RADIAL LIMITS OF HARMONIC FUNCTIONS ON THE UNIT DISC

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ABSTRACT. This note characterizes those functions on the unit circle that can arise as the radial limit function of a harmonic function on the unit disc.

Let $\mathbb{D}$ denote the unit disc and $\mathbb{T}$ the unit circle. A classical result of Alice Roth [8] (or see Chapter IV, §5 of Gaier [4]) characterizes those functions on $\mathbb{T}$ that can be expressed as $\lim_{r \rightarrow \infty} f(re^{i\theta})$ for some entire function f . She showed that these are precisely the Baire-one functions on $\mathbb{T}$ (that is, pointwise limits of sequences from $C(\mathbb{T})$) that are constant on each (connected) component of some relatively open dense subset of $\mathbb{T}$. More recently, Boivin and Paramonov [3] established (in particular) an analogue of Roth's result for harmonic functions on the plane: in this case the radial limit functions are characterized as those Baire-one functions on $\mathbb{T}$ that are first-degree polynomials of θ on each component of some relatively open dense subset of $\mathbb{T}$. The purpose of this note is to solve the corresponding problem for radial boundary limits of harmonic functions on the unit disc. We will say that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is *asymptotically mean-valued* at $e^{i\theta}$ if

$$\frac{1}{2t} \int_{(\theta-t, \theta+t)} f(e^{i\phi}) d\phi \rightarrow f(e^{i\theta}) \quad \text{as } t \rightarrow 0+.$$

Theorem 1. *Let $f : \mathbb{T} \rightarrow \mathbb{R}$. The following statements are equivalent:*

- (a) *there is a harmonic function h on $\mathbb{D}$ such that $h(rw) \rightarrow f(w)$ as $r \rightarrow 1-$ for each $w \in \mathbb{T}$;*
- (b) *f is Baire-one, and there is a relatively open dense subset J of $\mathbb{T}$ such that f is locally bounded and asymptotically mean-valued on J .*

Furthermore, if (b) holds, then (a) holds with the additional property that the function $w \mapsto \sup_{0 \leq r < 1} |h(rw)|$ is locally bounded on J .

Proof. To prove the theorem, we suppose firstly that condition (a) holds. Then f is obviously Baire-one. For each $k \in \mathbb{N}$, let J_k denote the interior, relative to $\mathbb{T}$, of the compact set

$$K_k = \{w \in \mathbb{T} : |h(rw)| \leq k \text{ when } 0 \leq r < 1\}.$$

Since $\bigcup_k K_k = \mathbb{T}$, it follows by a Baire category argument that the (relatively open) set $J = \bigcup_k J_k$ is dense in $\mathbb{T}$. Clearly f is locally bounded in J , so it remains to check that f is asymptotically mean-valued at an arbitrary point w of J . To do this, we choose k such that $w \in J_k$ and define u to be the Poisson integral in $\mathbb{D}$ of

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$f\chi_{J_k}$, where χ_A denotes the characteristic function with value 1 on a set A and 0 elsewhere. We extend u and h to $\overline{\mathbb{D}}$ by defining $u = f\chi_{J_k}$ and $h = f$ on $\mathbb{T}$. We denote by H_g^U the (generalized) solution to the Dirichlet problem on an open set U with boundary data g (see Chapter 6 of [1] for an account of this). Also, let $U_k = \{r\zeta : \zeta \in J_k \text{ and } 0 < r < 1\}$ and $h_\rho(z) = h(\rho z)$ when $0 < \rho < 1$. Then $h_\rho = H_{h_\rho}^{U_k}$ on U_k , so we can let $\rho \rightarrow 1-$ and appeal to dominated convergence to see that $h = H_h^{U_k}$ on U_k . Also, $u = H_u^{U_k}$ on U_k . Thus

$$u(rw) = (u - h)(rw) + h(rw) = H_{u-h}^{U_k}(rw) + h(rw) \rightarrow f(w)$$

as $r \rightarrow 1-$, since $u - h = 0$ on J_k . A converse of Fatou's theorem, due to Loomis [7], now allows us to conclude that f is asymptotically mean-valued at w . Thus condition (b) holds.

