MOMENTS OF THE PROBABILITY DENSITY FUNCTIONS OF ERROR TERMS IN DIVISOR PROBLEMS

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ABSTRACT. We derive an explicit formula for the moments of the probability density function of a class of functions. An application of this shows that the density function of the error term in the Piltz divisor problem is asymmetric.

1. INTRODUCTION

The error term $\Delta(x)$ in the Dirichlet divisor problem has been studied extensively by numerous authors. In the paper [3], Heath-Brown pioneered the study of the limit distribution of $\Delta(x)$. Among other things, he showed that $\Delta(x)$ has a limiting distribution and the probability density function decays rapidly. Heath-Brown's method actually applies to a rather general type of functions $F(t)$, which satisfy the following

Hypothesis (H). There exists a sequence $a_1(t), a_2(t), \dots$ of continuous real-valued functions of period 1 such that

$$\lim_{N \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \min\{1, |F(t) - \sum_{n \leq N} a_n(\gamma_n t)|\} dt = 0$$

where $\gamma_1, \gamma_2, \dots$ are nonzero constants which are linearly independent over $\mathbb{Q}$.

Remark. The condition of taking $\min\{1, |\cdot|\}$ in (H) is the weakest among all measurements $\min\{1, |\cdot|^\beta\}$ ($\beta > 0$), in view of Hölder's inequality (or see [4, p. 362]).

One of the main results in [3] is the following.

Theorem A ([3, Theorem 5]). *If $F(t)$ satisfies Hypothesis (H) with the $a_n(t)$'s satisfying the conditions:*

$$(C.1) \quad \int_0^1 a_n(t) dt = 0 \quad (n \in \mathbb{N}),$$

$$(C.2) \quad \sum_{n=1}^{\infty} \int_0^1 a_n(t)^2 dt < \infty,$$

$$(C.3) \quad \max_{t \in [0,1]} |a_n(t)| \ll n^{1-\mu} \text{ and } \lim_{n \rightarrow \infty} n^\mu \int_0^1 a_n(t)^2 dt = \infty$$

where $\mu > 1$ is a constant, then

$$T^{-1} \text{meas}\{t \in [0, T] : F(t) \in I\} \rightarrow \int_I f(\alpha) d\alpha$$

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as $T \rightarrow \infty$. Furthermore, the density function $f(\alpha)$ satisfies

$$\frac{d^k}{d\alpha^k} f(\alpha) \ll_{A,k} (1 + |\alpha|)^{-A}$$

for $k = 0, 1, 2, \dots$ and any constant A and $f(\alpha)$ extends to an entire function on $\mathbb{C}$.

From the high decaying rate of $f(\alpha)$, it is apparent that all the power moments of $f(\alpha)$ exist. But this does not imply that the power moments of $F(t)$ converge. Heath-Brown [3, Theorem 6] established that if $\lim_{T \rightarrow \infty} T^{-1} \int_0^T |F(t)|^K dt$ is finite, then for any positive integer $k < K$, $\lim_{T \rightarrow \infty} T^{-1} \int_0^T F(t)^k dt$ exists. In such cases, it is known that

$$\lim_{T \rightarrow \infty} T^{-1} \int_0^T F(t)^k dt = \int_{-\infty}^{\infty} \alpha^k f(\alpha) d\alpha.$$

The variance (second power moment) in [2] was also computed in this way.

In this note, we obtain a quite explicit formula for all the integral power moments of the density function, without the knowledge of the power moments of $F(t)$. As a consequence, we can say something about the symmetry of the density functions of some well-known error terms in number theory, including $\Delta(x)$ and $\Delta_3(x)$, the error terms in the Dirichlet and Pilz divisor problems respectively. (See Corollary 4.)

Theorem 1. *Under the assumptions in Theorem A, we have, for any $k \in \mathbb{N}$,*

$$(1.1) \quad \int_{-\infty}^{\infty} \alpha^k f(\alpha) d\alpha = \sum_{1 \leq r \leq k} \sum_{\substack{l_1, \dots, l_r \geq 1 \\ l_1 + \dots + l_r = k}} \frac{k!}{l_1! l_2! \dots l_r!} \sum_{1 \leq n_1 < \dots < n_r} \prod_{i=1}^r A(n_i, l_i),$$

where

$$(1.2) \quad A(n, l) = \int_0^1 a_n(t)^l dt.$$

The infinite series in (1.1) converges absolutely.

