POISSON INTEGRALS ASSOCIATED TO DUNKL OPERATORS FOR DIHEDRAL GROUPS

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(Communicated by Juha M. Heinonen)

ABSTRACT. In this paper we study the boundary behavior of Poisson integrals associated to Dunkl differential-difference operators for dihedral groups and the boundary integral representations for functions on the unit disc of $\mathbb{C}$ annihilated by the Laplace operator corresponding to these differential-difference operators.

1. NOTATION AND STATEMENT OF THE MAIN RESULT

For every integer k such that $k \geq 1$, let D_k be the dihedral group of order $2k$; that is, D_k consists of the rotations $z \mapsto ze^{\frac{2\pi il}{k}}$ and the reflections $z \mapsto \bar{z}e^{\frac{2\pi il}{k}}$, $0 \leq l \leq k-1$, $z \in \mathbb{C}$.

If $k \geq 1$ is a fixed integer and α is a fixed positive real number, we associate to the group D_k the weight function h defined by

$$h(z) = \left| \frac{z^k - \bar{z}^k}{2i} \right|^\alpha,$$

which is a product of powers of the linear functions on $\mathbb{R}^2 \cong \mathbb{C}$ whose zero-sets are the mirrors of the reflections in D_k .

The complex Dunkl operators are defined for a complex-valued function f of class C^1 on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by

$$T_h f(z) = \frac{\partial f(z)}{\partial z} + \alpha \sum_{l=0}^{k-1} \frac{f(z) - f(\bar{z}\omega^{2l})}{z - \bar{z}\omega^{2l}}$$

and

$$\bar{T}_h f(z) = \frac{\partial f(z)}{\partial \bar{z}} - \alpha \sum_{l=0}^{k-1} \frac{f(z) - f(\bar{z}\omega^{2l})}{z - \bar{z}\omega^{2l}} \omega^{2l}$$

where $\omega = e^{\frac{\pi i}{k}}$. As in [D1], let Δ_h denote the h -Laplacian operator, $\Delta_h = 4T_h \bar{T}_h$, and say that a complex-valued function f on $\mathbb{D}$ is h -harmonic if it is of class C^2 and $\Delta_h f = 0$ on $\mathbb{D}$.

An orthogonal basis for $L^2(h(e^{i\theta})^2 d\theta)$ on the unit circle was constructed in [D2], and this led to the definition of a Poisson kernel P which reproduces h -harmonic

Received by the editors October 21, 2003 and, in revised form, February 5, 2004.

2000 *Mathematics Subject Classification*. Primary 30E25, 31A10, 31A20; Secondary 31A05.

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polynomials in $\mathbb{D}$ from their boundary values, and which is given by

$$(1.1) \quad P(z, w) = \frac{1 - |z|^2 |w|^2}{B(\alpha, \alpha + 1) |1 - z\bar{w}|^2} \int_0^1 \frac{u^{\alpha-1} (1-u)^\alpha du}{[(1-u)|1 - z^k w^k|^2 + u|1 - z^k \bar{w}^k|^2]^\alpha}$$

for $z, w \in \mathbb{C}$ such that $|zw| < 1$ (see [D2, Theorems 1.3 and 2.1]).

Define the Poisson integral $P[f]$ of a function $f \in L^1(h(e^{i\theta})^2 d\theta)$ by

$$P[f](z) = c_\alpha \int_{-\pi}^{\pi} f(e^{i\theta}) P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta$$

for $z \in \mathbb{D}$, where $c_\alpha = \left(\int_{-\pi}^{\pi} h(e^{i\theta})^2 d\theta \right)^{-1}$.

Let S^1 denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and $C(S^1)$ be the space of all complex-valued continuous functions on S^1 . The main purpose of this paper is to prove the following result:

Theorem 1.1. *If $f \in C(S^1)$, then the function F defined by*

$$(1.2) \quad F(z) = \begin{cases} P[f](z), & \text{if } z \in \mathbb{D}, \\ f(z), & \text{if } z \in S^1, \end{cases}$$

is continuous on $\overline{\mathbb{D}}$.

Trivially, we have the following corollary.

