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SEMI-CONTINUITY OF METRIC PROJECTIONS IN ℓ_∞-DIRECT SUMS

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ABSTRACT. Let Y be a proximinal subspace of finite codimension of c_0. We show that Y is proximinal in ℓ_∞ and the metric projection from ℓ_∞ onto Y is Hausdorff metric continuous. In particular, this implies that the metric projection from ℓ_∞ onto Y is both lower Hausdorff semi-continuous and upper Hausdorff semi-continuous.

1. Preliminaries

Let X be a real Banach space. For x in X and $r > 0$, we denote by $B_X(x, r)$ ($B_X[x, r]$), the open (closed) ball in X, with x as center and r as radius. The closed unit ball of X will be denoted by B_X and the unit sphere of X by S_X. Also, X^* denotes the dual of X. The collection of norm attaining functionals in X^* would be denoted by $NA(X)$. That is, a functional f in X^* is in $NA(X)$ if and only if there exists x in S_X such that $f(x)$ is equal to $\|f\|$.

For a subspace Y of X, let

$$Y^\perp = \{f \in X^* : f(x) = 0 \ \forall \ x \in Y\}.$$

If A is a closed subset of X and x is in X, $d(x, A) = \inf\{\|x - y\| : y \in A\}$. If $\mathbb{C}(Y)$ denotes the class of non-empty, bounded and closed subsets of Y, then the Hausdorff metric on $\mathbb{C}(Y)$ is given by

$$h(A, B) \ = \ \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

for A and B in $\mathbb{C}(Y)$.

Let $D \subseteq X$ and F be a map from D into a collection of non-empty subsets of X. If x is in D, the set-valued map F is *lower semi-continuous* at x if given $\epsilon > 0$ and z in $F(x)$, there exists $\delta > 0$ such that for all y in $D \cap B(x, \delta)$, there exists w in $F(y) \cap B(z, \epsilon)$. If the choice of δ is independent of the choice of $z \in F(x)$, or equivalently

$$F(y) \cap B(z, \epsilon) \neq \emptyset, \quad \forall \ z \in F(x) \text{ and } \forall \ y \in D \cap B(x, \delta),$$

then following [3], we say F is *lower Hausdorff semi-continuous* at x. The set-valued map F is *upper Hausdorff semi-continuous* at x in D if given $\epsilon > 0$, there exists

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$\delta > 0$ such that $F(y) \subseteq F(x) + \epsilon B_X$, for all y in $D \cap B(x, \delta)$. The map F is said to be lower Hausdorff (upper Hausdorff) semi-continuous on the domain D if F is lower Hausdorff (upper Hausdorff) semi-continuous at each point $x \in D$.

If $F(x)$ belongs to $\mathbb{C}(Y)$ for all x in $D \subseteq X$ and x is in D, we say F is Hausdorff metric continuous at x in D if the single-valued map F from D into the metric space $(\mathbb{C}(Y), h)$ is continuous. We say F is Hausdorff metric continuous on D if F is Hausdorff metric continuous at all x in D.

All subspaces are assumed to be closed. Let Y be a subspace of X. For $x \in X$, let

$$P_Y(x) = \{y \in Y : \|x - y\| = d(x, Y)\}.$$

The subspace Y is said to be *proximinal* in X, if for each $x \in X$, the set $P_Y(x)$ is non-empty. It is easily verified that if Y is a proximinal subspace of X, then the set $P_Y(x)$ is bounded, closed and convex. The set-valued map $P_Y : X \to 2^Y$ is called the *metric projection* from X onto Y. A usual compactness argument shows that all finite-dimensional subspaces are proximinal.

We also need the notion of strong proximinality as defined in [7].

Definition 1.1. A proximinal subspace Y of a Banach space is called *strongly proximinal* if for each x in X and $\epsilon > 0$, there exists $\delta > 0$ such that

$$s(x, \delta) = \sup\{d(z, P_Y(x)) : z \in Y \text{ and } \|x - z\| < d(x, Y) + \delta\} \ < \ \epsilon.$$

Remark 1.2. It is easily verified that if Y is a strongly proximinal subspace of a Banach space X, then the metric projection P_Y is upper Hausdorff semi-continuous on X. However, a proximinal subspace Y, with P_Y upper Hausdorff semi-continuous, need not be strongly proximinal. For example, there exist proximinal hyperplanes that are not strongly proximinal (see Remark 1.2 of [7]). But the metric projection onto any proximinal hyperplane is upper Hausdorff semi-continuous.

