SOLUTIONS TO TWO QUESTIONS ABOUT THE WEYL ALGEBRAS

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ABSTRACT. Affirmative answers are given to the following two questions about the Weyl algebras: a question of J. Alev: *Does the first Weyl algebra contain a non-noetherian subalgebra?*, and a question of T. Lenagan: *Is there a uniserial module M of length 2 over the Weyl algebra A_n with a holonomic submodule U such that $V = M/U$ is non-holonomic?*

1. THE FIRST WEYL ALGEBRA CONTAINS A LEFT AND RIGHT NON-NOETHERIAN SUBALGEBRA

Let K be a field of characteristic zero. The (first) Weyl algebra $A_1 = \langle x, \partial \mid \partial x - x\partial = 1 \rangle$ is a simple noetherian domain. Any commutative subalgebra of the Weyl algebra A_1 is a finitely generated algebra (hence noetherian) [1, 7]. An old problem of Dixmier [7] which is still open (problem 4, p. 242) asks whether for each K -derivation δ of the Weyl algebra A_1 the commutative graded algebra $\text{gr}(N(\delta)) := \bigoplus_{i \geq 0} \ker(\delta^i) / \ker(\delta^{i-1})$ is finitely generated (an affirmative answer to this problem would imply that the filtered subalgebra $N(\delta) = \bigcup_{i \geq 0} \ker(\delta^i)$ of the Weyl algebra A_1 is a finitely generated noetherian algebra). Many ring theorists (including the author) believe that the answer is affirmative. Numerous ‘experimental facts’ support this belief.

For quite some time the following question of J. Alev was open: *Does the first Weyl algebra contain a non-noetherian subalgebra?* Lemma 1.1 gives an example of such a subalgebra.

Let $B := K[h][x; \sigma]$ be a skew polynomial algebra with coefficients from a polynomial algebra $K[h]$ in one variable h where the K -algebra automorphism σ of $K[h]$ is given by the rule $\sigma(h) = h - 1$. So, $B = \bigoplus_{i \geq 0} K[h]x^i$ and

$$(1) \quad a(h)x^i b(h)x^j = a(h)\sigma^i(b(h))x^{i+j} = a(h)b(h-i)x^{i+j}$$

for all $i, j \geq 0$, where $a(h), b(h) \in K[h]$.

The algebra B is a subalgebra of the Weyl algebra A_1 via the algebra monomorphism

$$B \rightarrow A_1, \quad x \mapsto x, \quad h \mapsto \partial x.$$

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For each $i \geq 1$, let $K[h]_{<i} := \{a \in K[h] \mid \deg(a) < i\}$ and $K[h]_{<0} := K$. Obviously, $\sigma(K[h]_{<i}) = K[h]_{<i}$ for all $i \geq 0$. It follows that $C := \bigoplus_{i \geq 0} K[h]_{<i} x^i$ is a subalgebra of B .

Lemma 1.1. *The subalgebra C of the Weyl algebra A_1 is not a finitely generated algebra and is not a left and right noetherian algebra.*

Proof. The algebra $C = \bigoplus_{i \geq 0} C_i$ is a positively graded algebra where $C_i := K[h]_{<i} x^i$ with

$$\begin{aligned} C_i C_j &= K[h]_{<i} \sigma^i(K[h]_{<j}) x^{i+j} \subseteq K[h]_{<i} K[h]_{<j} x^{i+j} \\ &\subseteq K[h]_{<i+j-1} x^{i+j} \subset K[h]_{<i+j} x^{i+j} = C_{i+j}, \end{aligned}$$

a strict inclusion for all $i, j \geq 1$. It follows that the algebra C is not finitely generated and that the ideal $I := \bigoplus_{i \geq 1} C_i$ of the algebra C is not a finitely generated left (or right) C -module. This means that C is not a left (or right) noetherian algebra. $\square$

2. NON-HOLONOMIC MODULES OF LENGTH TWO OVER THE WEYL ALGEBRA A_n

Let K be a field of characteristic zero. The Weyl algebra $A_n = A_n(K)$ is a K -algebra generated over the field K by $2n$ generators $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ subject to the defining relations:

$$\partial_i x_j - x_j \partial_i = \delta_{ij} \text{ (the Kronecker delta), } x_i x_j = x_j x_i, \partial_i \partial_j = \partial_j \partial_i.$$

The Weyl algebra A_n is canonically isomorphic to the ring of differential operators with polynomial coefficients $K[x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]$ ($x_i \leftrightarrow x_i, \partial_i \leftrightarrow \frac{\partial}{\partial x_i}$).

