BLOWUP FOR $u_t = \Delta u + |\nabla u|^2 u$ FROM $\mathbb{R}^n$ INTO $\mathbb{R}^m$

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ABSTRACT. In this note we consider the global regularity of smooth solutions $u = (u^1, \dots, u^m)$ to the vector-valued Cauchy problem

$$u_t = \Delta u + |\nabla u|^2 u \quad \text{in } \mathbb{R}^n \times [0, \infty), \quad u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^n.$$

We show that if $n, m \geq 3$, the gradient-blowup phenomenon occurs in finite time for suitably chosen u_0 vanishing at infinity. We also present a simple example of the L^∞ -blowup solutions for $|u_0| \equiv 1 + \epsilon$ for any $\epsilon > 0$, if $m \geq 2$.

1. INTRODUCTION

In this note we consider the global (in time) regularity of smooth solutions $u = (u^1, \dots, u^m)$ to the Cauchy problem

$$(1.1) \quad \begin{cases} u_t = \Delta u + |\nabla u|^2 u & \text{in } \mathbb{R}^n \times [0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases}$$

where

$$|\nabla u|^2 = \sum_{i=1}^m |\nabla u^i|^2.$$

More precisely, we are concerned with the existence and nonexistence of global smooth solutions to (1.1) in the case when $m \geq 2$. Here, recall that for every $u_0 \in C_b^1(\mathbb{R}^n, \mathbb{R}^m)$, there exists a unique, local in time, smooth solution u for a sufficiently small $T > 0$ depending only on $\|\nabla u\|_{L^\infty}$, which can be proved via the contraction mapping principle argument (see e.g. [3, Proposition 15.3.1]).

In the scalar-valued case ($m = 1$), all solutions can be extended globally. Indeed, since the transformation

$$v = \int_0^u e^{s^2/2} ds$$

satisfies $v_t = \Delta v$, one can always obtain a unique global smooth solution u to the single equation for every $u_0 \in L^\infty(\mathbb{R}^n, \mathbb{R})$.

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On the other hand, the vector-valued problem is completely different from the single case. For instance, consider the harmonic map heat flow into the sphere $S^{m-1} \subset \mathbb{R}^m$:

$$(1.2) \quad \begin{cases} u_t = \Delta u + |\nabla u|^2 u & \text{in } \mathbb{R}^n \times [0, \infty), \\ u(x, 0) = u_0(x) \in S^{m-1}. \end{cases}$$

It is well known that the problem can admit the formation of finite-time singularities. Indeed, Coron and Ghidaglia [2] proved that if $m = n + 1$ and $n \geq 3$, there does occur the gradient-blowup phenomenon in finite time for some rotationally symmetric initial data. Furthermore, for the two-dimensional domain it has been shown by Chang, Ding and Ye [1] that the Dirichlet problem from D^2 into S^2 also develops finite-time singularities. Hence, we would like to know what occurs for smooth local solutions to (1.1) with the exception of such a geometric constraint.

Our main result in this note is as follows.

Theorem 1.1. *Let $m \geq 2$. There is a family of initial data u_0 with $|u_0| = 1 + \epsilon$ for any $\epsilon > 0$ such that the L^∞ -blowup solutions to (1.1) occur in finite time.*

If $n, m \geq 3$, then there exists a family of initial data with $\|u_0\|_{L^\infty} \leq 1$ and $\lim_{|x| \rightarrow \infty} u_0(x) = 0$ such that the gradient of the solution blows up in finite time.

2. PROOF OF THEOREM 1.1

Suppose that u_0 is given by

$$(2.1) \quad u_0(x) = \mu_0 \left(\cos \left(a_0 + \sum_{j=1}^n a_j x_j \right), \sin \left(a_0 + \sum_{j=1}^n a_j x_j \right), 0, \dots, 0 \right),$$

where μ_0 and $\{a_j\}_{0 \leq j \leq n}$ are constants. By uniqueness, the solution u to (1.1) reduces to two functions $f = f(x, t)$ and $\varphi = \varphi(x, t)$ satisfying

$$\begin{cases} f_t = \Delta f + |\nabla f|^2 f + f(f^2 - 1)|\nabla \varphi|^2 & \text{in } \mathbb{R}^n \times [0, \infty), \\ \varphi_t = \Delta \varphi + 2\langle \nabla f, \nabla \varphi \rangle & \text{in } \mathbb{R}^n \times [0, \infty), \\ f(x, 0) = f_0(x), \quad \varphi(x, 0) = \varphi_0(x) & \text{in } \mathbb{R}^n, \end{cases}$$