A subspace Y of a Banach space X is called an *L-summand* of X if there is a subspace Z of X such that

$$X \ = \ Y \oplus Z$$

and for any x in X with $x = y + z$, where y is in Y and z is in Z, we have

$$\|x\| \ = \ \|y\| + \|z\|.$$

A subspace E of a Banach space X is said to be an M-ideal of X if $E^\perp$ is an L-summand of the dual space X^*. A Banach space that is an M-ideal in its second dual is called an *M-embedded* space.

A finite-dimensional normed linear space X is called *polyhedral* if B_X has only a finite number of extreme points. A Banach space X is called *polyhedral* if every finite-dimensional subspace of X is polyhedral. A well-known example of an infinite-dimensional polyhedral space is the sequence space c_0.

2. List of known results needed

We require a few known results about approximative properties of M-ideals and finite-dimensional polyhedral spaces. We quote them below with the appropriate references. All the results on M-ideals, which we list below, can be found in [9]. The following proposition couples Proposition 1.1 and Proposition 1.8 of Chapter II in [9].

Proposition 2.1. *Let Y be an M-ideal of a Banach space X. Then Y is proximinal in X and the metric projection P_Y from X onto Y is Hausdorff metric continuous on X.*

Proposition 2.2 (See Example 1.4 of Chapter III of [9])**.** *The sequence space c_0 is an M-ideal in its second dual space ℓ_∞ or equivalently, c_0 is an M-embedded space.*

Remark 2.3. M-ideals are strongly proximinal. In fact, they have a stronger proximinality property. M-ideals are known to have the 3-ball property (Theorem I.2.2, [9]). It was shown in [8] and [10] that if a subspace Y has the 3-ball property in X, then Y is L-proximinal. That is, for each x in X, we have

$$\|x\| = d(x, Y) + d(0, P_Y(x)).$$

Thus if Y is an M-ideal in X, then Y is L-proximinal. It is easily verified that L-proximinality implies strong proximinality.

We now move on to a few facts about finite-dimensional spaces. We first observe that the metric projection, onto even one-dimensional subspaces, need not be lower semi-continuous [2]. However, the following result of A. L. Brown, from [1], has an affirmative assertion in the polyhedral case.

Proposition 2.4. *Let X be a finite-dimensional polyhedral space and Y be a subspace of X. Then the metric projection P_Y from X onto Y is lower semi-continuous on X.*

We also need some standard facts about finite-dimensional subspaces, which can be derived using the usual compactness arguments. We prove one below.

Fact 2.5. *Let Y be a finite-dimensional subspace of a Banach space X, and assume that the metric projection P_Y is lower semi-continuous at some x in X. Then P_Y is lower Hausdorff semi-continuous at x.*

Proof. The set $P_Y(x)$ is compact since it is closed and bounded. Let $\epsilon > 0$ be given. Using the lower semi-continuity of P_Y at x, select for each z in $P_Y(x)$, a positive number δ_z such that for every y in $B_X(x, \delta_z)$, the set $P_Y(y)$ intersects the open ball $B_X(z, \epsilon/2)$. Select a finite subcover, say, $\{B_X(z_i, \epsilon/2) \cap P_Y(x) : 1 \leq i \leq k\}$, of the open cover $\{B_X(z, \epsilon/2) \cap P_Y(x) : z \in P_Y(x)\}$ of $P_Y(x)$. Set $\delta = \min\{\delta_{z_i} : 1 \leq i \leq k\}$. Choose any z in $P_Y(x)$ and i such that z is in $B_X(z_i, \epsilon/2)$. Now for any y in $B_X(x, \delta)$, we have $P_Y(y) \cap B_X(z_i, \epsilon/2)$ is non-empty and so $P_Y(y) \cap B_X(z, \epsilon)$ is non-empty. □

An easy compactness argument again proves the following statement.

Fact 2.6. *Any finite-dimensional subspace of a Banach space is strongly proximinal.*

The fact below now follows from Remark 1.2.

Fact 2.7. *If Y is a finite-dimensional subspace of a Banach space X, then the metric projection P_Y is upper Hausdorff semi-continuous on X.*

Finally, we make an easy observation connecting the three semi-continuity concepts we mentioned earlier.