Remarks. (i) As $A(n, 1) = 0$ for all n by (C.1), the first moment of $f(\alpha)$ always vanishes, that is,

$$\int_{-\infty}^{\infty} \alpha f(\alpha) d\alpha = 0.$$

(ii) The function $f(\alpha)$ is symmetric if and only if

$$\int_{-\infty}^{\infty} \alpha^k f(\alpha) d\alpha = 0$$

for all positive odd integers k . Hence it is known that the density function of $x^{-1/4} \Delta(x)$ is asymmetric, since the third power moment $X^{-1} \int_0^X (x^{-1/2} \Delta(x^2))^3 dx$ is nonzero (see [5]). For $\Delta_3(x)$, however, we cannot draw a similar conclusion in this way since none of the odd power moments higher than the first is known for $\Delta_3(x)$. Using our Theorem 1, we can show that

$$\int_{-\infty}^{\infty} \alpha^3 f(\alpha) d\alpha \neq 0$$

for the density function $f(\alpha)$ of $x^{-1/3}\Delta_3(x)$. Thus $x^{-1/3}\Delta_3(x)$ also has an asymmetric density function (see Corollary 4). More generally, concerning the symmetry of the density function $f(\alpha)$ of a certain $F(t)$, we have the following theorems.

Theorem 2. *If each $a_n(t)$ in Theorem A is of the form $a_n(t) = b(n)\cos(2\pi t + \phi_n)$ where $b(n), \phi_n \in \mathbb{R}$, then all the odd power moments of $f(\alpha)$ are equal to zero and hence $f(\alpha)$ is symmetric.*

Theorem 3. *Suppose for each $n \in \mathbb{N}$, $a_n(t)$ is of the form*

$$a_n(t) = \sum_{m=1}^{\infty} b_n(m) \cos(2\pi mt - \theta)$$

where $\theta \in [-\pi, \pi]$ and the series is absolutely convergent. If the assumptions in Theorem A hold, then

- (a) when $\theta = \pm\pi/2$, the density function $f(\alpha)$ is symmetric;
- (b) when $\theta \neq \pm\pi/2$ and when all $b_n(m)$ are nonnegative and satisfy $b_n(m_1)b_n(m_2)b_n(m_1+m_2) \neq 0$ for some $n, m_1, m_2 \in \mathbb{N}$, then the density function $f(\alpha)$ is asymmetric.

Theorem 2 is straightforward, since by (1.2), $A(n, l) = 0$ for any positive odd integer l , and for $l_1 + l_2 + \cdots + l_r$ equal to an odd integer, at least one l_i must be odd.

To deduce Theorem 3, we first evaluate each of $A(n, l)$. Let E_l be the set of all maps σ sending $(x_1, \dots, x_l)$ in $\mathbb{R}^l$ to one of $x_1 \pm \cdots \pm x_l$, and let $n_-(\sigma)$ denote the number of minus signs in $\sigma(x_1, \dots, x_l)$. Using the identity $\prod_{i=1}^l \cos A_i = 2^{1-l} \sum_{\sigma \in E_l} \cos(\sigma(A_1, \dots, A_l))$, we see that

$$A(n, l) = 2^{1-l} \sum_{\sigma \in E_l} \sum_{\substack{m_1, \dots, m_l \geq 1 \\ \sigma(m_1, \dots, m_l) = 0}} \prod_{i=1}^l b_n(m_i) \cdot \cos((l - 2n_-(\sigma))\theta).$$

For $\theta = \pm\pi/2$, it is clear that $A(n, l) = 0$ for all odd l . Hence for all positive odd k ,

$$\int_{-\infty}^{\infty} \alpha^k f(\alpha) d\alpha = 0,$$

since for $l_1 + \cdots + l_r = k$, at least one l_i is odd. Thus $f(\alpha)$ is symmetric. On the other hand, under the conditions in (b), the third power moment

$$(1.3) \quad \int_{-\infty}^{\infty} \alpha^3 f(\alpha) d\alpha = \frac{\cos \theta}{4} \sum_{n=1}^{\infty} \sum_{m_1 \pm m_2 \pm m_3 = 0} \prod_{i=1}^3 b_n(m_i) \neq 0.$$

Hence, the density function $f(\alpha)$ is asymmetric.

Remark. In view of the proof, the conditions imposed on the coefficients $b_n(m)$ in Theorem 3 (b) can be relaxed to the nonvanishing of the multiple sum in (1.3).

