DEGENERATIONS OF CUBIC THREEFOLDS AND MATROIDS

TAWANDA GWENA

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ABSTRACT. We present a surprising connection between cubic threefolds and the well-known regular matroid R_{10} by making use of intermediate Jacobians of cubic threefolds realized as Prym varieties. As a corollary we obtain a new proof of the nonrationality of generic cubic threefolds.

0. INTRODUCTION

This paper uses the methods of degenerations of Prym varieties of [ABH] to establish a surprising new connection between cubic threefolds and a very well-known regular matroid called R_{10} . As in [ABH] we will work over an arbitrary algebraically closed field of characteristic not 2.

Recall that the Jacobian $J\tilde{C}$ of a smooth curve C is an algebraic variety parametrizing line bundles on C of degree 0. Let $\mathcal{R}_g$ be the moduli space of smooth curves of genus $2g - 1$ with involution ι , and let $\overline{\mathcal{R}}_g$ be its compactification obtained by adding stable curves. Also, let $\mathcal{A}_g$ be the moduli space of principally polarized Abelian varieties of dimension g . The Prym map $P : \mathcal{R}_g \rightarrow \mathcal{A}_{g-1}$ is given by setting $P(\tilde{C}, \iota)$ to be the identity component of the kernel of the norm map $J\tilde{C}$ to $J(\tilde{C}/\iota)$. The Prym map extends to a rational map from $\overline{\mathcal{R}}_g$, the space of admissible double covers of curves, to $\overline{\mathcal{A}}_{g-1}^{\text{vor}}$, the toroidal compactification of $\mathcal{A}_{g-1}$ for the second Voronoi fan. Where this map is regular is given by [ABH], Theorem 3.2. We will adapt this result in Lemma 4.

A system of $m \geq n$ vectors R generating $\mathbb{R}^n$ is called a unimodular system or regular matroid if, when we write any n vectors of R as columns in terms of a basis $B \subset R$ we obtain a matrix that is totally unimodular, i.e., the maximal minors are either 0, 1 or -1 . Such a unimodular system of vectors can be represented by a matrix $(I_n | A)$ where the columns are the vectors. From the unimodular system of vectors above we can obtain a lattice dicing. The vectors define a family $H(R)$ of parallel hyperplanes $H(\mathbf{r}, z) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{r} = z\}$, $z \in \mathbb{Z}$, $\mathbf{r} \in R$. If $B \subset R$ is a basis for $\mathbb{R}^n$, then the intersection points of hyperplanes in $H(B)$ is a lattice L and $H(R)$ is then called a *lattice dicing*. The set of intersection points of hyperplanes of $H(R)$ is L if and only if R is a unimodular system of vectors. There are three basic types of unimodular systems of vectors: cographic, graphic, and the special one called R_{10} , which is neither graphic nor cographic ([O], Cor. 13.2.5). A celebrated theorem of Seymour on matroid decompositions shows that all unimodular systems

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of vectors can be obtained from amalgams of the three types mentioned above ([DG], [O, 13.2.4]).

Degenerating families of Jacobians of curves give rise to cographic unimodular systems of vectors (described in Section 1). Degenerating families of Prym varieties also give rise to cell decompositions, and under special situations, as in the next section, we do get a lattice dicing, hence a unimodular system of vectors. The question that arises is: Which unimodular systems of vectors do degenerations of Prym varieties give? In this paper we obtain R_{10} from degenerations of Prym varieties.

Recall that the well-known Segre Cubic is a singular cubic threefold with ten simple nodes as singularities. Mumford's construction ([CG], Appendix C) associates to a cubic threefold an intermediate Jacobian realized as a Prym variety. A family of cubic threefolds degenerating to the Segre Cubic gives a degenerating family of Prym varieties. Our main result is

Theorem 1. *The unimodular system of vectors associated to this family of Prym varieties is R_{10} .*

As a corollary of Theorem 1 we will easily obtain the following result:

Theorem 2 ([C]). *Fix an algebraically closed field k of characteristic not two. Then a generic cubic threefold over k is not birational to $\mathbb{P}^3$.*

In section 4 we will give the proof and a precise meaning of generic. This theorem was proved in [C] by a different degeneration method. In fact, Clemens and Griffiths ([CG]) showed that over $\mathbb{C}$ every smooth cubic threefold is not rational, and Murre extended the result to any algebraically closed field of characteristic not two ([M2]).