Remark 2.8. Let X and Y be Banach spaces, and let F be a set-valued map from X into Y with $F(x)$ in $\mathbb{C}(Y)$ for all x in X. Then F is Hausdorff metric continuous at x in X if and only if F is both lower Hausdorff semi-continuous and upper Hausdorff semi-continuous at x.

This remark follows from the fact that if E and G are in $\mathbb{C}(Y)$, then

$$h(E,G) < \epsilon \;\Leftrightarrow\; G \subseteq\; E + \epsilon B_Y \text{ and } G \cap B_Y(z,\epsilon) \neq \emptyset \;\forall\; z \in E.$$

The fact below now follows from the above observations and results of this section.

Fact 2.9. *Let X be a finite-dimensional polyhedral space and Y be a subspace of X or X be a Banach space and Y be an M-ideal in X. In either case, Y is strongly proximinal in X and the metric projection P_Y from X onto Y is Hausdorff metric continuous.*

3. Semi-continuity in direct sum spaces

In this section, we consider the ℓ_∞- direct sum, $X = X_1 \oplus_\infty X_2$, of two Banach spaces X_1 and X_2. If Y_1 and Y_2 are subspaces of X_1 and X_2 respectively, we set $Y = Y_1 \oplus_\infty Y_2$. For any x in X, we denote by x_i the unique elements of X_i, for $i \in \{1,2\}$, satisfying $x = x_1 + x_2$. Clearly,

$$\|x\| = \max\{\|x_1\|, \|x_2\|\}.$$

We set

$$d_i(x) = d(x_i, Y_i), \text{ for } \; i \in \{1,2\}.$$

We note that

$$d(x,Y) = \max\{d_1(x), d_2(x)\}$$

and if z is in X, then

$$|d_i(x) - d_i(z)| \leq \|x_i - z_i\| \text{ for } i \in \{1,2\}. \tag{1}$$

The following remark, with X and Y as above, is easy to verify.

Remark 3.1. Let Y_1 and Y_2 be proximinal subspaces of X_1 and X_2 respectively. Then Y is proximinal in X and

$$P_Y(x) = \begin{cases} P_{Y_1}(x_1) + P_{Y_2}(x_2) & \text{if} \quad d_1(x) = d_2(x), \\[2ex] B_{X_1}[x_1, d_2(x)] \cap Y_1 + P_{Y_2}(x_2) & \text{if} \quad d_1(x) < d_2(x), \\[2ex] P_{Y_1}(x_1) + B_{X_2}[x_2, d_1(x)] \cap Y_2 & \text{if} \quad d_1(x) > d_2(x). \end{cases}$$

Note that in all the above three cases, we have

$$P_Y(x) \supseteq P_{Y_1}(x_1) + P_{Y_2}(x_2).$$

□

We need the following fact in the sequel.

Fact 3.2. *Let E be a Banach space, F be a proximinal subspace of E and x be in $E \backslash F$. Let $\alpha \;>\; d(x,F) = d_x$. Then given $\epsilon > 0$, there exists $\delta > 0$ such that for any y in $B_E(x,\delta)$ and β satisfying $|\beta - \alpha| < \delta$, we have*

$$h(B_E[x,\alpha] \cap F, B_E[y,\beta] \cap F) \;\leq\; \epsilon.$$

Proof. Let $2\gamma = \alpha - d_x,\; K = \alpha + d_x + 2$ and $\delta = \min\{1, \gamma/2, \gamma\epsilon/(2K)\}$. Let y be in $B_E(x,\delta)$. If $d_y = d(y,F)$ and β is a scalar such that $|\alpha - \beta| < \delta$, then it is easily verified, using (1), that

$$|d_x - d_y| < \delta \text{ and } \beta - d_y \;>\; \gamma.$$

Select any t in $B_E[x,\alpha]\cap F$. We will construct an element v in $B_E[y,\beta]\cap F$ satisfying $\|t-v\|<\epsilon$. We have

$$\|y-t\| \;\le\; \|y-x\|+\|x-t\| \;\le\; \delta+\alpha\le\beta+2\delta.$$

Now select any w in $P_F(y)$, and let

$$v=\lambda t+(1-\lambda)w, \text{ where } \lambda=\frac{\beta-d_y}{\beta-d_y+2\delta}.$$

Then v is in F and

$$\begin{aligned}\|y-v\| &\le \lambda\|t-y\|+(1-\lambda)d_y\\ &\le \lambda(\beta+2\delta)+(1-\lambda)d_y\\ &= \lambda(\beta-d_y+2\delta)+d_y=\beta.\end{aligned}$$