Corollary 1.2. *If $f \in C(S^1)$, then*

$$\sup_{w \in S^1} |P[f](rw) - f(w)| \rightarrow 0 \quad (r \rightarrow 1).$$

It should be noted that the boundary behavior of Poisson integrals associated to Dunkl operators acting on functions on $\mathbb{R}^d$, $d \geq 2$, and corresponding to the abelian group $\mathbb{Z}_2^d$, was considered in Theorem 5.5.7 of [DX].

Theorem 1.1 will be used in Section 3 to study the boundary behavior of Poisson integrals and give conditions under which an h -harmonic function in the disc is the Poisson integral of some type of function or measure on the unit circle.

Remark 1.3. If $f \in L^1(h(e^{i\theta})^2 d\theta)$, then $P[f]$ is h -harmonic on $\mathbb{D}$. Indeed, as stated in the proof of Theorem 1.3 of [D2], if $\{\psi_n(z), \bar{z}\bar{\psi}_n(z) : n \geq 0\}$ denotes the orthonormal basis of $L^2(c_\alpha h_1(e^{i\theta})^2 d\theta)$ associated to $h_1(x + iy) := |y|^\alpha$, the Cauchy kernel associated to h is given by

$$(1.3) \quad C(z, w) = \sum_{n=0}^{\infty} \sum_{l=0}^{k-1} z^l \psi_n(z^k) \bar{w}^l \bar{\psi}_n(w^k) \quad (|zw| < 1),$$

with $\bar{T}_h(z^l \psi_n(z^k)) = 0$ for $0 \leq l \leq k-1$ and $n \geq 0$. Using (1.3) and the expression of the ψ_n in terms of Heisenberg polynomials (see [D1, Proposition 3.11] and [D2]), we easily get that for fixed $w \in S^1$, $\bar{T}_h C(z, w) = 0$ for all $z \in \mathbb{D}$, so that $z \mapsto P(z, w)$ is h -harmonic on $\mathbb{D}$, because $P(z, w) = C(z, w) + \bar{z}w C(w, z)$ and $\Delta_h(z C(z, w)) = z \Delta_h C(z, w) + 4 \bar{T}_h C(z, w)$ (see [D1, Proposition 2.2]). From this we easily deduce that $P[f]$ is h -harmonic on $\mathbb{D}$.

Thus Theorem 1.1 provides the solution of the analogue of the Dirichlet problem for the unit disc of $\mathbb{C}$ associated to the h -Laplacian operator. The uniqueness theorem that corresponds to this existence theorem is proved in Section 3.

2. PROOF OF THE MAIN RESULT

For $z \in \mathbb{D}$ and $\theta \in (-\pi, \pi]$, set

$$a(z, \theta) = |1 - z^k e^{ik\theta}|^2$$

and

$$b(z, \theta) = |1 - z^k e^{-ik\theta}|^2.$$

We consider the integral

$$\int_{-\pi}^{\pi} \int_0^1 \frac{u^{\alpha-1} (1-u)^\alpha du}{[(1-u)a(z, \theta) + ub(z, \theta)]^\alpha} (\sin^2 k\theta)^\alpha d\theta \quad (z \in \mathbb{D})$$

in two pieces: the first is taken over $A_1(z) = \{\theta \in (-\pi, \pi] : b(z, \theta) \leq 2a(z, \theta)\}$, and the second over the complement of $A_1(z)$, denoted by $A_2(z)$; for $j \in \{1, 2\}$ and $z \in \mathbb{D}$, set

$$(2.1) \quad I_j(z) = \int_{A_j(z)} \int_0^1 \frac{u^{\alpha-1} (1-u)^\alpha du}{[(1-u)a(z, \theta) + ub(z, \theta)]^\alpha} (\sin^2 k\theta)^\alpha d\theta.$$

Then we have the following

Lemma 2.1. *Set $K = 2\pi \left(\frac{1+\sqrt{2}}{2}\right)^{2\alpha}$. For any $z \in \mathbb{D}$,*

$$I_1(z) \leq \frac{K}{\alpha},$$

and if $z \in \mathbb{D}$ satisfies $|z| \geq \frac{1}{2}$, then

$$I_2(z) \leq K \left(\int_0^1 \frac{t^{\alpha-1}}{(1+t)^\alpha} dt + 2^{k+1} \int_1^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\frac{1}{2}}} dt \right).$$