1. DEGENERATIONS OF PRYM VARIETIES

We first review degenerations of Jacobians of curves in order to introduce the different types of unimodular systems of vectors. Suppose we have a degenerating family of Jacobians of curves of genus g with central fiber C , a stable curve. This degenerating family gives a cell decomposition of a real vector space ([ABH], Section 3.1). This decomposition depends only on C . Let $\Gamma = \Gamma(C)$ be the dual graph of C . The vertices $\{v_i\}_{i \in I}$ correspond to irreducible components of C and the edges $\{e_j\}_{j \in J}$ correspond to the nodes. We have a chain complex $C_\bullet$ with

$$C_0(\Gamma, \mathbb{Z}) = \bigoplus_{i \in I} v_i \mathbb{Z}, \quad C_1(\Gamma, \mathbb{Z}) = \bigoplus_{j \in J} e_j \mathbb{Z}.$$

Let $H_1(\Gamma, \mathbb{Z})$ be the first homology group. There is an embedding $H_1(\Gamma, \mathbb{Z}) \hookrightarrow C_1(\Gamma, \mathbb{Z})$. The cell decomposition obtained by intersecting the subspace $H_1(\Gamma, \mathbb{R}) \subset C_1(\Gamma, \mathbb{R})$ with the standard cubes in $C_1(\Gamma, \mathbb{R})$ is called a *cographic dicing* and the associated unimodular system is called cographic. The cell decomposition associated to the degeneration of curves with central fiber C is this cographic dicing defined by C . Conversely, given a cographic unimodular system we can construct a degenerating family of Jacobians of curves that have that unimodular system. If a cographic unimodular system of vectors with $m \geq n$ vectors in $\mathbb{R}^n$ is $(I_n | A)$, then $(-A^T | I_{m-n})$ is also a unimodular system of vectors. These are the *graphic* unimodular systems of vectors.

We now review degenerations of Prym varieties from [ABH], Section 3. A degenerating family of Prym varieties produces a cell decomposition of a real vector

space ([ABH], 3.2 (PP6)). If the map is regular at the central fiber, then we obtain the simplest type of cell decomposition, which is a lattice dicing. We construct it as follows.

Definition 3. Let the dual graph of $\tilde{C}$ (a curve with involution ι) be $\tilde{\Gamma}$. We obtain a cell decomposition by the following: We define a map

$$\begin{aligned}\pi^- : H_1(\tilde{\Gamma}, \mathbb{Z}) &\longrightarrow H_1(\tilde{\Gamma}, \tfrac{1}{2}\mathbb{Z}) \\ h &\longmapsto \tfrac{1}{2}(h - \iota(h)).\end{aligned}$$

Let $X^- := \pi^-(H_1(\tilde{\Gamma}, \mathbb{Z}))$. The space $X^- \otimes \mathbb{R}$ is contained in $C_1(\tilde{\Gamma}, \mathbb{R})$. Each edge e_j of $\tilde{\Gamma}$ defines a coordinate function z_j in $C_1(\tilde{\Gamma}, \mathbb{R})$. Let $m_j = 1$ if $z_j : X^- \rightarrow \mathbb{Z}$ is surjective and $m_j = 2$ if $z_j : X^- \rightarrow \frac{1}{2}\mathbb{Z}$ is surjective. The functions $m_j z_j$ define a cell decomposition of $X^- \otimes \mathbb{R}$.

We adapt the main results of [ABH] and [V] to give a condition for where the map $\overline{\mathcal{R}}_g \rightarrow \overline{\mathcal{A}}_{g-1}^{vor}$ is regular.

