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AN EXTREMAL FUNCTION FOR THE CHANG-MARSHALL INEQUALITY OVER THE BEURLING FUNCTIONS

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Abstract. S.-Y. A. Chang and D. E. Marshall showed that the functional $\Lambda(f) = (1/2\pi)\int_0^{2\pi} \exp\{|f(e^{i\theta})|^2\}d\theta$ is bounded on the unit ball $\mathcal{B}$ of the space $\mathcal{D}$ of analytic functions in the unit disk with $f(0) = 0$ and Dirichlet integral not exceeding one. Andreev and Matheson conjectured that the identity function $f(z) = z$ is a global maximum on $\mathcal{B}$ for the functional Λ. We prove that Λ attains its maximum at $f(z) = z$ over a subset of $\mathcal{B}$ determined by kernel functions, which provides a positive answer to a conjecture of Cima and Matheson.

1. Introduction

Let $\mathcal{D}$ be the Dirichlet space of functions f analytic on the unit disk $\mathbb{D}$, with $f(0) = 0$ and a finite Dirichlet integral

$$\|f\|_{\mathcal{D}}^2 = \frac{1}{\pi}\int\int_{\mathbb{D}} |f'(z)|^2 dxdy.$$

It is well known that $\mathcal{D}$ is a Hilbert space with inner product

$$\langle f, g\rangle_{\mathcal{D}} = \frac{1}{\pi}\int\int_{\mathbb{D}} f'(z)\overline{g'(z)}dxdy.$$

Let $\mathcal{B} = \{f \in \mathcal{D} : \|f\|_{\mathcal{D}} \leq 1\}$ be its closed unit ball.

We shall be concerned with functionals Λ_Φ on $\mathcal{B}$ defined by

$$\Lambda_\Phi(f) = \frac{1}{\pi}\int_0^{2\pi} \Phi(|f(e^{i\theta})|)d\theta,$$

for $f \in \mathcal{B}$ and $\Phi : (-\infty, \infty) \to \mathbb{R}$ being a continuous convex nondecreasing function. A function f is a **maximum** for Λ_Φ if $f \in \mathcal{B}$ and $\Lambda_\Phi(f) \geq \Lambda_\Phi(g)$ for all $g \in \mathcal{B}$.

Chang and Marshall [3] proved that if $\Phi_\alpha(t) = e^{\alpha t^2}$ for $\alpha > 0$, then Λ_{Φ_α} is bounded on $\mathcal{B}$ if and only if $\alpha \leq 1$. In their proof they compared functions in $\mathcal{B}$ to the *Beurling functions*

$$B_a(z) = \frac{\log \frac{1}{1-\overline{a}z}}{\sqrt{\log \frac{1}{1-|a|^2}}},$$

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for $a \in \mathbb{D} \setminus \{0\}$, where the branch of the logarithm is chosen so that $B_a(a)$ is real. The denominator assures that $\|B_a\|_{\mathcal{D}} = 1$. Up to a normalizing factor, the B_a are the kernel functions for $\mathcal{D}$. We shall denote by $\mathcal{B}_0$ the set of all Beurling functions and by $\widetilde{\mathcal{B}}_0$ its closed convex hull.

A shorter proof of this fact has since been found by Marshall [9]. A significantly more general and stronger inequality has been found by Essén [7]. Andreev and Matheson [1] showed that the identity function $f(z) = z$ is a local maximum for Λ_{Φ_1} on $\mathcal{B}$ and conjectured that it is also a global maximum. Cima and Matheson [4] showed that the identity function is a local maximum on the set $\mathcal{B}_0$ and that the functional Λ_{Φ_1} attains its maximum on $\widetilde{\mathcal{B}}_0$. On the other hand, they showed that Λ_{Φ_1}, when restricted to $\mathcal{B}$, is not weakly continuous at 0, and thus it is an open question whether there exists a global maximum for Λ_{Φ_1} on $\mathcal{B}$. Matheson and Pruss [10] studied the regularity of the extremal functions. We refer the reader to their paper for an excellent discussion of this and other related problems and for a list of open problems.

Our principle result is:

Theorem 1.1. *The inequality*

$$\Lambda_{\Phi_1}(f) < \Lambda_{\Phi_1}(z) \tag{1.1}$$

holds true for all $f \in \widetilde{\mathcal{B}}_0$.

Our result proves Conjecture 1 of Cima and Matheson in [4].