In this section, $K = \mathbb{C}$ is the field of complex numbers. The Weyl algebra $A_n = A_n(\mathbb{C})$ is a simple Noetherian domain of Gelfand-Kirillov dimension $\text{GK}(A_n) = 2n$, $A_n = A_1 \otimes \dots \otimes A_1$, n times. J. Bernstein proved that $\text{GK}(M) \geq n$ for any nonzero finitely generated A_n -module ([4] or [5], Chap. 1).

A finitely generated A_n -module M is called *holonomic* if $\text{GK}(M) = n$. In the case of the first Weyl algebra A_1 all *simple* modules were classified by R. Block, [6] (a different approach is given in [3]) and all of them are holonomic A_1 -modules. Let $M_1, \dots, M_n$ be nonzero simple A_1 -modules. Then their tensor product $\bigotimes_{i=1}^n M_i$ is a simple holonomic A_n -module. It was believed that all simple A_n -modules ($n \geq 2$) are holonomic, but T. Stafford gave an example of a simple non-holonomic A_n -module, [10].

Theorem 2.1 (Theorem 1.1, [10]). *For $2 \leq i \leq n$, pick $\lambda_i \in \mathbb{C}$ that are linearly independent over the field of rational numbers $\mathbb{Q}$. Then the element*

$$\alpha = x_1 + \left(\sum_2^n \lambda_i \partial_i x_i \right) \partial_1 + \sum_2^n (x_i + \partial_i)$$

generates a maximal left ideal of A_n . In particular, the simple A_n -module $A_n/A_n\alpha$ has Gelfand-Kirillov dimension $2n - 1$.

Denote by $St_n(\alpha)$ the Stafford's module $A_n/A_n\alpha$ from Theorem 2.1.

Question of Lenagan ([8]). *Is there a uniserial module M of length 2 over the Weyl algebra A_n with a holonomic submodule U such that $V = M/U$ is non-holonomic.*

Theorem 2.2 gives a positive answer to this question.

Given $\mu \in \mathbb{C} \setminus \mathbb{Z}$, then the A_1 -module

$$S(\mu) = A_1/A_1(\partial_1 x_1 - \mu)$$

is simple, and two such modules are isomorphic, i.e., $S(\mu) \simeq S(\mu')$, iff $\mu - \mu' \in \mathbb{Z}$ ([3], Theorem 3.2).

Then the tensor product of the modules above

$$S(\mu_1, \dots, \mu_n) = \bigotimes_{i=1}^n S(\mu_i) \simeq A_n/A_n(\partial_1 x_1 - \mu_1, \dots, \partial_n x_n - \mu_n)$$

is a *simple holonomic* A_n -module and two such modules are isomorphic if and only if the corresponding tensor multiples are isomorphic as A_1 -modules (that is, $\mu_i - \mu_j \in \mathbb{Z}$ for all i, j).

Theorem 2.2. *Let the A_n -modules $St_n(\alpha)$ and $S(\mu_1, \dots, \mu_n)$ be as above ($n \geq 2$). Then*

$$\dim \operatorname{Ext}_{A_n}^1(St_n(\alpha), S(\mu_1, \dots, \mu_n)) = \infty.$$

Remark. G. Perets [9] gave an example of a uniserial module M of length 2 over the Weyl algebra A_n with a *non-holonomic* submodule U such that $V = M/U$ is *holonomic*.

