THE KUNZE-STEIN PHENOMENON ASSOCIATED WITH JACOBI TRANSFORMS

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(Communicated by Andreas Seeger)

ABSTRACT. Recently A. D. Ionescu (2000) established the endpoint estimate for the Kunze-Stein phenomenon, which states that if G is a noncompact connected semisimple Lie group of real rank one with finite center, then

$$L^{2,1}(G) * L^{2,1}(G) \subseteq L^{2,\infty}(G).$$

In this paper, we will prove the corresponding result for the Jacobi transform. Our method is analytical, in which we do not use the structure of Lie groups.

1. INTRODUCTION

The Kunze-Stein phenomenon on semisimple Lie groups states that if G is a connected semisimple Lie group with finite center and $p \in [1, 2)$, then (see [1, 2])

$$(1.1) \quad L^p(G) * L^2(G) \subseteq L^2(G),$$

i.e., there exists a constant C such that, if f is in $L^p(G)$ and h is in $L^2(G)$, then $f * h$ is in $L^2(G)$ and $\|f * h\|_2 \leq C\|f\|_p\|h\|_2$. (For function spaces X , Y and Z , we shall use $X * Y \subseteq Z$ to mean both the set inclusion and the associated norm inequality.) If G is of real rank one, Cowling [2] proved that for $p \in [1, 2)$ and $(u, v, w) \in [1, \infty]^3$ such that $1 + 1/w \leq 1/u + 1/v$, then

$$(1.2) \quad L^{p,u}(G) * L^{p,v}(G) \subseteq L^{p,w}(G).$$

The relation (1.2) is not true for the endpoint $p = 2$. Recently, Ionescu [3] proved the endpoint estimate for the Kunze-Stein phenomenon, which states that if G is a noncompact connected semisimple Lie group of real rank one with finite center, then

$$(1.3) \quad L^{2,1}(G) * L^{2,1}(G) \subseteq L^{2,\infty}(G).$$

The convolution associated with Jacobi transforms also has the Kunze-Stein phenomenon similar to (1.1) and (1.2) (see (3.1) and (3.2) below). In this paper we will prove the corresponding endpoint estimate for Jacobi transforms.

Received by the editors July 19, 2003 and, in revised form, February 24, 2004.

2000 *Mathematics Subject Classification*. Primary 33C45; Secondary 43A90, 42B25, 22E30.

Key words and phrases. Kunze-Stein phenomenon, Jacobi transform, Lorentz space.

This research was supported by the National Natural Science Foundation of China, Projects 10001002 and 10371004.

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For certain discrete values of α and β , Jacobi transforms have an interpretation as spherical transforms on noncompact symmetric spaces of rank one. In this group-theoretic context, our result is the same as the endpoint estimate in [2] restricted to K -bi-invariant functions.

In Ionescu's proof the structure of Lie groups is applied, which could not be used here. Our proof just uses the exact estimate of the convolution kernel for Jacobi functions.

2. PRELIMINARIES

Throughout this paper, we assume that $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha \neq -\frac{1}{2}$. For $\lambda \in \mathbb{C}$ and $t \geq 0$, the Jacobi function $\phi_\lambda(t)$ of order (α, β) is defined by

$$\phi_\lambda(t) = F\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda); \alpha + 1; -\sinh^2 t\right),$$

where $\rho = \alpha + \beta + 1$, and F denotes the Gaussian hypergeometric function. The system $\{\phi_\lambda : \lambda \geq 0\}$ is a continuous orthogonal system on $\mathbb{R}^+$ with respect to the weight function

$$\Delta(t) = (2 \sinh t)^{2\alpha+1} (2 \cosh t)^{2\beta+1}, t > 0.$$

The Jacobi transform $f \rightarrow \hat{f}$ is defined by

$$\hat{f}(\lambda) = \int_0^\infty f(t) \phi_\lambda(t) \Delta(t) dt$$

for all functions f on $\mathbb{R}^+$ and complex numbers λ for which the right-hand side is well-defined.

Let f be a suitable function on $\mathbb{R}^+$, and let $s > 0$. The generalized translation $T_s f$ is defined by

$$T_s f(t) = \int_0^\infty f(u) K(s, t, u) \Delta(u) du,$$

where the convolution kernel is given by

$$\begin{aligned} K(s, t, u) = & \frac{2^{-2\rho} \Gamma(\alpha + 1) (\cosh s \cosh t \cosh u)^{\alpha - \beta - 1}}{\pi^{1/2} \Gamma(\alpha + 1/2) (\sinh s \sinh t \sinh u)^{2\alpha}} \\ (2.1) \quad & \cdot (1 - B^2)^{\alpha - 1/2} F(\alpha + \beta, \alpha - \beta; \alpha + 1/2; (1 - B)/2) \end{aligned}$$

if $|s - t| < u < s + t$, and $K(s, t, u) = 0$ otherwise. Here

$$(2.2) \quad B = \frac{\cosh^2 s + \cosh^2 t + \cosh^2 u - 1}{2 \cosh s \cosh t \cosh u}.$$