Corollary 4. *Let $\Delta(x)$ and $\Delta_3(x)$ be the error terms in the Dirichlet divisor problem and the Pilz divisor problem respectively. Then both of their density functions are asymmetric.*

The corollary follows from the fact that the Voronoi series approximations for $x^{-1/4}\Delta(x)$ and $x^{-1/3}\Delta_3(x)$, as can be seen in [3], for example, are of the type in Theorem 3(b).

In [3], it is also shown that the odd power moments

$$\int_0^X \Delta(x)^k dx \sim \beta_k X^{1+k/4},$$

for $1 \leq k \leq 9$, with undetermined constants β_k . Recently Zhai [6] reported that these constants are all positive (except $\beta_1 = 0$), and he conjectured that this is also the case for all odd $k \geq 11$. We now, however, give the following two examples to illustrate the possible peculiar features of the density function.

Example 1. Let K be any positive integer and let

$$a_n(t) = \mu(n)^2 n^{-3/4} (\cos(2\pi t) + \cos(4\pi Kt)).$$

Define $F(t) = \sum_{n=1}^{\infty} a_n(\sqrt{nt})$. We shall see that the k th power moment of f is equal to zero for $k = 1, 3, \dots, 2K - 1$ but is positive for $k = 2K + 1$.

Let $1 \leq l \leq 2K + 1$ be odd and $m_i \in \{1, 2K\}$. Each $\sigma(m_1, \dots, m_l)$ ($\sigma \in E_l$) can be written as

$$\sigma(m_1, \dots, m_l) = \pm 2K \sigma_1(1, \dots, 1) \pm \sigma_2(1, \dots, 1)$$

for certain $\sigma_1 \in E_r$ and $\sigma_2 \in E_s$ with $r + s = l$. Since $|\sigma_2(1, \dots, 1)| \leq s \leq l$, $\sigma(m_1, \dots, m_l) = 0$ only if $l = 2K + 1$, $r = 1$ and $\sigma_2(1, \dots, 1) = 2K$. Hence $\int_0^1 a_n(t)^l dt$ is equal to zero for all odd $l \leq 2K - 1$ and is positive for $l = 2K + 1$.

Example 2. We construct below a function $F(t) = \sum_{n=1}^{\infty} a_n(\sqrt{nt})$ whose third power moment is positive while its fifth power moment is negative.

Let

$$a_n(t) = \mu(n)^2 n^{-3/4} (\cos(2\pi t - \pi/4) + b \cos(4\pi t - \pi/4) + \cos(16\pi t - \pi/4))$$

where $b > 0$ is a large number to be determined later. Direct computation shows that $A(n, 3) = \frac{3}{4} b \mu^2(n) n^{-9/4} \cos(\pi/4)$, and the third power moment equals

$$\frac{3b}{4\sqrt{2}} \frac{\zeta(9/4)}{\zeta(9/2)} > 0.$$

To evaluate the fifth power moment, we note that $A(n, 2) = (1 + \frac{1}{2}b^2)\mu^2(n)n^{-3/2}$. From Theorem 1, the fifth power moment is thus equal to

$$\sum_{n \neq n'} \frac{5!}{3!2!} A(n, 2) A(n', 3) + \sum_{n=1}^{\infty} A(n, 5).$$

The first double sum is $O(b^3)$. Up to permutations, for each $\sigma \in E_5$, $\sigma(m_1, \dots, m_5)$ is of the form

$$\pm(m_1 + m_2 + m_3 + m_4 - m_5) \quad \text{or} \quad \pm(m_1 + m_2 + m_3 - m_4 - m_5) \quad \text{or} \quad m_1 + \dots + m_5.$$

When $m_i \in \{1, 2, 8\}$, the first form equals 0 only for the case $m_1 = \dots = m_4 = 2$ and $m_5 = 8$; the second form takes zero with at most three m_i 's equal to 2, while the last form never vanishes. This yields

$$A(n, 5) = \left(\frac{5b^4}{16} \cos\left(\frac{3\pi}{4}\right) + O(b^3) \right) \frac{\mu^2(n)}{n^{15/4}}.$$

Therefore, when b is large enough, the fifth power moment of the corresponding density function is equal to

$$-\frac{5}{16\sqrt{2}} \frac{\zeta(15/4)}{\zeta(15/2)} b^4 + O(b^3) < 0.$$

2. SOME PREPARATIONS

We first prove a lemma for later use, and meanwhile, show that the infinite sum in (1.1) is absolutely convergent.

