HARMONIC ANALYSIS ON DISCRETE ABELIAN GROUPS

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ABSTRACT. Let G be an Abelian group and let $\mathbb{C}^G$ denote the linear space of all complex-valued functions defined on G equipped with the product topology. We prove that the following are equivalent.

- (i) Every nonzero translation invariant closed subspace of $\mathbb{C}^G$ contains an exponential; that is, a nonzero multiplicative function.
- (ii) The torsion free rank of G is less than the continuum.

1. INTRODUCTION

Let G be a locally compact Abelian group and let $C(G)$ denote the linear space of all complex-valued continuous functions defined on G equipped with the topology of uniform convergence on compact sets. A nonzero function $m \in C(G)$ is called an exponential if m is multiplicative; that is, if $m(x + y) = m(x) \cdot m(y)$ holds for every $x, y \in G$. It is easy to see that the translates of a function $f \in C(G)$ span a one-dimensional subspace of $C(G)$ if and only if f is a constant multiple of an exponential. We say that *harmonic (or spectral) analysis holds on G* if every nonzero translation invariant closed subspace of $C(G)$ contains an exponential. If the translation invariant closed subspaces of $C(G)$ are spanned by polynomial-exponential functions, then we say that *harmonic (or spectral) synthesis holds on G* . (A function is polynomial-exponential if it belongs to the algebra generated by the continuous additive functions and by the exponentials.)

A theorem of R. J. Elliot [2] claims that harmonic synthesis holds on every discrete Abelian group. As it turned out, Elliot's proof was defective. In fact, by a recent result of L. Székelyhidi [6], the statement is false. Székelyhidi proves that harmonic synthesis fails on $\mathbb{R}$ (equipped with the discrete topology) or, more generally, on every Abelian group containing the free Abelian group of rank ω . The statement that even harmonic analysis may fail on some free Abelian groups was proved in [5].

In this note our aim is to characterize those Abelian groups on which harmonic analysis holds. Let $r_0(G)$ denote the torsion free rank of G ; that is, the cardinality of a maximal independent system of elements of infinite order. In other words, $r_0(G)$ is the maximal cardinality κ such that G contains the free Abelian group of

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rank κ as a subgroup. The cardinality of continuum is denoted by 2^ω . Our main result is the following.

Theorem 1. *Harmonic analysis holds on a discrete Abelian group G if and only if $r_0(G) < 2^\omega$.*

In particular, harmonic analysis holds on every discrete Abelian group of cardinality less than 2^ω and fails to hold on every torsion-free Abelian group of cardinality $\geq 2^\omega$.

As a special case we find that harmonic analysis holds on the free Abelian group of rank ω . (This was first proved in [4].) Since, by [6], harmonic synthesis fails to hold on this group, we find that harmonic analysis does not imply harmonic synthesis in general.

2. SUFFICIENCY

In this section we show that if $r_0(G) < 2^\omega$, then harmonic analysis holds on G . This will be shown by using duality.

The set of functions from G to $\mathbb{C}$ will be denoted by $\mathbb{C}^G$. We shall say that an operator $D : \mathbb{C}^G \rightarrow \mathbb{C}^G$ is a difference operator if there are elements $a_i \in \mathbb{C}$ and $x_i \in G$ ($i = 1, \dots, n$) such that

$$(Df)(x) = \sum_{i=1}^n a_i f(x + x_i)$$

for every $f \in \mathbb{C}^G$ and $x \in G$. The set of all difference operators will be denoted by $\mathcal{D}$. If $A \in \mathcal{D}$ and $a \in \mathbb{C}$, then we define $aA \in \mathcal{D}$ by $(aA)f = a(Af)$. If $A, B \in \mathcal{D}$, then the sum and product of A, B are defined by $(A+B)f = Af + Bf$ and $(AB)f = A(Bf)$. It is easy to check that under these operators $\mathcal{D}$ becomes a commutative algebra. Let T_g denote the translation operator $T_g f(x) = f(x + g)$. Then T_0 is the identity operator; also, it is the unit element of $\mathcal{D}$. Every difference operator is the linear combination of translation operators.