For suitable functions f and g on $\mathbb{R}^+$, the convolution $f * g$ is defined by

$$\begin{aligned} f * g(t) &= \int_0^\infty (T_t f)(s) g(s) \Delta(s) ds \\ (2.3) \quad &= \int_0^\infty \int_0^\infty f(u) g(s) K(s, t, u) \Delta(s) \Delta(u) ds du; \end{aligned}$$

it has the property

$$(f * g)^\wedge(\lambda) = \hat{f}(\lambda) \hat{g}(\lambda)$$

whenever these Jacobi transforms are well-defined.

We begin with the following lemma on upper and lower bounds for the kernel $K(s, t, u)$.

Lemma 2.1. For $|s - t| < u < s + t$,

$$(2.4) \quad K(s, t, u) \sim e^{-\rho(s+t+u)} \left(\frac{(1+s)(1+t)(1+u)}{stu} \right)^{2\alpha} \cdot \left(\frac{(t+s+u)(t+s-u)(u-s+t)(u+s-t)}{(1+t+s+u)(1+t+s-u)(1+u-s+t)(1+u+s-t)} \right)^{\alpha-1/2},$$

where we use the notion $f \sim g$ to mean that there exist positive constants C_1 and C_2 such that $C_1 g \leq f \leq C_2 g$ everywhere.

Proof. By (2.2), we have

$$\begin{aligned} 1 - B^2 &= \frac{(\cosh^2(t+s) - \cosh^2 u)(\cosh^2 u - \cosh^2(t-s))}{4(\cosh s \cosh t \cosh u)^2} \\ &= \frac{\sinh(t+s+u) \sinh(t+s-u) \sinh(u-s+t) \sinh(u+s-t)}{4(\cosh s \cosh t \cosh u)^2}. \end{aligned}$$

Then (2.4) follows easily from (2.1) and the fact that for $x \geq 0$, $\sinh x \sim \frac{x}{1+x}e^x$ and $\cosh x \sim e^x$.

Let $(\mathbb{R}^+, \mu)$ be the measure space on $\mathbb{R}^+$ with the measure $d\mu(t) = \Delta(t)dt$, and let $L^p(\mathbb{R}^+, \mu)$ ($1 \leq p \leq \infty$) be the function spaces as in the obvious way. Then the convolution of $f \in L^1(\mathbb{R}^+, \mu)$ and $g \in L^p(\mathbb{R}^+, \mu)$ ($1 \leq p \leq \infty$) satisfies

$$(2.5) \quad \|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Here $\|\cdot\|_p$ is the norm in $L^p(\mathbb{R}^+, \mu)$.

We can also define $\|\cdot\|_{p,q}^*$ for the measurable functions on $(\mathbb{R}^+, \mu)$ and the corresponding Lorentz spaces $L^{p,q}(\mathbb{R}^+, \mu)$ as in [7] and [8]. Notice that if χ_E is the characteristic function of a measurable set $E \subset \mathbb{R}^+$, we have

$$(2.6) \quad \|\chi_E\|_{2,q}^* = \mu(E)^{1/2} = \|\chi_E\|_2.$$

□

3. THE KUNZE-STEIN PHENOMENON

The convolution operators defined by (2.3) also have the Kunze-Stein phenomenon (see (7.29) in [4]), which states that for $1 \leq p < 2$,

$$(3.1) \quad L^p(\mathbb{R}^+, \mu) * L^2(\mathbb{R}^+, \mu) \subseteq L^2(\mathbb{R}^+, \mu).$$

This result can be strengthened with the help of Lorentz spaces. Let T be the set $\{(u, v, w) \in [1, \infty]^3 : 1 + 1/w \leq 1/u + 1/v\}$. Then we have

Proposition 3.1. If $1 < p < 2$ and (u, v, w) is in T , then

$$(3.2) \quad L^{p,u}(\mathbb{R}^+, \mu) * L^{p,v}(\mathbb{R}^+, \mu) \subseteq L^{p,w}(\mathbb{R}^+, \mu).$$

Proposition 3.1 follows from Theorem 3.2 below and a bilinear interpolation theorem ([2, Theorem 1.2]). It can also be proved as in [2].

The main purpose of this paper is to prove the following endpoint estimate.

Theorem 3.2. The convolution defined by (2.3) satisfies

$$L^{2,1}(\mathbb{R}^+, \mu) * L^{2,1}(\mathbb{R}^+, \mu) \subseteq L^{2,\infty}(\mathbb{R}^+, \mu).$$

We will need the following lemma (see Lemma 3 in [3]).