Proof. Let $z \in \mathbb{D}$. We have $|\sin k\theta| \leq \frac{1}{2}(\sqrt{a(z, \theta)} + \sqrt{b(z, \theta)})$, so that for any $\theta \in A_1(z)$, $(\sin^2 k\theta)^\alpha \leq \left(\frac{1+\sqrt{2}}{2}\right)^{2\alpha} a(z, \theta)^\alpha$ and

$$I_1(z) \leq K \int_0^1 u^{\alpha-1} du = \frac{K}{\alpha}.$$

If $\theta \in A_2(z)$ and $u \in [0, 1]$, then $(\sin^2 k\theta)^\alpha \leq K' b(z, \theta)^\alpha$ where $K' = \left(\frac{1+\frac{1}{\sqrt{2}}}{2}\right)^{2\alpha}$ and $(1-u)a(z, \theta) + ub(z, \theta) \geq a(z, \theta) + u\frac{b(z, \theta)}{2}$, so that

$$\begin{aligned} I_2(z) &\leq K' \int_{A_2(z)} \frac{b(z, \theta)^\alpha}{a(z, \theta)^\alpha} \int_0^1 \frac{u^{\alpha-1}}{\left(1 + u\frac{b(z, \theta)}{2a(z, \theta)}\right)^\alpha} du d\theta \\ &= \frac{K}{2\pi} \int_{A_2(z)} \int_0^{c(z, \theta)} \frac{t^{\alpha-1}}{(1+t)^\alpha} dt d\theta, \end{aligned}$$

where $c(z, \theta) = \frac{b(z, \theta)}{2a(z, \theta)} > 1$. Then

$$I_2(z) \leq K \int_0^1 \frac{t^{\alpha-1}}{(1+t)^\alpha} dt + \frac{K}{2\pi} I_3(z)$$

with

$$\begin{aligned} I_3(z) &= \int_{A_2(z)} \int_1^{c(z,\theta)} \frac{t^{\alpha-1}}{(1+t)^\alpha} dt d\theta \\ &= \int_1^\infty \frac{t^{\alpha-1}}{(1+t)^\alpha} \lambda(A_2(z) \cap \{\theta \in (-\pi, \pi] : c(z, \theta) \geq t\}) dt, \end{aligned}$$

where λ denotes the Lebesgue measure on $(-\pi, \pi]$.

For any $t \geq 1$, we have

$$\begin{aligned} \lambda(\{\theta \in (-\pi, \pi] : c(z, \theta) \geq t\}) &\leq \lambda(\{\theta \in (-\pi, \pi] : b(z, \theta) \geq a(z, \theta)(1+t)\}) \\ &\leq \lambda(\{\theta \in (-\pi, \pi] : a(z, \theta) \leq \frac{4}{1+t}\}) \end{aligned}$$

because $b(z, \theta) \leq 4$. Set $z = \rho e^{i\phi}$, $M = 4^{k+1}$, and if A is a Borel set of $\mathbb{R}$, let χ_A be the characteristic function of A ; if $\rho \geq \frac{1}{2}$, then

$$\begin{aligned} &\lambda\left(\left\{\theta \in (-\pi, \pi] : a(z, \theta) \leq \frac{4}{1+t}\right\}\right) \\ &\leq \lambda\left(\left\{\theta \in (-\pi, \pi] : \left|\frac{1}{\rho^k} - e^{ik(\theta+\phi)}\right|^2 \leq \frac{M}{1+t}\right\}\right) \\ &= \lambda\left(\left\{u \in (-\pi, \pi] : \left|\frac{1}{\rho^k} - e^{iku}\right|^2 \leq \frac{M}{1+t}\right\}\right) \\ &= \frac{1}{k} \int_{-k\pi}^{k\pi} \chi_{\left\{\theta \in (-k\pi, k\pi] : \left|\frac{1}{\rho^k} - e^{i\theta}\right|^2 \leq \frac{M}{1+t}\right\}} d\theta \\ &= \int_{-\pi}^{\pi} \chi_{\left\{\theta \in (-\pi, \pi] : \left|\frac{1}{\rho^k} - e^{i\theta}\right|^2 \leq \frac{M}{1+t}\right\}} d\theta \\ &\leq \lambda\left(\left\{\theta \in (-\pi, \pi] : |1 - e^{i\theta}| \leq 2\sqrt{\frac{M}{1+t}}\right\}\right) \\ &\leq 2\pi\sqrt{\frac{M}{1+t}}, \end{aligned}$$