The linear space $\mathbb{C}^G$ endowed with the product topology is a locally convex topological vector space. It is easy to see that the continuous linear functionals on $\mathbb{C}^G$ are of the form $Lf = \sum_{i=1}^n a_i f(x_i)$, where $a_i \in \mathbb{C}$ and $x_i \in G$ ($i = 1, \dots, n$). Let V be a translation invariant closed subspace of $\mathbb{C}^G$, and put $\mathcal{I} = \{D \in \mathcal{D} : Df = 0 \text{ for every } f \in V\}$. Then $\mathcal{I}$ is an ideal of $\mathcal{D}$, and for every $f \in \mathbb{C}^G$ we have $f \in V \iff Df = 0$ ($D \in \mathcal{I}$). Indeed, if $f \notin V$, then the local convexity of the space implies the existence of a continuous linear functional L such that $Lf \neq 0$ and $Lg = 0$ for $g \in V$. Thus there are elements $a_i \in \mathbb{C}$ and $x_i \in G$ ($i = 1, \dots, n$) satisfying $\sum_{i=1}^n a_i f(x_i) \neq 0$ and $\sum_{i=1}^n a_i g(x_i) = 0$ for every $g \in V$. Since V is translation invariant, we obtain $D = \sum_{i=1}^n a_i T_{x_i} \in \mathcal{I}$ and $Df \neq 0$. Clearly, if V is nonzero, then $\mathcal{I} \neq \mathcal{D}$.

In order to prove that every nonzero translation invariant closed subspace contains an exponential, it is enough to show that if $\mathcal{I}$ is a proper ideal, then there is an exponential m such that $Dm = 0$ for every $D \in \mathcal{D}$. The situation is that of Hilbert's Nullstellensatz, and the statement for the free group of rank ω can be proved by generalizing Hilbert's theorem to the polynomial ring of countably many variables [4]. In the general case we shall apply the proof of Hilbert's theorem instead of generalizing it. The next lemma is a variant of [3, Theorem 5.2, p. 32]

and is contained in Problem 10(b) of [1, Chapter V. §3, p. 373]. For the sake of completeness we provide a simple proof.

Lemma 2. *Suppose that K is a field, k is a subfield, and X is a subset of K such that $\max(|X|, \omega) < |k|$, and $K = k[X]$; that is, K is generated as a ring by k and X . Then K is an algebraic extension of k .*

Proof. It is enough to prove that every element of X is algebraic over k . Suppose this is not true, and let $Y \neq \emptyset$ be a maximal algebraically independent subset of X . If $L = k(Y)$, then $k \subset L \subset K$, and every element of K is algebraic over L . Let p_z denote the monic minimum polynomial of $z \in K$ over L . Its coefficients are elements of L ; that is, they are rational functions of the elements of Y with coefficients from k . Let $G_z \in k[Y]$ be the product of the denominators of the coefficients of p_z . In the algebraic closure $\bar{L}$ of L the polynomial p_z splits into linear factors. The roots of these factors are the conjugates of z over L . Clearly, the elementary symmetric polynomials of these conjugates are of the form F/G_z , where $F \in k[Y]$. Therefore, every symmetric polynomial of the conjugates of z with coefficients from k is of the form $F/(G_z)^N$, where $F \in k[Y]$ and N is a positive integer.

Let y_0 be a fixed element of Y (note that $Y \neq \emptyset$ by assumption). For every $z \in K$, the set Γ_z of elements $c \in k$ such that the polynomial G_z is divisible by $y_0 - c$ is finite. Therefore, $|\bigcup_{x \in X} \Gamma_x| \leq \max(|X|, \omega) < |k|$, and thus there exists an element $c \in k$ such that $y_0 - c$ is not a factor of any of the polynomials G_x ($x \in X$).

Now $1/(y_0 - c) \in L \subset K = k[X]$, and thus $1/(y_0 - c) = p(x_1, \dots, x_n)$, where p is a polynomial of the elements $x_1, \dots, x_n \in X$ with coefficients from k . Let $L_1 = L(x_1) \dots (x_n)$; then L_1 is a finite extension of L . Let ϕ_j ($j = 1, \dots, N$) be a list of the isomorphisms mapping L_1 into $\bar{L}$ and fixing the elements of L . Then

$$N/(y_0 - c) = \sum_{j=1}^N p(\phi_j(x_1) \dots \phi_j(x_n)).$$

For each $i = 1, \dots, n$, the right-hand side is a symmetric polynomial of the conjugates of x_i over L , and thus it is of the form F/G , where $F, G \in k[Y]$, and G divides a power of $\prod_{i=1}^n G_{x_i}$. Since L is a pure transcendental extension of k , it follows from $N/(y_0 - c) = F/G$ that G is divisible by $y_0 - c$, and thus at least one of the polynomials G_{x_i} must be divisible by $y_0 - c$. This, however, contradicts the choice of c , which completes the proof. $\square$