Now

$$\begin{aligned}\|t-v\| &= (1-\lambda)\|t-w\| = \tfrac{2\delta}{\beta-d_y+2\delta}\|t-w\|\\ &< \tfrac{2\delta}{\gamma}(\|t-x\|+\|x-y\|+\|y-w\|)\\ &\le \tfrac{2\delta}{\gamma}(\alpha+\delta+d_y)\\ &\le \tfrac{2\delta}{\gamma}(\alpha+d_x+2\delta)\\ &\le \tfrac{2\delta}{\gamma}K<\epsilon.\end{aligned}$$

Similarly, for any s in $B_E[y,\beta]\cap F$, we can get v' in $B_E[x,\alpha]\cap F$ satisfying $\|s-v'\|<\epsilon$, and this completes the proof of the fact. $\square$

Now we can prove the main result of this section.

Theorem 3.3. *Let Y_i be a proximinal subspace of the normed linear space X_i for $i\in\{1,2\}$, and let $Y=Y_1\oplus_\infty Y_2$. If P_{Y_i} is lower Hausdorff semi-continuous on X_i for $i\in\{1,2\}$, then P_Y is lower Hausdorff semi-continuous on $X=X_1\oplus_\infty X_2$.*

Proof. By Remark 3.1, Y is proximinal in X. Fix x in X and let $\epsilon>0$ be given. Using the lower Hausdorff semi-continuity of the maps P_{Y_i} at x_i, we can get $\delta>0$ such that

$$z\in X,\ \|x-z\|<\delta\Rightarrow B_{X_i}(p_i,\epsilon)\cap P_{Y_i}(z_i)\neq\emptyset,\ \text{for } i\in\{1,2\}. \tag{2}$$

Case 1. $d_1(x)=d_2(x)$.

In this case, we have $P_Y(x)=P_{Y_1}(x_1)\oplus_\infty P_{Y_2}(x_2)$. Select any $p_i\in P_{Y_i}(x_i)$ for i in $\{1,2\}$ and z in X with $\|x-z\|<\delta$. Using (2), we can pick r_i from $B_{X_i}(p_i,\epsilon)\cap P_{Y_i}(z_i)$ for i in $\{1,2\}$. By Remark 3.1, r_1+r_2 is in $P_Y(z)$. Since $\|(p_1+p_2)-(r_1+r_2)\|<\epsilon$, it follows that P_Y is lower Hausdorff semi-continuous at x.

Case 2. $d_1(x)\neq d_2(x)$.

We discuss only the case $d_1(x) < d_2(x)$, the proof for the other case being similar. Let $2\gamma = d_2(x) - d_1(x)$. Replacing x by x_1 and α by $d_2(x)$ in Fact 3.2, we can get $\delta > 0$ such that if $\|x - z\| < \delta$, then

$$d_2(z) - d_1(z) \ > \ \gamma$$

and

$$h(B_{X_1}[x_1, d_2(x)] \cap Y_1, B_{X_1}[z_1, d_2(z)] \cap Y_1) \ < \ \epsilon.$$

Without loss of generality, we assume that δ is so chosen that (2) is also satisfied.

We have, by Remark 3.1,

$$P_Y(w) \ = \ B_{X_1}[w_1, d_2(w)] \cap Y_1 + P_{Y_2}(w_2)$$

for all w in X with $\|x - w\| < \delta$. Choose any z in X with $\|x - z\| < \delta$. If t is in $B_{X_1}[x_1, d_2(x)] \cap Y_1$ and s in $P_{Y_2}(x_2)$, using the above inequality and (2), we select r in $B_{X_1}[z_1, d_2(z)] \cap Y_1$ and p in $P_{Y_2}(z_2)$ satisfying $\|t - r\| < \epsilon$ and $\|s - p\| < \epsilon$. Clearly $r + p$ is in $P_Y(z)$, and this completes the proof for this case. □

We now prove a similar result for upper Hausdorff semi-continuity.