so that

$$I_3(z) \leq 2^{k+2}\pi \int_1^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\frac{1}{2}}} dt,$$

which concludes the proof of the lemma. $\square$

Proof of Theorem 1.1. By formula (1.1), $P[f]$ is continuous on $\mathbb{D}$. Now let $z_0 \in S^1$. Since

$$(2.2) \quad c_\alpha \int_{-\pi}^{\pi} P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta = 1 \quad (z \in \mathbb{D}),$$

we have, for any $z \in \mathbb{D}$,

$$P[f](z) - f(z_0) = c_\alpha \int_{-\pi}^{\pi} [f(e^{i\theta}) - f(z_0)] P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta.$$

Let $\varepsilon > 0$. There exists $\delta \in (0, 1]$ such that for any $w \in S^1$ satisfying $|w - z_0| \leq \delta$, $|f(w) - f(z_0)| \leq \frac{\varepsilon}{2}$.

Using formula (1.1) and the fact that for any $z \in \mathbb{D}$ satisfying $|z - z_0| \leq \frac{\delta}{2}$, then if $|e^{i\theta} - z_0| \geq \delta$, we have $|1 - ze^{-i\theta}| \geq \frac{\delta}{2}$, we obtain, if $|z - z_0| \leq \frac{\delta}{2}$:

$$(2.3) \quad |P[f](z) - f(z_0)| \leq \frac{\varepsilon}{2} + 2c_\alpha \|f\|_\infty \frac{1 - |z|^2}{B(\alpha, \alpha + 1)(\frac{\delta}{2})^2} I(z)$$

with $\|f\|_\infty = \sup_{w \in S^1} |f(w)|$ and

$$\begin{aligned} I(z) &= \int_{|e^{i\theta} - z_0| \geq \delta} \int_0^1 \frac{u^{\alpha-1}(1-u)^\alpha du}{[(1-u)|1 - z^k e^{ik\theta}|^2 + u|1 - z^k e^{-ik\theta}|^2]^\alpha} (\sin^2 k\theta)^\alpha d\theta \\ &\leq I_1(z) + I_2(z), \end{aligned}$$

where $I_1(z)$ and $I_2(z)$ are given by formula (2.1). Then it follows from Lemma 2.1 that $I(z)$ is bounded on $\{z \in \mathbb{D} : |z - z_0| \leq \frac{\delta}{2}\}$ by a constant depending only on α and k . Consequently, (2.3) implies that there is $\eta \in (0, \frac{\delta}{2}]$ such that for any $z \in \mathbb{D}$ satisfying $|z - z_0| \leq \eta$, we have $|P[f](z) - f(z_0)| \leq \varepsilon$, which completes the proof of the theorem. $\square$

3. APPLICATIONS

In this section, we study the boundary behavior of Poisson integrals and establish that under certain conditions an h -harmonic function in the disc is the Poisson integral of some type of function or measure on the unit circle.

Let $1 \leq p \leq \infty$ and $f \in L^p(h(e^{i\theta})^2 d\theta)$. For fixed $r \in [0, 1)$, for all $z, w \in S^1$, we have

$$(3.1) \quad 0 \leq P(rz, w) \leq \frac{1 - r^2}{(1 - r)^2} \frac{1}{(1 - r^k)^{2\alpha}},$$

so that the function $P[f]_r$ defined on S^1 by $P[f]_r(z) = P[f](rz)$ is bounded on S^1 and hence is in $L^p(h(e^{i\theta})^2 d\theta)$.

If $1 < p < \infty$, then, using Hölder's inequality and (2.2), we get

$$|P[f]_r(w)|^p \leq c_\alpha \int_{-\pi}^{\pi} |f(e^{it})|^p P(rw, e^{it}) h(e^{it})^2 dt \quad (w \in S^1),$$

which is also true if $p = 1$. It now follows from Fubini's theorem that

$$(3.2) \quad \|P[f]_r\|_{L^p(h(e^{i\theta})^2 d\theta)} \leq \|f\|_{L^p(h(e^{i\theta})^2 d\theta)},$$

when $1 \leq p < \infty$, since

$$(3.3) \quad P(rw, w') = P(rw', w), \quad w, w' \in S^1.$$

Note that (3.2) holds when $p = \infty$.