Theorem 3. *If G is an Abelian group satisfying $r_0(G) < 2^\omega$, then harmonic analysis holds in G .*

Proof. As we saw already, it is enough to show that if $\mathcal{I}$ is a proper ideal, then there exists an exponential m such that $Dm = 0$ for every $D \in \mathcal{I}$. We may assume that $\mathcal{I}$ is maximal. Then $K = \mathcal{D}/\mathcal{I}$ is a field. It is easy to see that $c \mapsto c \cdot T_0$ ($c \in \mathbb{C}$) is an isomorphism from $\mathbb{C}$ into $\mathcal{D}$. Therefore, $k = \{c \cdot T_0 : c \in \mathbb{C}\}$ is a subfield of $\mathcal{D}$ isomorphic to $\mathbb{C}$. Let ϕ denote the natural ring homomorphism from $\mathcal{D}$ onto K . Then $c \cdot T_0 \mapsto \phi(c \cdot T_0)$ ($c \in \mathbb{C}$) is an isomorphism from k into K , since $c \neq 0$ implies $c \cdot T_0 \notin \mathcal{I}$ and $\phi(c \cdot T_0) \neq 0$. Thus $k' = \phi(k)$ is a subfield of K isomorphic to $\mathbb{C}$. If we identify k' with $\mathbb{C}$, then K becomes an extension of $\mathbb{C}$.

We put $m(g) = \phi(T_g)$ for every $g \in G$. Then $m : G \rightarrow K$ is multiplicative since

$$m(g + h) = \phi(T_{g+h}) = \phi(T_g T_h) = \phi(T_g) \cdot \phi(T_h) = m(g)m(h)$$

for every $g, h \in G$. Also, $m(g) \neq 0$ for every g since

$$[m(g) = 0] \implies [T_g \in \mathcal{I}] \implies [T_0 \in \mathcal{I}] \implies [\mathcal{I} = \mathcal{D}],$$

which is not the case. If $D \in \mathcal{I}$, $D = \sum_{i=1}^k c_i T_{g_i}$, then we have

$$\begin{aligned} Dm(g) &= \sum_{i=1}^k c_i \cdot m(g + g_i) = \sum_{i=1}^k c_i \cdot m(g) \cdot m(g_i) = m(g) \cdot \sum_{i=1}^k c_i \cdot \phi(T_{g_i}) \\ &= m(g) \cdot \phi(D) = 0. \end{aligned}$$

In order to complete the proof it is enough to show that $K = \mathbb{C}$. Clearly, $K = \mathbb{C}[m(G)]$. In other words, K is the ring generated by $\mathbb{C}$ and by the elements $m(g)$ ($g \in G$).

Let tG denote the torsion group of G (that is, the set of elements of finite order), and let $X \subset G$ be a set containing exactly one element of each coset of tG . We show that $|X| < 2^\omega$. Suppose $|X| \geq 2^\omega$; then $|G/tG| = |X| \geq 2^\omega$. Now G/tG is a torsion free group, and thus it contains a free Abelian group of rank $|G/tG| \geq 2^\omega$. Let Y be a set of generators of this free subgroup with $|Y| \geq 2^\omega$. Let ψ denote the natural homomorphism from G onto G/tG , and select a point from each set $\psi^{-1}(\{y\})$ ($y \in Y$). The set Z of these points is an independent system of elements of infinite order. Since $|Z| \geq 2^\omega$, it follows that $r_0(G) \geq 2^\omega$, contradicting the assumption on G . Thus $|X| < 2^\omega$, as we stated.

Every element of G can be written in the form $x + h$ where $x \in X$ and $h \in tG$. If $h \in tG$ and $nh = 0$, then $1 = m(0) = m(nh) = m(h)^n$ and thus $m(h)$ is an n^{th} root of unity. In particular, $m(h) \in \mathbb{C}$ for every $h \in tG$. Therefore we have $K = \mathbb{C}[m(G)] = \mathbb{C}[m(X)]$. Since $|m(X)| \leq |X| < 2^\omega = |\mathbb{C}|$, it follows from Lemma 2 that K is an algebraic extension of $\mathbb{C}$. As $\mathbb{C}$ is algebraically closed, we obtain $K = \mathbb{C}$, which completes the proof. $\square$

3. NECESSITY

Lemma 4. *If harmonic analysis holds on the group G , then the same is true for every subgroup of G .*