Lemma 4 ([V], Theorem 0.1, [ABH], Theorem 3.2). *Let $(\tilde{C}, \iota)$ be given. If there do not exist two connected subgraphs Γ_0, Γ_1 of the dual graph of $\tilde{C}$ such that $\iota(\Gamma_i) = \Gamma_i$ ($i = 0, 1$), connected by $2n \geq 4$ interchanged edges, then the rational map $\overline{\mathcal{R}}_6 \rightarrow \overline{\mathcal{A}}_5^{vor}$ is regular in the neighborhood of $(\tilde{C}, \iota)$ and for the image point, the cell decomposition defined in ([ABH], 3.2, (PP6)) equals the decomposition defined above.*

2. PRYM VARIETIES OF CUBIC THREEFOLDS

The Prym variety associated to a smooth cubic threefold X in $\mathbb{P}^4$ is constructed as follows. The lines in X are parametrized by a Fano surface F . Choose a generic line ℓ in X corresponding to a point in F , and let $\tilde{C}_\ell$ be the subvariety of lines in X incident to ℓ . The blowup of X along ℓ has the structure of a conic bundle over $\mathbb{P}^2$ whose discriminant curve C_ℓ is a smooth plane quintic. The curve $\tilde{C}_\ell$ is an unramified double cover of C_ℓ . The pair $(\tilde{C}_\ell, \iota)$ is a point in $\mathcal{R}_6$, and the Prym variety $P(\tilde{C}_\ell, \iota)$ is independent of ℓ ([M1], Section 1). This construction of $(\tilde{C}_\ell, \iota)$ can be extended to cubic threefolds with nodes. Now we construct a singular plane quintic and a double cover for the Segre Cubic.

Definition 5. For any field of characteristic $\neq 2$ the Segre Cubic $\mathcal{S}$ in $\mathbb{P}^4$ is given by the equation

$$\left\{ (x_0, \dots, x_4) : \sum_{i \neq j \neq k} 2x_i x_j x_k + \sum_{i \neq j} x_i^2 x_j = 0, \quad i, j, k \in \{0, \dots, 4\} \right\}.$$

One node is given by the coordinates $(1 : 1 : 1 : -1 : -1)$, and the other nine are obtained by permuting these coordinates.

We can obtain $\mathcal{S}$ as follows: Let $p_1, \dots, p_5$ be points in general position in $\mathbb{P}^3$, and for each pair of points let ℓ_{ij} be the line joining p_i and p_j . Blow up $\mathbb{P}^3$ at the five points, and then blow down the proper transforms of the ten lines ℓ_{ij} . The image is $\mathcal{S}$ in $\mathbb{P}^4$ ([H], 3.1-3.2).

There are 15 planes contained in $\mathcal{S}$. Ten of them come from proper transforms of the planes in $\mathbb{P}^3$ containing three points p_i, p_j and p_k . These are labeled Π_{ijk}

($i < j < k$). The other five are images of the exceptional divisors coming from the blowup. These are labeled Π_i ($1 \leq i \leq 5$). Each of the 15 planes contains 4 nodes.

$\mathcal{S}$ contains at least 6 two-dimensional families of lines, R_i ($1 \leq i \leq 5$), which are proper transforms of lines through p_i and R_0 , which is the family of twisted cubics through the five points. Each line in R_i goes through the five planes Π_{ijk} ($i < j < k$). Each line in R_0 goes through Π_i ([SR], VIII 2.32).

Choose a line ℓ in R_1 that does not go through a node and goes through exactly 5 planes, and project from it onto $\mathbb{P}^2$. The image of each of the five planes is a line in the degeneracy locus for $\mathcal{S}$. Each line meets each of the other four lines at a node. Each node is where the preimage is a plane containing a node on $\mathcal{S}$. Let C_0 be the plane quintic consisting of these 5 lines. We label each line L_i , $1 \leq i \leq 5$. The dual graph of C_0 is the complete graph on five vertices K_5 .

We now describe a double cover for the curve C_0 above. The preimage of each point in C_0 consists of two lines, excluding ℓ , which are points in an étale double cover $\tilde{C}_0$ interchanged by the involution ι . The whole cover can be described as follows: Start with ten copies of $\mathbb{P}^1$, L_i^ϵ where $1 \leq i \leq 5$ and $0 \leq \epsilon \leq 1$. Each line L_i^ϵ has four points marked on it, $p_{i,j}^\epsilon$ where $1 \leq j \leq 5$ and $j \neq i$. In the following the notation $(p_{i,j}^0 \sim p_{j,i}^1)$ means that the points $p_{i,j}^0$ and $p_{j,i}^1$ are identified. The double curve $\tilde{C}_0$ is

$$\tilde{C}_0 = \left(\prod_{i,\epsilon} L_i^\epsilon \right) / (p_{i,j}^0 \sim p_{j,i}^1).$$

The dual graph $\Gamma_0 = \Gamma(\tilde{C}_0)$ appears in Figure 1. If the vertices of $\Gamma(C_0)$ are labeled v_j , then the vertices a_j and b_j both map to v_j . The map of edges is given by the map of vertices.