Lemma 1. *For any integer $k \geq 1$, the multiple sum in (1.1) converges absolutely. Let c_k be the value of this multiple sum. Then,*

$$\lim_{N \rightarrow \infty} \lim_{T \rightarrow \infty} T^{-1} \int_0^T \left(\sum_{n \leq N} a_n(\gamma_n t) \right)^k dt = c_k.$$

To prove Lemma 1, we approximate each $a_n(t)$ by the trigonometric polynomial $\sigma_{n,M}(t)$, which is the convolution of a_n with the Fejér kernel, that is,

$$(2.1) \quad \sigma_{n,M}(t) = \int_0^1 a_n(t+v) \frac{1}{M} \left(\frac{\sin \pi M v}{\sin \pi v} \right)^2 dv = \sum_{|m| \leq M} c_{n,m} e(mt),$$

where $c_{n,m} = (1 - |m|/M) \hat{a}_n(m)$ and $e(y) = e^{2\pi i y}$. It is well known that

$$(2.2) \quad \|a_n - \sigma_{n,M}\| \rightarrow 0 \text{ as } M \rightarrow \infty.$$

Here $\|\cdot\|$ denotes the supremum norm on $[0, 1]$. Moreover, from (2.1) and (C.3), we see that

$$(2.3) \quad \|\sigma_{n,M}\| \leq \|a_n\| \ll 1 \quad \text{for } n \in \mathbb{N}.$$

Using $a^k - b^k \ll_k |a - b|(|a|^{k-1} + |b|^{k-1})$ and (2.3), we obtain

$$\begin{aligned} & \left(\sum_{n \leq N} a_n(\gamma_n t) \right)^k - \left(\sum_{n \leq N} \sigma_{n,M}(\gamma_n t) \right)^k \\ & \ll_k \left(\sum_{n \leq N} \|a_n\| \right)^{k-1} \sum_{n \leq N} |a_n(\gamma_n t) - \sigma_{n,M}(\gamma_n t)| \\ & \ll_k N^{k-1} \sum_{n \leq N} \|a_n - \sigma_{n,M}\|. \end{aligned}$$

Let us write $m_T(f) = T^{-1} \int_0^T f(t) dt$ for short. Then

$$(2.4) \quad \begin{aligned} & m_T \left(\left(\sum_{n \leq N} a_n(\gamma_n t) \right)^k \right) - m_T \left(\left(\sum_{n \leq N} \sigma_{n,M}(\gamma_n t) \right)^k \right) \\ & \ll_k N^{k-1} \sum_{n \leq N} \|a_n - \sigma_{n,M}\|. \end{aligned}$$

Let

$$(2.5) \quad S_M(n, l) = \int_0^1 \sigma_{n,M}(t)^l dt,$$

and define

$$(2.6) \quad S_M^{(k)}(N) = \sum_{1 \leq r \leq k} \sum_{\substack{l_1, \dots, l_r \geq 1 \\ l_1 + \dots + l_r = k}} \frac{k!}{l_1! \cdots l_r!} \sum_{n_1 < \dots < n_r \leq N} \prod_{i=1}^r S_M(n_i, l_i).$$

Then we have

Lemma 2. *Given any positive integers k, N and M ,*

$$m_T((\sum_{n \leq N} \sigma_{n,M}(\gamma_n t))^k) \rightarrow S_M^{(k)}(N) \quad \text{as } T \rightarrow \infty.$$

The rate of convergence depends on N, M, k and the γ_n 's.

The proof of this will be given at the end of this section.

Now we prove that the right-hand side of (1.1) is absolutely convergent. By (1.2) and (C.3), $|A(n, l)| \ll A(n, 2)$ for any $l \geq 2$ and by (C.1), $A(n, 1) = 0$. Hence,

$$\sum_{n_1, \dots, n_r} \prod_{i=1}^r |A(n_i, l_i)| \ll \sum_{n_1, \dots, n_r} \prod_{i=1}^r A(n_i, 2) = (\sum_n A(n, 2))^r < \infty,$$

by (C.2), and the right-hand side of (1.1) converges absolutely.