Proof. Let H be a subgroup of G , and let $V \neq 0$ be a closed invariant subspace of $\mathbb{C}^H$. Then $W = \{f \in \mathbb{C}^G : (T_g f)|_H \in V \text{ for every } g \in G\}$ is a closed invariant subspace of $\mathbb{C}^G$. Let U be a subset of G containing exactly one element of each coset of H . If $f \in V$, $f \neq 0$, then the function $g : G \rightarrow \mathbb{C}$ defined by $g(u + x) = f(x)$ ($u \in U$, $x \in H$) belongs to W , and thus $W \neq 0$. By assumption, W contains an exponential m . Then $(m|_H) \in V$, showing that harmonic analysis holds on H . $\square$

Lemma 5. *Let $\mathcal{I}$ be a proper ideal of $\mathcal{D}$. Then there exists a function $f : G \rightarrow \mathbb{C}$ such that f is not identically zero and $Df = 0$ for every $D \in \mathcal{I}$.*

Proof. Since $\mathcal{D}$ is a linear space over $\mathbb{C}$, $\mathcal{I}$ is a linear subspace of $\mathcal{D}$ and $T_0 \notin \mathcal{I}$, there is a linear map $\psi : \mathcal{D} \rightarrow \mathbb{C}$ such that $\psi(D) = 0$ for every $D \in \mathcal{I}$, and $\psi(T_0) = 1$. We define

$$f(g) = \psi(T_g)$$

for every $g \in G$. Then f maps G into $\mathbb{C}$. Since $f(0) = \psi(T_0) = 1$, f is not identically zero.

Let $D = \sum_{i=1}^n a_i T_{g_i}$ be an element of $\mathcal{I}$, and let $g \in G$. Then $\psi(T_g D) = 0$ by $T_g D \in \mathcal{I}$, and thus

$$\begin{aligned} (Df)(g) &= \sum_{i=1}^n a_i f(g_i + g) = \sum_{i=1}^n a_i \cdot \psi(T_{g_i+g}) \\ &= \psi\left(\sum_{i=1}^n a_i T_g T_{g_i}\right) \\ &= \psi(T_g \cdot D) = 0. \end{aligned}$$

Thus $Df = 0$ for every $D \in \mathcal{I}$, which proves the lemma. $\square$

If $r_0(G) \geq 2^\omega$, then G contains the free Abelian group of rank 2^ω as a subgroup. If harmonic analysis held on G then, by Lemma 4, it would hold on each subgroup as well. Thus the next lemma will complete the proof of Theorem 1. Note that the statement of the lemma is just a reformulation of Problem 10(c) of [1, Chapter V. §3, p. 373]. Again, we provide the simple proof.

Lemma 6. *Let G be a free Abelian group of rank $\geq 2^\omega$. Then harmonic analysis does not hold on G .*

Proof. Let X be the set of generators of G . Since $|X| = |G| \geq 2^\omega$, there exists a surjective map $\phi : X \rightarrow \mathbb{C}(t)$. Then there is a unique homomorphism $\Phi : \mathcal{D} \rightarrow \mathbb{C}(t)$ such that $\Phi(T_x) = \phi(x)$ for every $x \in X$. Put

$$\mathcal{J} = \{D \in \mathcal{D} : \Phi(D) = 0\}.$$

Since $\mathbb{C}(t)$ is a field, $\mathcal{J}$ is a maximal ideal, and the factor ring $\mathcal{D}/\mathcal{J}$ is isomorphic to $\mathbb{C}(t)$. The set of functions

$$V = \{f \in \mathbb{C}^G : Df = 0 \ (D \in \mathcal{J})\}$$

is a closed translation invariant subspace of $\mathbb{C}^G$ and, by Lemma 5, V is nonzero. We claim that V does not contain any exponential function. Suppose $m \in V$ is an exponential. It is easy to check that the map

$$\Psi\left(\sum_{i=1}^n a_i T_{b_i}\right) = \sum_{i=1}^n a_i m(b_i)$$

is a homomorphism from $\mathcal{D}$ into $\mathbb{C}$, and $Dm = 0$ if and only if $\Psi(D) = 0$. The ideal $\mathcal{I} = \{D \in \mathcal{D} : Dm = 0\}$ contains $\mathcal{J}$ and hence $\mathcal{I} = \mathcal{J}$ by the maximality of $\mathcal{J}$. Thus $\mathcal{J}$ is the kernel of Ψ . Therefore, $\mathcal{D}/\mathcal{J}$ is isomorphic to $\mathbb{C}$, which is a contradiction. $\square$

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