2.1. Proof of Theorem 2.2. Let D be a ring with an automorphism σ and a central element a . The *generalized Weyl algebra* $A = D(\sigma, a)$ of degree 1 is the ring generated by D and two indeterminates X and Y subject to the defining relations [2]:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y, \forall \alpha \in D, YX = a \text{ and } XY = \sigma(a).$$

The algebra $A = \bigoplus_{n \in \mathbb{Z}} A_n$ is a $\mathbb{Z}$ -graded algebra where $A_n = Dv_n$, $v_n = X^n$ ($n > 0$), $v_n = Y^{-n}$ ($n < 0$), and $v_0 = 1$. It follows from the defining relations that

$$v_n v_m = (n, m) v_{n+m}$$

for some $(n, m) \in D$.

The first Weyl algebra is a generalized Weyl algebra

$$A_1 \simeq K[H](\sigma, a = H), \quad x \leftrightarrow X, \quad \partial \leftrightarrow Y, \quad \partial x \leftrightarrow H,$$

with $D = K[H]$, the polynomial algebra in one variable, and $\sigma : H \rightarrow H - 1$. We say that an element

$$u = d_i v_i + d_{i+1} v_{i+1} + \dots + d_j v_j \in A_1, \quad \text{all } d_k \in D, \quad d_i \neq 0, \quad d_j \neq 0,$$

has *length* $l(u) = j - i$. Clearly, $l(uv) = l(u) + l(v)$ for $u, v \in A_1$.

For $\mathbf{k} := (k_1, \dots, k_n) \in \mathbb{Z}^n$, let $v_{\mathbf{k}} = v_{k_1}(1) \cdots v_{k_n}(n)$ where for $1 \leq i \leq n$ and $m \geq 0$: $v_m(i) = (x_i)^m$, $v_{-m}(i) = (\partial_i)^m$, $v_0(i) = 1$. The Weyl algebra A_n is the tensor product of the first Weyl algebras $A_1^{(i)} = \mathbb{C}\langle x_i, \partial_i \rangle = \mathbb{C}[H_i](\sigma_i, H_i)$. It follows from the definition of generalized Weyl algebras that

$$A_n = \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} A_{\mathbf{k}}$$

is a $\mathbb{Z}^n$ -graded algebra ($A_{\mathbf{k}} A_{\mathbf{e}} \subset A_{\mathbf{k}+\mathbf{e}}, \forall \mathbf{k}, \mathbf{e} \in \mathbb{Z}^n$) where $A_{\mathbf{k}} = K[H_1, \dots, H_n] v_{\mathbf{k}}$ and $K[H_1, \dots, H_n]$ is the polynomial algebra in n variables. The $\mathbb{Z}^n$ -grading above of the Weyl algebra A_n is the tensor product of the $\mathbb{Z}$ -gradings of the tensor multiples $A_1^{(i)}$.

The A_1 -module $S(\mu)$ ($\mu \notin \mathbb{Z}$) is a $\mathbb{Z}$ -graded module:

$$\begin{aligned} S(\mu) &= \bigoplus_{i \in \mathbb{Z}} S(\mu)_i, \quad S(\mu)_i = \mathbb{C}e_i, \quad e_i = v_i + A_1(H - \mu), \\ He_i &= (\mu + i)e_i, \quad Xe_k = e_{k+1}, \quad Ye_{-k} = e_{-k-1} \quad (k \geq 0), \\ Xe_{-m} &= (\mu - m)e_{-m+1}, \quad Ye_m = (\mu + m - 1)e_{m-1} \quad (m > 0). \end{aligned}$$

We say that a nonzero element

$$s = s_i e_i + s_{i+1} e_{i+1} + \cdots + s_j e_j \in S(\mu), \quad \text{all } s_k \in \mathbb{C}, \quad s_i \neq 0, \quad s_j \neq 0,$$

has *length* $l(s) = j - i$. Then $l(us) = l(u) + l(s)$ for $s \in S(\mu)$ and $u = d_m v_m + \cdots + d_n v_n \in A_1$ such that none of the scalars from the set $\mu + \mathbb{Z}$ is a root of the polynomial $d_m d_n \neq 0$.