Theorem 3.1. (a) Let $1 \leq p < \infty$ and $f \in L^p(h(e^{i\theta})^2 d\theta)$. Then

$$\|P[f]_r - f\|_p \rightarrow 0 \quad (r \rightarrow 1),$$

where the L^p -norm is with respect to $h(e^{i\theta})^2 d\theta$.

(b) If $f \in L^\infty(h(e^{i\theta})^2 d\theta)$, then, as $r \rightarrow 1$, the functions $P[f]_r$ converge to f in the weak-star topology on $L^\infty(h(e^{i\theta})^2 d\theta)$.

(c) Let μ be a complex Borel measure on S^1 , and $P[\mu]$ be the Poisson integral of the measure μ on S^1 ; that is,

$$P[\mu](z) = c_\alpha \int_{S^1} P(z, w) d\mu(w) \quad (z \in \mathbb{D}).$$

Then, as $r \rightarrow 1$, the measures $P[\mu](re^{i\theta})h(e^{i\theta})^2d\theta$ converge to μ in the weak-star topology of the dual space of $C(S^1)$.

(d) If $P[\mu](z) = 0$ for all $z \in \mathbb{D}$, then $\mu = 0$.

Proof. To prove (a), we may assume $f \in C(S^1)$, because of (3.2) and the density of continuous functions in $L^p(h(e^{i\theta})^2d\theta)$. In this case, the result follows easily from Corollary 1.2.

If $f \in L^\infty(h(e^{i\theta})^2d\theta)$, $g \in L^1(h(e^{i\theta})^2d\theta)$ and μ is a complex Borel measure on S^1 , then by (3.3) and Fubini's theorem,

$$\int_{-\pi}^{\pi} g(e^{i\theta})P[f]_r(e^{i\theta})h(e^{i\theta})^2d\theta = \int_{-\pi}^{\pi} f(e^{i\theta})P[g]_r(e^{i\theta})h(e^{i\theta})^2d\theta$$

and

$$\int_{-\pi}^{\pi} g(e^{i\theta})P[\mu](re^{i\theta})h(e^{i\theta})^2d\theta = \int_{S^1} P[g]_r(w) d\mu(w),$$

so that part (b) follows from part (a) applied to g , and if in addition g is continuous on S^1 , Corollary 1.2 applied to g completes the proof of (c).

Part (d) follows from part (c) and the uniqueness assertion of the Riesz representation theorem. $\square$

The following proposition is the uniqueness theorem corresponding to Theorem 1.1.

Proposition 3.2. *Let f be a complex-valued continuous function on $\overline{\mathbb{D}}$. Assume f is h -harmonic on $\mathbb{D}$. Then*

$$f(z) = c_\alpha \int_{-\pi}^{\pi} f(e^{i\theta})P(z, e^{i\theta})h(e^{i\theta})^2d\theta, \quad z \in \mathbb{D}.$$

Proof. Without loss of generality, we may assume that f is real. Put $u = f - F$ where F is the function defined by formula (1.2). By Theorem 1.1 and Remark 1.3, u is continuous on $\overline{\mathbb{D}}$ and h -harmonic on $\mathbb{D}$. Thus, by Theorem 4.2 in [R], we obtain

$$\max_{\overline{\mathbb{D}}} (u) = \max_{S^1} (u) = 0,$$

so that $u \leq 0$ on $\mathbb{D}$. The same argument shows that $-u \leq 0$. Hence $f = F$ on $\mathbb{D}$, and the proof is complete. $\square$

Theorem 3.3. *Assume u is a complex-valued h -harmonic function on $\mathbb{D}$. Write $u_r(w) = u(rw)$ for $r \in [0, 1)$ and $w \in S^1$.*

(a) *If $1 < p \leq \infty$, then u is the Poisson integral of a function $f \in L^p(h(e^{i\theta})^2d\theta)$ if and only if*

$$(3.4) \quad \sup_{0 \leq r < 1} \|u_r\|_{L^p(h(e^{i\theta})^2d\theta)} < \infty.$$

