequality

$$h^0(\mathcal{L}_0) - h^0(\alpha_*\omega_{S_1}) + h^0(\mathcal{L}_1) - h^1(\mathcal{L}_0) \leq 0.$$

Noting that $h^0(\mathcal{L}_0) \geq 2$ and using Lemma 2.6, we have

$$2 + (g(C) - 2)(b_1 - 1) \leq p_g(S_1) \leq \frac{9}{2}K_X^3 - 2p_g(X) + 6.$$

Also noting that $b_1 \geq b_0$ and using Miyaoka's inequality, one can get

$$(g(C) - 2)(b_0 - 1) \leq 322p_g(X) + 324b_0 + 324g - 320,$$

$$(g(C) - 326)b_0 \leq 322p_g(X) + 325g - 322.$$

If $g(C) > 326$, we have

$$b_0 \leq \frac{322p_g(X)}{g(C) - 326} + 325 + \frac{105628}{g(C) - 326}.$$

It is easy to see $b_0 \leq p_g(X) + 653$ provided $g(C) \geq 648$. $\square$

Theorem 2.8. *There exists a computable constant K_4 such that for all X , a Gorenstein minimal projective 3-fold of general type with at worst locally factorial terminal singularities, satisfying $\dim \phi_1(X) = 2$ and $\kappa(W) = -\infty$, we have $g(C) \leq 647$ provided $p_g(X) \geq K_4$.*

Proof. If $b_0 \leq 2$, then Corollary 2.5 gives $g(C) \leq 111$ provided $p_g(X) \geq K_3$. If $b_0 > 2$ and $g(C) \geq 648$, we have $b_0 \leq p_g(X) + 653$ by Proposition 2.7. Applying 2.2 and the Miyaoka-Yau inequality, we obtain

$$\begin{aligned} \frac{2}{3}(g(C) - 1)[p_g(X) - 2] &\leq 72[p_g(X) + b_0 + g(C) - 1] \\ &\leq 144p_g(X) + 72g(C) + 72 \cdot 652. \end{aligned}$$

If $p_g(X) > 110$, we can obtain

$$g(C) \leq 217 \cdot \frac{p_g(X)}{p_g(X) - 110} + \frac{70414}{p_g(X) - 110}.$$

One can easily find a constant K'_4 such that $g(C) \leq 217$ provided $p_g(X) \geq K'_4$. This contradicts the assumption $g(C) \geq 648$. Thus, when $p_g(X) \geq K_4 := \max\{K_3, K'_4\}$, $g(C) \leq 647$. $\square$

Theorems 1.5, 2.3 and 2.8 imply Theorem 0.1.

Example 2.9. We give a family of examples where X is canonically fibred by curves of genus 3 and the canonical map is of constant moduli. The original idea comes from [1]. Let W be a smooth projective surface with $p_g(W) = 0$ and let C_0 be a smooth curve of genus 2. Denote $X_0 := W \times C_0$. $p_1 : X_0 \rightarrow W$ and $p_2 : X_0 \rightarrow C_0$ are projections. Suppose we have a divisor H on W such that $|2H|$ is base point free, $h^0(W, 2H) \geq 2$ and that $|K_W + H|$ gives a generically finite map. Let $\theta = P - Q$ be a divisor on C_0 such that $2\theta \sim 0$. On X_0 , denote $\delta := p_1^*(H) + p_2^*(\theta)$. Then we can see that δ determines a smooth double cover $\pi : X \rightarrow X_0$ with

$$\begin{aligned} K_X &= \pi^*(K_{X_0} + \delta), \\ p_g(X) &= h^0(X_0, p_1^*(K_W + H)). \end{aligned}$$

Because $h^0(C_0, K_{C_0} + \theta) = 1$, we can see that Φ_{K_X} factors through π and p_1 . If $K_W + H$ is nef, then X is minimal. Denote $f := p_1 \circ \pi$. Then f is the derived fibration from ϕ_1 and a general fiber of f is a curve of genus 3.

2.10. Open problem. It is quite interesting to consider a parallel problem. Let X be a Gorenstein minimal projective 3-fold with at worst locally factorial terminal singularities. Suppose ϕ_1 is generically finite onto its image. Is the generic degree of ϕ_1 universally upper bounded? We don't know any work on this problem. It is well known that the upper bound is 16 in the surface case.

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