where $f_0(x) \equiv \mu_0$ and $\varphi_0(x) := a_0 + \sum_{j=1}^n a_j x_j$. Let $\mu = \mu(t)$ be a solution to the ODE:

$$\mu_t = (a_1^2 + \dots + a_n^2)\mu(\mu^2 - 1), \quad \mu(0) = \mu_0.$$

Then it is obvious that

$$u(x, t) = \mu(t) \left(\cos \left(a_0 + \sum_{j=1}^n a_j x_j \right), \sin \left(a_0 + \sum_{j=1}^n a_j x_j \right), 0, \dots, 0 \right)$$

is a solution to (1.1). Hence, if $|\mu_0| > 1$ and $a_1^2 + \dots + a_n^2 \neq 0$, the solution u blows up in finite time.

Next, we shall show the existence of the gradient-blowup solutions. Let k be an integer $\leq \min\{m, n\}$. Suppose that u_0 is a smooth data such that

$$(2.2) \quad u_0(x) = \frac{f_0(r)}{r} (x_1, \dots, x_k, 0, \dots, 0), \quad r = \sqrt{x_1^2 + \dots + x_k^2},$$

which must vanish at the origin for the sake of compatibility. By uniqueness, one deduces that

$$(2.3) \quad u(x, t) = \frac{f(r, t)}{r}(x_1, \dots, x_k, 0, \dots, 0).$$

Then it follows from straightforward calculations that f satisfies

$$(2.4) \quad \begin{cases} f_t = f_{rr} + \frac{k-1}{r}f_r - \frac{k-1}{r^2}f + \left(f_r^2 + \frac{k-1}{r^2}f^2\right)f =: \tau(f), \\ f(r, 0) = f_0(r), \quad f(0, t) = 0 \end{cases}$$

for $r, t \geq 0$.

Here, let us establish the following comparison principle.

Lemma 2.1. *Let*

$$f, \tilde{f} \in C^{2,1}((0, \infty) \times (0, T)) \cap C_b([0, \infty) \times [0, T]).$$

Suppose that f and $\tilde{f}$ satisfy

$$(2.5) \quad \begin{cases} e^{f^2/2}(f_t - \tau(f)) \geq e^{\tilde{f}^2/2}(\tilde{f}_t - \tau(\tilde{f})), \\ f(r, 0) \geq \tilde{f}(r, 0), \quad f(0, t) \geq \tilde{f}(0, t) \end{cases}$$

for $(r, t) \in [0, \infty) \times [0, T]$. Then $f \geq \tilde{f}$ on $[0, \infty) \times [0, T]$.

Proof. Define

$$G(y) = \int_0^y e^{s^2/2} ds$$

and its inverse function $\mathbf{g} = G^{-1}$. If we set

$$h = G(f) - G(\tilde{f}) = \int_{\tilde{f}}^f e^{s^2/2} ds,$$

then (2.5) corresponds to

$$h_t - h_{rr} - \frac{k-1}{r}h_r + \frac{k-1}{r^2}\psi h \geq 0, \quad h(r, 0) \geq 0, \quad h(0, t) \geq 0,$$

where

$$\psi(r, t) = 1 - \int_0^1 \left(2\mathbf{g}(\theta G(f) + (1-\theta)G(\tilde{f}))^2 + \mathbf{g}(\theta G(f) + (1-\theta)G(\tilde{f}))^4 \right) d\theta.$$

Then the classical maximum principle argument yields that $h \geq 0$, from which it follows that $f \geq \tilde{f}$. $\square$

Let $\lambda = \lambda(t)$ solve the ODE: $\lambda_t = -1/\lambda$, $\lambda(0) = \lambda_0 > 0$, i.e., $\lambda(t) = \sqrt{\lambda_0^2 - 2t}$. Let us prove that the following ansatz:

$$(2.6) \quad \tilde{f}(r, t) = \frac{2\mu_0\lambda r}{\lambda^2 + r^2} \quad (0 < \mu_0 \leq 1)$$

is a blowup subsolution to (2.4), provided that λ_0 is small enough and for some μ_0 . Notice that $\tilde{f}_r(0, t) \rightarrow \infty$ as $t \rightarrow T = \lambda_0^2/2$ and also that $\|\tilde{f}(\cdot, t)\|_{L^\infty} = \mu_0$. We have