Theorem 3.4. *Let X_i be a Banach space, Y_i a strongly proximinal subspace of X_i, for $i \in \{1, 2\}$. If $X = X_1 \oplus_\infty X_2$ and $Y = Y_1 \oplus_\infty Y_2$, then the metric projection P_Y, from X onto Y, is upper Hausdorff semi-continuous.*

Proof. By Remark 3.1, Y is proximinal in X, and by Remark 1.2, the metric projection from X_i onto Y_i is upper Hausdorff semi-continuous, for i in $\{1, 2\}$. Fix x in X and let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$\|x - z\| < \delta \Rightarrow \ P_{Y_i}(z_i) \subseteq P_{Y_i}(x_i) + \epsilon B_{X_i}, \tag{3}$$

for i in $\{1, 2\}$.

Case 1. $d_1(x) = d_2(x)$.

In this case, we have $P_Y(x) = P_{Y_1}(x_1) \oplus_\infty P_{Y_2}(x_2)$. Since Y_i is strongly proximinal in X_i, we can select $\eta > 0$ such that

$$s(x_i, \eta) < \epsilon \text{ for } \ i \in \{1, 2\}, \tag{4}$$

where $s(x_i, \eta)$ is given by Definition 1.1. We now choose $0 < \delta < \eta/4$ so that (3) holds and consider any z with $\|x - z\| < \delta$. If $d_1(z) = d_2(z)$, then $P_Y(z) = P_{Y_1}(z_1) + P_{Y_2}(z_2)$ and clearly by (3),

$$P_Y(z) \subseteq P_Y(x) + \epsilon B_X$$

in this case.

Now assume that $d_1(z) < d_2(z)$. Since

$$|d_i(x) - d_i(z)| \leq \|x - z\| < \eta/4, \text{ for } i \in \{1, 2\},$$

we have

$$|d_2(z) - d_1(x)| \leq |d_2(z) - d_2(x)| + |d_2(x) - d_1(x)| = |d_2(z) - d_2(x)| < \eta/4. \tag{5}$$

Now, by Remark 3.1, $P_Y(z) = B_{X_1}[z_1, d_2(z)] \cap Y_1 + P_{Y_2}(z_2)$. Select any t in $B_{X_1}[z_1, d_2(z)] \cap Y_1$. Then, using (5), we have

$$\|t - x_1\| \leq \|t - z_1\| + \|z_1 - x_1\| \leq \ d_2(z) + \eta/4 \leq d_1(x) + \eta/2.$$

By (4), $s(x_1, \eta) < \epsilon$ and so we have $d(t, P_{Y_1}(x_1)) < \epsilon$. Thus there exists r in $P_{Y_1}(x_1)$ satisfying $\|t - r\| < \epsilon$ and

$$B_{X_1}[z_1, d_2(z)] \cap Y_1 \subseteq P_{Y_1}(x_1) + \epsilon B_{X_1}.$$

Since, by (3),

$$P_{Y_2}(z_2) \subseteq P_{Y_2}(x_2) + \epsilon B_{X_2},$$

we conclude that

$$P_Y(z) \subseteq P_Y(x) + \epsilon B_X.$$

If $d_2(z) < d_1(z)$, we argue just as above to conclude that P_Y is upper Hausdorff semi-continuous.

Case 2. $d_1(x) \neq d_2(x)$.

We discuss only the case $d_1(x) < d_2(x)$, the proof for the other case being similar. Let $2\gamma = d_2(x) - d_1(x)$. Replacing x by x_1 and α by $d_2(x)$ in Fact 3.2, we can get $\delta > 0$ such that if $\|x - z\| < \delta$, then

$$d_2(z) - d_1(z) \ > \ \gamma$$

and

$$h(B_{X_1}[x_1, d_2(x)] \cap Y_1, B_{X_1}[z_1, d_2(z)] \cap Y_1) \ < \ \epsilon.$$

Without loss of generality, we assume that δ is so chosen that (3) is also satisfied.

We have

$$P_Y(w) \ = \ B_{X_1}[w_1, d_2(w)] \cap Y_1 + P_{Y_2}(w_2)$$

for all w in X with $\|x - w\| < \delta$. Select any z in X with $\|x - z\| < \delta$. If t is in $B_{X_1}[z_1, d_2(z)] \cap Y_1$ and s in $P_{Y_2}(z_2)$, using the above inequality and (3), we select r in $B_{X_1}[x_1, d_2(x)] \cap Y_1$ and p in $P_{Y_2}(x_2)$ satisfying $\|t - r\| < \epsilon$ and $\|s - p\| < \epsilon$. Clearly $r + p$ is in $P_Y(x)$ and $P_Y(z) \subseteq P_Y(x) + \epsilon B_X$. □

Remark 3.5. Let X be an ℓ_∞-direct sum of two non-zero Banach spaces X_1 and X_2 and Y_i be a proximinal, proper subspace of X_i, for $i \in \{1, 2\}$. It was recently shown in [4] that if P_Y is upper Hausdorff semi-continuous on X, where $Y = Y_1 \oplus_\infty Y_2$, then Y_i must be strongly proximinal in X_i, for $i \in \{1, 2\}$. This clearly implies that Theorem 3.4 does not hold if, for any one of the two values of i, strong proximinality of Y_i is replaced by the strictly weaker assumption that Y_i is proximinal and P_{Y_i} is upper Hausdorff semi-continuous.