2. Proof of Theorem 1.1

It is natural to set $B_0(z) = z$ (see [4]). A function $\Phi(x)$ continuous on $-\infty < x < \infty$ is said to be *convex* if $\Phi((x+y)/2) \le [\Phi(x) + \Phi(y)]/2$, and *strictly convex* if strict inequality holds whenever $x \ne y$. Theorem 1.1 is a consequence of the following result.

Theorem 2.1. *Let* $\Phi(x)$ *be a convex nondecreasing function on* $-\infty < x < \infty$. *For all* $a_0, a \in \mathbb{D} \setminus \{0\}$ *such that* $0 \le |a_0| < |a| < 1$, *we have*

$$\int_0^{2\pi} \Phi(\log |B_a(re^{i\theta})|) d\theta \le \int_0^{2\pi} \Phi(\log |B_{a_0}(re^{i\theta})|) d\theta, \tag{2.1}$$

$0 < r < 1$. *If* Φ *is strictly convex, then the inequality is strict for all* r.

Proof. Our proof is based on the deep results of Albert Baernstein [2, Theorem 1] on integral means of univalent functions (see also Chapter 7 of Duren's book [5]). In particular, we need the following proposition [2, Proposition 3].

Proposition 2.2. *For* $g, h \in L^1(-\pi, \pi)$, *the following statements are equivalent.*

(a) For each function $\Phi(s)$ *convex and nondecreasing on* $-\infty < s < \infty$,

$$\int_{-\pi}^{\pi} \Phi(g(x)) dx \le \int_{-\pi}^{\pi} \Phi(h(x)) dx.$$

(b) For each $t \in \mathbb{R}$,

$$\int_{-\pi}^{\pi} [g(x) - t]^+ dx \le \int_{-\pi}^{\pi} [h(x) - t]^+ dx.$$

(c) $g^*(\theta) \le h^*(\theta)$, $0 \le \theta \le \pi$.

Here for each $r \in (r_1, r_2)$ and $u(re^{i\theta}) \in L^1(0, 2\pi)$ the *Baernstein star-function* of u is defined as

$$u^*(re^{i\theta}) = \sup_{|E|=2\theta} \int_E u(re^{it})dt, \tag{2.2}$$

$0 \le \theta \le \pi$, where $|E|$ denotes the Lebesgue measure of the set $E \subset [-\pi, \pi]$.

In view of Proposition 2.2, we want first to show that

$$\int_{-\pi}^{\pi} \log^+ \left[\frac{|B_a(re^{i\theta})|}{\rho}\right] d\theta \le \int_{-\pi}^{\pi} \log^+ \left[\frac{B_{a_0}(re^{i\theta})}{\rho}\right] d\theta, \tag{2.3}$$

$0 < r < 1$, for each $\rho > 0$ and all a and a_0 such that $0 \le |a_0| < |a| < 1$. Notice that

$$\int_{-\pi}^{\pi} \log^+ \left[\frac{|B_{a'}(re^{i\theta})|}{\rho}\right] d\theta = \int_{-\pi}^{\pi} \log^+ \left[\frac{|B_{a''}(re^{i\theta})|}{\rho}\right] d\theta$$

whenever $|a'| = |a''|$. Hence we may assume from now on that $0 \le a_0 < a < 0$.

We can apply Jensen's theorem to obtain

$$\int_{-\pi}^{\pi} \log^+ \left[\frac{|B_a(re^{i\theta})|}{\rho}\right] d\theta = \int_{-\pi}^{\pi} N(r, \rho e^{i\phi}) d\phi, \tag{2.4}$$

since $B_a(0) = 0$. It is easy to see that B_a is a univalent function in the unit disk $\mathbb{D}$, $B_a(0) = 0$ and $B_a'(0) = a/A$, where $A = \{\log[1/(1-|a|^2)]\}^{1/2}$, for each $0 < a < 1$, with a continuous extension to the closed unit disk $\overline{\mathbb{D}}$, and if $\alpha = \rho e^{i\phi} \neq 0$ is in the range D_a of B_a, then

$$N(r, \alpha) = \int_0^r \frac{n(t, \alpha)}{t} dt = \log^+ \left[\frac{r}{|\alpha|}\right] = \log^+ \left[\frac{r}{|B_a^{-1}(\alpha)|}\right], \tag{2.5}$$

$0 < r < 1$. Let $u_a(\zeta) = -\log|B_a^{-1}(\zeta)|$ be the Green's function of D_a with pole at 0. Extend it to a continuous function in the punctured plane by setting $u_a(\zeta) = 0$, $\zeta \notin D_a$. The formula (2.5) takes the form

$$N(r, \zeta) = [u_a(\zeta) + \log r]^+,$$

$0 < r < 1$, for arbitrary ζ, and equation (2.4) becomes

$$\int_{-\pi}^{\pi} \log^+ \left[\frac{|B_a(re^{i\theta})|}{\rho}\right] d\theta = \int_{-\pi}^{\pi} [u_a(\rho e^{i\phi}) + \log r]^+ d\phi. \tag{2.6}$$