Lemma 3.3. *There exists a positive constant C such that for all characteristic functions f that are supported in $[1, \infty)$, we have*

$$\int_0^\infty f(t)e^{\rho t} dt \leq C\|f\|_2 = C\|f\|_{2,1}^*.$$

Proof of Theorem 3.2. In view of the general theory of Lorentz spaces, just as in [3], it suffices to prove that

$$(3.3) \quad \int_0^\infty \int_0^\infty \int_0^\infty f(s)g(t)h(u)K(s, t, u)\Delta(s)\Delta(t)\Delta(u)dsdtdu \leq C\|f\|_{2,1}^*\|g\|_{2,1}^*\|h\|_{2,1}^*$$

whenever f, g, h are characteristic functions of open sets with finite measure.

First we observe that we only need to prove (3.3) for functions f, g, h that are supported in $[1, \infty)$. In fact, we split $f = f_0 + f_1$, $g = g_0 + g_1$ and $h = h_0 + h_1$, where f_0, g_0 and h_0 are supported in $[0, 1]$ and f_1, g_1 and h_1 are supported in $[1, +\infty)$. Then the integral in (3.3) is divided into 8 integrals in the obvious way. By (2.5) and (2.6), any of the integrals containing one of the local parts, say f_0 , is bounded by

$$\|f_0\|_1\|g\|_2\|h\|_2 \leq C\|f_0\|_2\|g\|_2\|h\|_2 \leq C\|f\|_{2,1}^*\|g\|_{2,1}^*\|h\|_{2,1}^*.$$

Now we suppose that f, g, h are supported in $[1, \infty)$. If $\alpha \geq 1/2$, by Lemma 2.1 it is easy to see that for $s \geq 1, t \geq 1, u \geq 1$ and $|s - t| < u < s + t$ we have

$$K(s, t, u) \leq Ce^{-\rho(s+t+u)};$$

hence

$$(3.4) \quad K(s, t, u)\Delta(s)\Delta(t)\Delta(u) \leq Ce^{\rho(s+t+u)}.$$

By (3.4) and Lemma 3.3 we get (3.3) in the case of $\alpha \geq 1/2$.

If $-1/2 < \alpha < 1/2$, we only need to prove (3.3) in which the integral is taken over the domain $S = \{(s, t, u) : s \leq t \leq u \leq s + t, s, t, u \geq 1\}$, i.e.,

$$(3.5) \quad \iiint_S f(s)g(t)h(u)K(s, t, u)\Delta(s)\Delta(t)\Delta(u)dsdtdu \leq C\|f\|_{2,1}^*\|g\|_{2,1}^*\|h\|_{2,1}^*.$$

For $(s, t, u) \in S$, we have

$$\begin{aligned} & \frac{(t + s + u)(t + s - u)(u - s + t)(u + s - t)}{(1 + t + s + u)(1 + t + s - u)(1 + u - s + t)(1 + u + s - t)} \\ & \geq C \frac{t + s - u}{1 + t + s - u}, \end{aligned}$$

so

$$K(s, t, u)\Delta(s)\Delta(t)\Delta(u) \leq Ce^{\rho(s+t+u)} \left(\frac{t + s - u}{1 + t + s - u} \right)^{\alpha-1/2}.$$

Substitute the above estimate into (3.5), and break up the integral into two parts over $s + t - u \geq 1$ and over $s + t - u \leq 1$. We can treat the integral over $s + t - u \geq 1$ by using Lemma 3.3 in the same way as in the case $\alpha \geq 1/2$. Now it remains to estimate the integral

$$\iiint_{1 \leq s \leq t \leq u \leq s+t \leq u+1} f(s)g(t)h(u)e^{\rho(s+t+u)}|t + s - u|^{\alpha-1/2}dsdtdu.$$

To this end, make the change of variable $t = v - s$. Then by Hölder's inequality the absolute value of the integral in the variable s ,

$$\int_{1 \leq s \leq v-s \leq u} f(s) e^{\rho s} g(v-s) e^{\rho(v-s)} ds,$$

is bounded by $\|f\|_2 \|g\|_2$ uniformly in v . So applying Lemma 3.3, we see that the integral in (3.5) is bounded by

$$\|f\|_2 \|g\|_2 \int_1^\infty \left(\int_u^{u+1} (v-u)^{\alpha-1/2} dv \right) h(u) e^{\rho u} du,$$

and consequently by $C \|f\|_{2,1}^* \|g\|_{2,1}^* \|h\|_{2,1}^*$ since $\alpha - 1/2 > -1$. This completes the proof of the theorem. $\square$

Finally, I would like to thank the referee for his valuable comments and suggestions.

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