3. PROOF OF THE MAIN THEOREM

We now prove Theorem 1. First we give a degenerating family of cubic threefolds and the associated double covers of quintic curves. Let ℓ_0 be a line in $\mathcal{S}$ (chosen as in Section 2). Choose a smooth cubic threefold X such that it also contains ℓ_0 . We have a pencil of cubic threefolds that contain ℓ_0 :

$$X_{a,b} = aX + b\mathcal{S}, \quad (a : b) \in \mathbb{P}^1.$$

By restricting to some open subset S of $\mathbb{P}^1$ such that the only singular threefold is $\mathcal{S}$ over $0 = (0 : 1)$ we get a family $\mathcal{X}/S$ of cubic threefolds. Let $U = S - \{(0 : 1)\}$. For each $s \in U$ there is a pair $(\tilde{C}_s, \iota)$. The family of such pairs is a flat family over U . By direct calculation the limit is $(\tilde{C}_0, \iota)$.

Lemma 6. *Let $(\tilde{C}_0, \iota)$ be as above. Then the map $\overline{\mathcal{R}}_6 \rightarrow \overline{\mathcal{A}}_5^{vor}$ is regular in the neighborhood of $(\tilde{C}_0, \iota)$.*

Proof. Here we will use Lemma 4. Let Γ_0 be as above. Suppose Γ_1 and Γ_2 are subgraphs with $\iota(\Gamma_i) = \Gamma_i$ ($i = 1, 2$). Then the first possible case is that Γ_1 has 2 vertices and Γ_2 has 8 vertices. If the vertices in Γ_1 correspond to the lines L_1^0 and L_1^1 (using the notation from Section 2), then the lines L_1^0 and L_1^1 do not meet. So on the dual graph their corresponding vertices do not have an edge between them. Then Γ_1 is not connected and so the conditions of Lemma 4 are satisfied. The second possible case is if Γ_1 and Γ_2 have 4 and 6 vertices, respectively. Suppose

FIGURE 1. The edges are named as follows: $e_1 = (b_3, a_2)$, $e_2 = (a_4, b_2)$, $e_3 = (a_5, b_3)$, $e_4 = (a_5, b_4)$, $e_5 = (a_5, b_1)$, $e_6 = (a_4, b_3)$, $e_7 = (b_3, a_1)$, $e_8 = (b_2, a_5)$, $e_9 = (a_1, b_4)$, $e_{10} = (b_2, a_1)$. The rest of the edges are named as follows: if edge e_i is (a_j, b_k) ((b_j, a_k)) respectively) the edge e'_i is (b_j, a_k) , $((a_j, b_k)$ respectively).

The tree used to form the basis of $H_1(\Gamma, \mathbb{Z})$ is given by the edges $e_6 = (a_4, b_3)$, $e_7 = (b_3, a_1)$, $e_8 = (b_2, a_5)$, $e_9 = (a_1, b_4)$, $e_{10} = (b_2, a_1)$ and $e'_{10} = (a_2, b_1)$, $e'_7(a_3, b_1)$, $e'_9 = (b_1, a_4)$, $e'_8 = (a_2, b_5)$.

without loss of generality that the lines L_1^0 , L_1^1 , L_2^0 and L_2^1 are in Γ_1 . This would imply that Γ_1 is not connected, because the connected subgraph with L_1^0 and L_2^1 is not connected with the subgraph with L_1^1 and L_2^0 . Therefore the map is regular around $\mathcal{S}$. $\square$

It follows that the associated family of Prym varieties has the compactified Prym variety of $(\tilde{C}_0, \iota)$ as its limit over 0.