Consider the tensor product of the A_1 -modules $S(\mu_i)$, $i = 1, \dots, n$, above ($\mu_i \notin \mathbb{Z}$):

$$S = S(\mu_1, \dots, \mu_n) = \bigotimes_{i=1}^n S(\mu_i).$$

It is a $\mathbb{Z}^n$ -graded A_n -module

$$S = \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} S_{\mathbf{k}}, \quad S_{\mathbf{k}} = S(\mu_1)_{k_1} \otimes \cdots \otimes S(\mu_n)_{k_n}, \quad \mathbf{k} = (k_1, \dots, k_n),$$

and a $\mathbb{Z}$ -graded A_n -module with respect to the *total* $\mathbb{Z}$ -grading

$$S = \bigoplus_{j \in \mathbb{Z}} S_j, \quad S_j = \bigoplus_{i=1}^n \{ \bigotimes_{i=1}^n S(\mu_i)_{m_i} \mid m_1 + \cdots + m_n = j \}.$$

Fix $k \in \{1, \dots, n\}$. A nonzero element $s \in S$ can be uniquely written as a sum

$$s = s_i e_i(k) + s_{i+1} e_{i+1}(k) + \cdots + s_j e_j(k), \quad s_i \neq 0, \quad s_j \neq 0,$$

where the coefficients s_t are from the tensor product $\bigotimes_{m \neq k} S(\mu_m)$ and $\{e_i(k)\}$ is the basis of the A_1 -module $S(\mu_k)$. The k -*length* of the element s is defined as $l_k(s) = j - i$.

Let α be from Theorem 2.1 and $0 \neq s \in S$. Now it is easy to see that

$$l_k(\alpha s) = l_k(\alpha) + l_k(s) = 2 + l_k(s) \geq 2, \quad \text{for any } k > 1.$$

So, the linear map

$$\alpha_S : S \rightarrow S, \quad s \rightarrow \alpha s,$$

is injective and the cokernel $\text{coker}(\alpha_S) \equiv S/\alpha S$ of the map α_S has

$$(2) \quad \dim \text{coker}(\alpha_S) = \infty$$

since $\text{im}(\alpha_S) \cap W = 0$ where $W = S(\mu_1) \otimes \cdots \otimes S(\mu_{k-1}) \otimes \bar{1} \otimes S(\mu_{k+1}) \otimes \cdots \otimes S(\mu_n)$, $\bar{1} = 1 + A_1(H_k - \mu_k)$, and $\dim(W) = \infty$.

Using the short exact sequence

$$0 \rightarrow A_n \xrightarrow{\cdot \alpha} A_n \rightarrow St(\alpha) \rightarrow 0,$$

we get $\text{Ext}_{A_n}^1(St_n(\alpha), S) = \text{coker}(\alpha_S)$; hence, by (2), $\dim \text{Ext}_{A_n}^1(St_n(\alpha), S) = \infty$. This finishes the proof of Theorem 2.2. $\square$

For a polynomial $f \in K[H_1, \dots, H_n]$ let $V(f)$ be the set

$$\{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : f(\lambda_1, \dots, \lambda_n) = 0\}.$$

Corollary 2.3. *Suppose that $u = \sum_{\mathbf{i} \in \mathbb{Z}^n} u_{\mathbf{i}} v_{\mathbf{i}} \in A_n$ is not a homogeneous element where all $u_{\mathbf{i}} \in K[H_1, \dots, H_n]$, the A_n -module $S = \bigotimes_{i=1}^n S(\mu_i)$ is as above (i.e. all $\mu_i \in \mathbb{C} \setminus \mathbb{Z}$) and*

$$(3) \quad \{\mathbb{Z}^n \cup ((\mu_1, \dots, \mu_n) + \mathbb{Z}^n)\} \cap V(u_{\mathbf{i}}) = \emptyset, \text{ for all } \mathbf{i}.$$

Then the linear map $u_S : S \rightarrow S$, $s \mapsto us$, is injective and $\dim \operatorname{coker}(u_S) = \infty$.

Proof. The element u is not homogeneous, so there exists k such that $l_k(u) \geq 1$. Now, by (3),

$$l_k(us) = l_k(u) + l_k(s) > 0 \text{ for any } 0 \neq s \in S.$$

So, the linear map u_S is injective and $\operatorname{im}(u_S) \cap W = 0$ where W is as above. Hence $\dim \operatorname{coker}(u_S) = \infty$. $\square$

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