$$\begin{aligned} \tau(\tilde{f}) &= 2\mu_0\lambda \left(\frac{2r(-3\lambda^2 + r^2)}{(\lambda^2 + r^2)^3} + \frac{(k-1)(\lambda^2 - r^2)}{r(\lambda^2 + r^2)^2} - \frac{k-1}{r(\lambda^2 + r^2)} \right. \\ &\quad \left. + \frac{4\mu_0^2\lambda^2 r}{\lambda^2 + r^2} \left(\frac{(\lambda^2 - r^2)^2}{(\lambda^2 + r^2)^4} + \frac{k-1}{(\lambda^2 + r^2)^2} \right) \right) \\ &= \frac{2\mu_0\lambda}{r(\lambda^2 + r^2)^5} \left((\lambda^2 + r^2)^2 (2r^2(-3\lambda^2 + r^2) + (k-1)(\lambda^4 - r^4)) \right. \\ &\quad \left. - (k-1)(\lambda^2 + r^2)^2 + 4\mu_0^2\lambda^2 r((\lambda^2 - r^2)^2 + (k-1)(\lambda^2 + r^2)^2) \right) \\ &= \frac{2\mu_0\lambda r}{(\lambda^2 + r^2)^5} \left(-2r^2(\lambda^2 + r^2)^2((k+2)\lambda^2 + (k-2)r^2) \right. \\ &\quad \left. + 4\mu_0^2\lambda^2(k\lambda^4 + 2(k-2)\lambda^2 r^2 + kr^4) \right) \\ &= \frac{2\mu_0\lambda r}{(\lambda^2 + r^2)^5} \left(-2r^2((k+2)\lambda^6 + (3k+2)\lambda^4 r^2 + (3k-2)\lambda^2 r^4 + (k-2)r^6) \right. \\ &\quad \left. + 4\mu_0^2\lambda^2(k\lambda^4 + 2(k-2)\lambda^2 r^2 + kr^4) \right) \\ &= \frac{2\mu_0\lambda r}{(\lambda^2 + r^2)^5} \left(4k\mu_0^2\lambda^6 + 2(4(k-2)\mu_0^2 - (k+2)\lambda^2)\lambda^4 r^2 \right. \\ &\quad \left. + (4k\mu_0^2 - 2(3k+2)\lambda^2)\lambda^2 r^4 - 2(3k-2)\lambda^2 r^6 - 2(k-2)r^8 \right). \end{aligned}$$

Since

$$\tilde{f}_t = \frac{2\mu_0 r(\lambda^2 - r^2)}{\lambda(\lambda^2 + r^2)^2} = \frac{2\mu_0 r(\lambda^8 + 2\lambda^6 r^2 - 2\lambda^2 r^6 - r^8)}{\lambda(\lambda^2 + r^2)^5},$$

we have

$$\begin{aligned} \tilde{f}_t - \tau(\tilde{f}) &= -\frac{2\mu_0 r}{\lambda(\lambda^2 + r^2)^5} \left((-1 + 4k\mu_0^2)\lambda^8 + 2(-1 + 4(k-2)\mu_0^2 - (k+2)\lambda^2)\lambda^6 r^2 \right. \\ &\quad \left. + 2(2k\mu_0^2 - (3k+2)\lambda^2)\lambda^4 r^4 + 2(1 - (3k-2)\lambda^2)\lambda^2 r^6 + (1 - 2(k-2)\lambda^2)r^8 \right). \end{aligned}$$

Taking $\lambda_0 \leq 1/\sqrt{3k-2}$, and if one chooses $\mu_0 \in (0, 1]$ such that

$$\mu_0^2 \geq \max \left\{ \frac{1}{4k}, \frac{(k+2)\lambda_0^2 + 1}{4(k-2)}, \frac{(3k+2)\lambda_0^2}{2k} \right\},$$

then $\tilde{f}$ is a blowup subsolution to (2.4). On the other hand, since the constant one is a global supersolution, if $\tilde{f}_0 \leq f_0 \leq 1$, then by Lemma 2.1 the solution with $f = f_0$ satisfies $f \leq \tilde{f} \leq 1$. Therefore, the derivative f_r must blow up at $r = 0$ in finite time. Hence, this completes the proof of Theorem 1.1.

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