Let us now compute the corresponding dicing. By the above lemma, we only need to compute $X^- \subset C_1(\Gamma, \mathbb{Z})$ as in Definition 3 and show it is R_{10} . The following is a basis for $H_1(\Gamma, \mathbb{Z})$:

$$\begin{aligned}
 h_1 &= e'_6 + e'_7 - e'_9 + e_6 + e_7 - e_9, & h_7 &= e_5 - e'_9 + e_6 + e_7 + e_{10} + e_8, \\
 h_2 &= e'_1 + e'_7 + e'_9 + e_6 + e_7 + e_{10}, & h_8 &= e_4 + e_9 + e_{10} - e_8, \\
 h_3 &= e_1 - e'_{10} - e'_9 + e_6, & h_9 &= e_3 + e_7 + e_{10} - e_8, \\
 h_4 &= e_2 + e_6 + e_7 + e_{10}, & h'_8 &= e'_4 + e'_9 + e'_{10} - e'_8, \\
 h_5 &= e'_2 - e'_{10} - e'_9 + e_6 + e_7 - e_9, & h'_9 &= e'_3 + e'_7 + e'_{10} - e'_8, \\
 h_6 &= e'_5 - e'_8 - e'_{10} - e'_9 + e_6 + e_7, & &
 \end{aligned}$$

The basis for X^- is as follows:

$$\begin{aligned}\ell_1 &= \frac{1}{2}(h_2 - \iota(h_2)) = \frac{1}{2}(h_3 - \iota(h_3)), \\ \ell_2 &= \frac{1}{2}(h_4 - \iota(h_4)) = \frac{1}{2}(h_5 - \iota(h_5)), \\ \ell_3 &= \frac{1}{2}(h_9 - \iota(h_9)) = \frac{1}{2}(h'_9 - \iota(h'_9)), \\ \ell_4 &= \frac{1}{2}(h_8 - \iota(h_8)) = \frac{1}{2}(h'_8 - \iota(h'_8)), \\ \ell_5 &= \frac{1}{2}(h_6 - \iota(h_6)) = \frac{1}{2}(h_7 - \iota(h_7)),\end{aligned}$$

and $X^- = \langle \ell_1, \dots, \ell_5 \rangle$. The dicing is obtained by seeing how the edges e_j restrict to X^- . The matrix for the dicing of $X^- \otimes \mathbb{R}$ is (a_{ij}) where a_{ij} is defined to be 1 if ℓ_i contains e_j . The matrix we get is

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right),$$

which is R_{10} . This concludes the proof of Theorem 1. $\square$

4. APPLICATION TO CUBIC THREEFOLDS

As a corollary of Theorem 1 we get Theorem 2. The theorem we will use in the proof is

Theorem 7 ([M2], Thm. 3.11). *Let $\text{char}(k) \neq 2$. Let X be a nonsingular cubic threefold in $\mathbb{P}^4$, defined over k . If there exists a birational transformation between X and $\mathbb{P}^3$, then the canonically polarized Prym variety $(P(X), \Xi)$ associated with X is isomorphic, as a polarized abelian variety, to a product of canonically polarized Jacobian varieties of curves.*

The proof of Theorem 2 now follows easily. If the threefolds in the family from section 3 were rational, then by Theorem 7 the family of degenerating Pryms would be a family of degenerating Jacobians or product of Jacobians. Recall from section 1 that a degenerating family of curves would give a cographic unimodular system. But by Theorem 1 we get R_{10} , which is not cographic. This gives a contradiction. $\square$

Generic, as used in Theorem 2, means that in every one-parameter family of cubic threefolds degenerating to the Segre Cubic as in the proof of Theorem 1, there are only finitely many smooth rational cubic threefolds. Using the fact that the locus of products of canonically polarized Jacobians of curves in $\mathcal{A}_5$ is closed, it can be shown that the smooth rational threefolds are contained in a Zariski proper closed subset of the parameter space of smooth cubic threefolds.

In previous proofs of Theorem 2, which used degenerations, the limiting (generalized) Prym was an extension of a nontrivial Abelian variety by a torus. In [C] the Prym is an extension by a torus k^* , and in [B] the torus is $(\mathbb{C}^*)^2$. In this proof the extension is the maximal possible, as shown in the following proposition.

Proposition 8. *The generalized Prym associated to the double cover $(\tilde{C}_0, \iota)$ is $(k^*)^5$.*

Proof. This follows from the fact that the generalized Prym is an extension of an Abelian variety by a torus ([ABH], Section 1). The dimension of the torus is given by $h_1(\tilde{C}_0, \mathbb{Z}) - h_1(C_0, \mathbb{Z}) = 5$. The abelian part must therefore be 0. $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GEORGIA 30602