Conversely, suppose that condition (b) holds, and let $\{J_j\}$ be the components of J . We fix j temporarily and write J_j as $\{e^{i\theta} : |\theta - \theta_j| < a_j\}$ for some θ_j and a_j . Let U_j denote the sector $\{rw : w \in J_j \text{ and } 0 < r < 1\}$, and define

$$\begin{aligned} F_{j,k} &= \{re^{i\theta} : 0 < r < 1, |\theta - \theta_j| \leq a_j(1 - 2^{-k})\} \quad (k \geq 1), \\ S_{j,k}(\rho) &= \{re^{i\theta} : 0 \leq r \leq \rho, |\theta - \theta_j| = a_j(1 - 2^{-k})\} \quad (0 \leq \rho < 1; k \geq 1) \end{aligned}$$

and

$$L(j, k) = \{e^{i\theta} : a_j(1 - 2^{-k}) < |\theta - \theta_j| \leq a_j(1 - 2^{-k-1})\} \quad (k \geq 0).$$

For a given $k \geq 2$ it follows from Dini's theorem (applied on $\overline{F}_{j,k-1}$) that

$$H_{\chi_{L(j,k)}}^{U_j \setminus S_{j,k}(\rho)} \rightarrow 0 \quad \text{uniformly on } F_{j,k-1} \text{ as } \rho \rightarrow 1-,$$

so we can choose $\rho_{j,k} \in (0, 1)$ such that

$$\left(\sup_{L(j,k)} |f| \right) H_{\chi_{L(j,k)}}^{U_j \setminus S_{j,k}(\rho_{j,k})} < 2^{-k} \quad \text{on } F_{j,k-1}.$$

We also define $\rho_{j,0} = \rho_{j,1} = 0$. Let

$$V_j = U_j \setminus \left(\bigcup_{k=2}^{\infty} S_{j,k}(\rho_{j,k}) \right).$$

Then, assigning f the value 0 off $\mathbb{T}$, we have

$$H_{|f|}^{V_j} \leq \sum_{k=0}^{\infty} \left(\sup_{L(j,k)} |f| \right) H_{\chi_{L(j,k)}}^{U_j \setminus S_{j,k}(\rho_{j,k})},$$

so $H_f^{V_j}$ defines a harmonic function h_j on V_j that is bounded on $V_j \cap F_{j,k}$ for each k . Let $w \in J_j$ and choose $r_w > 0$ such that the set $W_w = \{z \in \mathbb{D} : |z - w| < r_w\}$ satisfies $W_w \subset V_j$ and $K_w \subset J_j$, where $K_w = \overline{W}_w \cap \mathbb{T}$. We know from the asymptotic mean value property of f and an elementary version of Fatou's theorem (Theorem 3.1 of [6]) that the Poisson integral u in $\mathbb{D}$ of the boundary function $f\chi_{K_w}$ has radial limit $f(w)$ at w . Hence, defining $h_j = f$ and $u = f$ on K_w , we see that

$$(1) \quad h_j(rw) = H_{h_j-u}^{W_w}(rw) + u(rw) \rightarrow f(w) \quad \text{as } r \rightarrow 1-.$$

Now let $\rho_j : J_j \rightarrow (1 - j^{-1}, 1)$ be a continuous function such that

$$\rho_j \left(e^{i\theta_j \pm ia_j(1-2^{-k})} \right) > \rho_{j,k} \quad (k \in \mathbb{N})$$

and $\rho_j(e^{i\theta}) \rightarrow 1$ as $\theta \rightarrow \theta_j \pm a_j$, and let

$$E_j = \{rw : w \in J_j \text{ and } \rho_j(w) \leq r < 1\} \quad (j \in \mathbb{N}).$$

Clearly E_j is a relatively closed subset of $\mathbb{D}$ such that $E_j \subset V_j$. The set

$$E_0 = \left\{ rw : w \in \mathbb{T} \setminus J \text{ and } \frac{1}{2} \leq r < 1 \right\}$$

is also relatively closed in $\mathbb{D}$ and, since f is Baire-one, we can choose a continuous function $h_0 : E_0 \rightarrow \mathbb{R}$ such that

$$(2) \quad h_0(rw) \rightarrow f(w) \text{ as } r \rightarrow 1- \quad (w \in \mathbb{T} \setminus J).$$

We now define a relatively closed subset of $\mathbb{D}$ by the disjoint union $E = \bigcup_j E_j$ and put $v = h_j$ on E_j ($j \geq 0$). We note that $E_0^\circ = \emptyset$ and that the set $\mathbb{D}^* \setminus E$ is connected and locally connected, where $\mathbb{D}^*$ denotes the one-point compactification of $\mathbb{D}$. (We refer to Section 3.2 of [5] for a discussion of local connectedness in this context.) Since the function $z \mapsto 1 - |z|$ is positive and superharmonic on $\mathbb{D}$, we can apply Corollary 3.21 of [5] (based on work of Armitage and Goldstein [2]) to see that there is a harmonic function h on $\mathbb{D}$ such that

$$(3) \quad |h(z) - v(z)| < 1 - |z| \quad (z \in E).$$

It follows from (2) and (3) that $h(rw) \rightarrow f(w)$ as $r \rightarrow 1-$ when $w \in \mathbb{T} \setminus J$, and from (1) and (3) that the same limit behaviour holds when $w \in J$. Hence condition (a) holds. Furthermore, the function $w \mapsto \sup_r |h(rw)|$ is locally bounded on J since h_j is bounded on $E_j \cap F_{j,k}$ for each j and k . $\square$

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