(b) *u is the Poisson integral of a complex Borel measure on S^1 if and only if*

$$(3.5) \quad \sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |u(re^{i\theta})| h(e^{i\theta})^2d\theta < \infty.$$

(c) *u is positive if and only if u is the Poisson integral of a bounded positive Borel measure on S^1 .*

(d) *u is the Poisson integral of a function $f \in L^1(h(e^{i\theta})^2d\theta)$ if and only if the u_r converge in $L^1(h(e^{i\theta})^2d\theta)$.*

(e) u is the Poisson integral of a function $f \in C(S^1)$ if and only if the u_r converge uniformly.

Proof. Inequality (3.2) shows that (3.4) is a necessary condition. If $u = P[\mu]$, where μ is a complex Borel measure on S^1 , then, using Fubini's theorem, (3.3) and (2.2), we obtain

$$\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |u(re^{i\theta})| h(e^{i\theta})^2 d\theta \leq \|\mu\|,$$

where $\|\mu\|$ denotes the total variation of μ , so that (3.5) holds.

Since for fixed $r \in [0, 1)$, $\Delta_h(u(rz)) = r^2(\Delta_h u)(rz) = 0$ on $\mathbb{D}$, we have by Proposition 3.2,

$$(3.6) \quad u(rz) = c_\alpha \int_{-\pi}^{\pi} u(re^{i\theta}) P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta, \quad z \in \mathbb{D}.$$

Assume u satisfies (3.4) or (3.5). Let $\{r_n\}$ be a sequence in $[0, 1)$ that increases to 1. If $1 < p \leq \infty$, let $f_n = u_{r_n}$; by (3.4), the sequence $\{f_n\}$ is bounded in $L^p(h(e^{i\theta})^2 d\theta)$, so that it follows from the Banach-Alaoglu theorem that there is a subsequence $\{f_{n_j}\}$, $n_1 < n_2 < \dots$, that converges weak-star to an element f of $L^p(h(e^{i\theta})^2 d\theta)$. Since for any $z \in \mathbb{D}$, $P(z, \cdot) \in L^q(h(e^{i\theta})^2 d\theta)$, $q = p/(p-1)$, we obtain, using (3.6),

$$u(z) = \lim_{j \rightarrow \infty} u(r_{n_j} z) = \lim_{j \rightarrow \infty} c_\alpha \int_{-\pi}^{\pi} f_{n_j}(e^{i\theta}) P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta = P[f](z)$$

for all $z \in \mathbb{D}$.

The proof of (b) is the same except that now the measures $u(r_n e^{i\theta}) h(e^{i\theta})^2 d\theta$ have bounded norms, so that there is a subsequence that converges weak-star to a complex Borel measure on S^1 .

If $u \geq 0$, the measures $u(r_n e^{i\theta}) h(e^{i\theta})^2 d\theta$ are positive and have bounded norms, because (3.6) yields

$$c_\alpha \int_{-\pi}^{\pi} u(r_n e^{i\theta}) h(e^{i\theta})^2 d\theta = u(0).$$

Then, as in the proof of (b), there is a subsequence that converges weak-star to a finite Borel measure μ on S^1 , and the measure μ is now positive.

We have already proved that if u is the Poisson integral of a function $f \in L^1(h(e^{i\theta})^2 d\theta)$ (respectively $f \in C(S^1)$), then the u_r converge to f in $L^1(h(e^{i\theta})^2 d\theta)$ (respectively uniformly on S^1). Conversely, if the u_r converge to a function f in $L^1(h(e^{i\theta})^2 d\theta)$, then, using (3.6) and (3.1), we get

$$u(z) = \lim_{r \rightarrow 1} u(rz) = \lim_{r \rightarrow 1} c_\alpha \int_{-\pi}^{\pi} u(re^{i\theta}) P(z, e^{i\theta}) h(e^{i\theta})^2 d\theta = P[f](z)$$

for all $z \in \mathbb{D}$.

If the u_r converge uniformly, they converge to a continuous g , and then $u = P[g]$ by a similar argument. $\square$

I would like to thank Professor E. H. Youssfi for proposing to me this problem as well as his suggestions and indications during the preparation of this paper.

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