By (2.3) we get that for any $l \geq 1$,

$$(2.7) \quad |S_M(n, l)| \ll 1, \quad |A(n, l)| \ll 1 \quad \text{and} \quad S_M(n, l) - A(n, l) \ll_l \|a_n - \sigma_{n,M}\|.$$

Using $|\prod_{i=1}^r x_i - \prod_{i=1}^r y_i| \ll_r \sum_{i=1}^r |x_i - y_i|$ when $|x_i|, |y_i| \ll 1$, we obtain from (2.6) and (2.7) that (recall that c_k is the sum in the right side of (1.1))

$$\begin{aligned} & S_M^{(k)}(N) - c_k \\ & \ll_k \sum_{1 \leq r \leq k} \sum_{\substack{l_1, \dots, l_r \geq 1 \\ l_1 + \dots + l_r = k}} \frac{k!}{l_1! \dots l_r!} \sum_{n_1 < \dots < n_r \leq N} \sum_{i=1}^r |S_M(n_i, l_i) - A(n_i, l_i)| \\ & + \sum_{1 \leq r \leq k} \sum_{\substack{l_1, \dots, l_r \geq 1 \\ l_1 + \dots + l_r = k}} \frac{k!}{l_1! \dots l_r!} \sum_{\substack{n_1, \dots, n_{r-1} \\ n_r > N}} \prod_{i=1}^r |A(n_i, l_i)| \\ (2.8) \quad & \ll_k N^{k-1} \sum_{n \leq N} \|a_n - \sigma_{n,M}\| + \sum_{n > N} A(n, 2). \end{aligned}$$

For any $\epsilon > 0$, in view of (C.2), we can fix an $N = N(\epsilon) > 1$ such that

$$\sum_{n > N} \int_0^1 a_n(t)^2 dt < \epsilon/2.$$

According to (2.2) we take $M = M(N, \epsilon) \geq 1$ for which

$$\|a_n - \sigma_{n,M}\| < \frac{\epsilon}{2} N^{-k} \quad \text{for all } n \leq N.$$

For this value of M , both (2.8) and (2.4) are $\ll_k \epsilon$. By the triangle inequality, (2.8), (2.4) and Lemma 2, we conclude that

$$m_T((\sum_{n \leq N} a_n(\gamma_n t))^k) - c_k \ll_k 3\epsilon,$$

for all $T \geq T_0(\epsilon, \gamma_1, \dots, \gamma_N)$. The proof of Lemma 1 is complete.

It remains to prove Lemma 2. From (2.5) and (2.1) it follows that

$$(2.9) \quad S_M(n, l) = \sum_{\substack{|m_1|, \dots, |m_l| \leq M \\ m_1 + \dots + m_l = 0}} c_{n, m_1} \dots c_{n, m_l}.$$

By (2.1),

$$\begin{aligned} & m_T((\sum_{n \leq N} \sigma_{n,M}(\gamma_n t))^k) \\ &= \sum_{n_1, \dots, n_k \leq N} \sum_{|m_1|, \dots, |m_k| \leq M} c_{n_1, m_1} \cdots c_{n_k, m_k} m_T(\prod_{i=1}^k e(m_i \gamma_{n_i} t)). \end{aligned}$$

Now $m_T(\prod_{i=1}^k e(m_i \gamma_{n_i} t))$ tends to 0 unless the linear combination $\sum_{i=1}^k m_i \gamma_{n_i} = 0$, in which case the limit is 1. Hence $m_T((\sum_{n \leq N} \sigma_{n,M}(\gamma_n t))^k)$ converges as $T \rightarrow \infty$. Indeed, as N and M are fixed integers, for any $\epsilon > 0$,

$$|m_T((\sum_{n \leq N} \sigma_{n,M}(\gamma_n t))^k) - L| < \epsilon$$

for all $T \geq T_0(\epsilon, N, M, \gamma_1, \dots, \gamma_N)$, where

$$(2.10) \quad L = \sum_{n_1, \dots, n_k \leq N} \sum^* c_{n_1, m_1} \cdots c_{n_k, m_k}$$

and the summation $\sum^*$ runs over $|m_1|, \dots, |m_k| \leq M$ for which $\sum_{i=1}^k m_i \gamma_{n_i} = 0$. The proof of Lemma 2 will be complete after checking that $L = S_M^{(k)}(N)$.

Using the linear independence over $\mathbb{Q}$ of the γ_n 's, when $\sum_{i=1}^k m_i \gamma_{n_i} = 0$, the γ_n 's can be partitioned into r subsums (for some $r \leq k$) in which each subsum is of the form $\gamma_{n_j} \sum m_v$ with $\sum m_v = 0$, and the γ_{n_j} 's of different subsums are distinct. It follows that L in (2.10) is equal to $S_M^{(k)}(N)$, by (2.9) and (2.6).