The following theorem now follows from Remark 2.8 and Theorems 3.3 and 3.4.

Theorem 3.6. *Let X_i be a Banach space, Y_i a strongly proximinal subspace of X_i with the metric projection from X_i onto Y_i Hausdorff metric continuous, for $i \in \{1, 2\}$. If $X = X_1 \oplus_\infty X_2$ and $Y = Y_1 \oplus_\infty Y_2$, then the metric projection P_Y from X onto Y is Hausdorff metric continuous.*

4. Proximinal subspaces of finite codimension of c_0

If Y is a proximinal subspace of finite codimension in a normed linear space X, then the annihilator $Y^\perp$ of Y is contained in $NA(X)$, the class of norm attaining functionals on X (see [5] and [6]). Let Y be a proximinal subspace of finite codimension in c_0. Since $NA(c_0)$ is the set of finite sequences in ℓ_1 and $Y^\perp$ is finite dimensional, there exists a positive integer k such that for any $f = (f_n)$ in $Y^\perp$, f_n is zero for all $n \geq k$. In the rest of this section, the subspace Y and positive integer k are fixed as above.

Let $\{e_n : n \geq 1\}$ denote the natural basis of c_0. For any sequence $x = (x_n)$ of scalars, we set $\tilde{x} = \sum_{n=1}^{k} x_n e_n$. Also, we set

$$X_1 = \text{sp}\,\{e_1, e_2, \cdots, e_k\},$$
$$X_2 = \{(x_n) \in \ell_\infty : x_n = 0 \text{ for } 1 \leq n \leq k\},$$
$$Y_1 = \{\tilde{x} : x \in Y\}$$

and finally

$$Y_2 = \{(x_n) \in c_0 : x_n = 0 \text{ for } 1 \leq n \leq k\}.$$

Then clearly Y_i is a subspace of X_i for $i = 1, 2$ and

$$X = X_1 \oplus_\infty X_2.$$

Also, note that if x is in c_0, then

$$x \in Y \Leftrightarrow \tilde{x} \in Y \Leftrightarrow \tilde{x} \in Y_1.$$

It is now clear that $Y = Y_1 \oplus_\infty Y_2$.

Now, following the same proof for c_0 an M-ideal in ℓ_∞, we get Y_2 to be an M-ideal in X_2. Since X_1 is a finite-dimensional subspace of c_0, it is a polyhedral space. By Fact 2.9, Y_i is a strongly proximinal subspace of X_i with the metric projection P_{Y_i} from X_i onto Y_i Hausdorff metric continuous for $i \in \{1, 2\}$. It is now clear that the main theorem of this article, given below, follows immediately from Theorem 3.6.

Theorem 4.1. *Let Y be a proximinal subspace of finite codimension in c_0. Then Y is proximinal in ℓ_∞ and the metric projection from ℓ_∞ onto Y is Hausdorff metric continuous.*

Remark 4.2. Let Y be a subspace of codimension k in c_0, and assume $Y^\perp$ is the span of a linearly independent set $\{f_1, f_2, \cdots, f_k\}$. Then it follows from Example 1.4 (a) of [9] that Y is an M-ideal in ℓ_∞ if and only if Y is an M-ideal in c_0 if and only if f_i belongs to $\{e_n : n \geq 1\}$ for each i, $1 \leq i \leq k$. We recall, from [6], that Y is proximinal in c_0 (and hence in ℓ_∞) if and only if $Y^\perp$ is contained in $NA(c_0)$ or equivalently, every element of $Y^\perp$ is a sequence of ℓ_1 with only a finite number of nonzero entries. Thus, there are plenty of proximinal subspaces of finite codimension of c_0 that are not M-ideals in ℓ_∞ and for these, Theorem 4.1 cannot be derived from Proposition 2.1.

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