3. PROOF OF THEOREM 1

We follow the argument in [3] to show that for any $\epsilon > 0$,

$$(3.1) \quad \left| \int_{-\infty}^{\infty} \alpha^k f(\alpha) d\alpha - T^{-1} \int_0^T (\sum_{n \leq N} a_n(\gamma_n t))^k dt \right| < \epsilon$$

for $N \geq N(\epsilon)$ and $T \geq T(\epsilon, N, \gamma_1, \dots, \gamma_N)$. The proof is then completed by applying Lemma 1.

For simplicity, denote

$$S_N(t) = \sum_{n \leq N} a_n(\gamma_n t) \quad \text{and} \quad \mathcal{L}(\alpha, N) = \prod_{n \leq N} \int_0^1 e(\alpha a_n(t)) dt.$$

Following from [3, (3.1) and (3.5)] with $S_N(t)$ in place of $F(t)$, we have

$$(3.2) \quad \left| m_T(p(S_N(t))) - \int_{-A}^A \hat{p}(\alpha) \mathcal{L}(\alpha, N) d\alpha \right| < 3\epsilon$$

for $A \geq A(\epsilon, \hat{p})$, $N \geq N(\epsilon, A)$ and $T \geq T(\epsilon, A, N)$. Here $p(\cdot)$ is any twice continuously differentiable function such that both p and its Fourier transform $\hat{p}$ are absolutely integrable. From the proof of [3, Theorem 4] (see [3, p. 400]), replacing by a greater $A(\epsilon)$ and $N(\epsilon, A)$ if necessary, we obtain

$$(3.3) \quad \left| \int_{-A}^A \hat{p}(\alpha) \mathcal{L}(\alpha, N) d\alpha - \int_{-\infty}^{\infty} p(\alpha) f(\alpha) d\alpha \right| < \epsilon$$

for $A \geq A(\epsilon, p)$, $N \geq N(\epsilon, A)$. Choose now a twice continuously differentiable p of compact support such that $p(\alpha) = \alpha^k$ for $|\alpha| \leq B$, and $|p(\alpha)| \leq |\alpha|^k$ for $|\alpha| > B$, as in the proof of [3, Theorem 6] (see [3, p. 397]). Then, for $B > 2c_{k+1}\epsilon^{-1}$, $N \geq N_0(\epsilon)$, $T \geq T_0(\epsilon, N)$, we have

$$|m_T(p(S_N(t))) - m_T(S_N(t)^k)| \leq B^{-1}m_T(S_N(t)^{k+1}) \leq 2c_{k+1}B^{-1} < \epsilon,$$

by Lemma 1. The fast decay of $f(\alpha)$ (see Theorem A) yields

$$\left| \int_{-\infty}^{\infty} p(\alpha)f(\alpha) d\alpha - \int_{-\infty}^{\infty} \alpha^k f(\alpha) d\alpha \right| < \epsilon.$$

This proves (3.1), by adding the last two inequalities with (3.3) and (3.2).

REFERENCES

1. P.M. Bleher, *Distribution of the error term in the Weyl asymptotics for the Laplace operator on a two-dimensional torus and related lattice problems*, Duke Math. J. **70** (1993), 655–682. MR1224102 (94g:11082)
2. P.M. Bleher, Z. Cheng, F.J. Dyson and J.L. Lebowitz, *Distribution of the error term for the number of lattice points inside a shifted circle*, Commun. Math. Phys. **154** (1991), 433–469. MR1224087 (94g:11081)
3. D.R. Heath-Brown, *The distribution and moments of the error term in the Dirichlet divisor problem*, Acta Arith. **60** (1992), 389–415. MR1159354 (93e:11114)
4. Y.-K. Lau, *On the existence of limiting distributions of some number-theoretic error terms*, J. Number Theory **94** (2002), 359–374. MR1916279 (2003e:11104)
5. K.-M. Tsang, *Higher-power moments of $\Delta(x)$, $E(t)$ and $P(x)$* , Proc. London Math. Soc. (3) **65** (1992), 65–84. MR1162488 (93c:11082)
6. W. Zhai, *On higher-power moments of $\Delta(x)$ II*, Acta Arith. **114** (2004), 35–54. MR2067871

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