Let $u_{a_0}(\zeta) = -\log|B_{a_0}^{-1}(\zeta)|$ for $\zeta \in D_{a_0}$, and let $u_{a_0}(\zeta) = 0$ elsewhere. In view of (2.6) the inequality (2.3) can be recast in the form

$$\int_{-\pi}^{\pi} [u_a(\rho e^{i\phi}) + \log r]^+ d\phi \le \int_{-\pi}^{\pi} [u_{a_0}(\rho e^{i\phi}) + \log r]^+ d\phi,$$

$0 < r < 1$, $0 < \rho < \infty$. By Proposition 2.2, this is implied by the inequality

$$u_a^*(\rho e^{i\phi}) \le u_{a_0}^*(\rho e^{i\phi}), \tag{2.7}$$

$0 < \rho < \infty$, $0 \le \phi \le \pi$.

The function $u(\zeta)$ is continuous in $0 < |\zeta| < \infty$, it is positive and harmonic in D_a, and identically zero outside D_a. Thus it is subharmonic in $0 < |\zeta| < \infty$. Hence by [2, Theorem A] and the definition (2.2) of the star-function, u_a^* is subharmonic in the open upper half-plane and continuous in the closed upper half-plane, except at the origin.

Since $B_a^{-1}(\zeta) = (1 - e^{-A\zeta})/a$, then, near the origin, u_a has the form

$$u_a(\zeta) = -\log|\zeta| - \log\frac{A}{a} + u_{1a}(\zeta), \tag{2.8}$$

where u_{1a} is harmonic and $u_{1a}(0) = 0$. Thus

$$u_a^*(\rho e^{i\phi}) + 2\phi\log\rho \to -2\phi\log\frac{A}{a}$$

as $\rho \to 0$ for $0 \le \phi \le \pi$. Similarly, near the origin, u_{a_0} has the form

$$u_{a_0}(\zeta) = -\log|\zeta| - \log\frac{A_0}{a_0} + u_{1a_0}(\zeta),$$

where u_{1a_0} is harmonic and $u_{1a_0}(0) = 0$. Thus

$$u_{a_0}^*(\rho e^{i\phi}) + 2\phi\log\rho \to -2\phi\log\frac{A_0}{a_0}$$

as $\rho \to 0$ for $0 \le \phi \le \pi$. It follows that

$$[u_a^*(\rho e^{i\phi}) - u_{a_0}^*(\rho e^{i\phi})] \to -2\phi\log\frac{a_0A}{aA_0}$$

as $\rho \to 0$ for $0 \le \phi \le \pi$. It is easy to see that $a_0A/(aA_0) > 1$ for $a_0 < a$ and hence that $-2\pi\log\frac{a_0A}{aA_0} \le -2\phi\log\frac{a_0A}{aA_0} \le 0$ for $a_0 < a$.

Hence $(u_a^* - u_{a_0}^*)$ is subharmonic in the upper half-plane and continuous in its closure except at the origin, where it has a bounded discontinuity: for $\phi = 0$,

$$\lim_{\rho\to 0}(u_a^*(\rho) - u_{a_0}^*(\rho)) = 0,$$

and for $\phi = \pi$,

$$\lim_{\rho\to 0}(u_a^*(-\rho) - u_{a_0}^*(-\rho)) = -2\pi\log\frac{a_0A}{aA_0}.$$

We want to show that $(u_a^* - u_{a_0}^*) < 0$ in the open upper half-plane. Since $u_a^* - u_{a_0}^*$ is discontinuous at the origin, we cannot apply the maximum principle for subharmonic functions to $u_a^* - u_{a_0}^*$ at this point. The proof of the inequality $(u_a^* - u_{a_0}^*) < 0$ for $\Im\zeta > 0$ will be based on the following four steps (a)–(d).

(a) On the positive real axis, by definition, $u_a^*(\zeta) = v^*(\zeta) = 0$ for $\zeta > 0$.

(b) Next let d_a be the distance from 0 to the complement of D_a. It is obvious that $\Re(1 - ae^{i\theta})^{-1} > 0$. Since the branch of the logarithm was chosen so that $B_a(a)$ is real, then

$$|B_a(e^{i\theta})| = \frac{1}{A}\{[\log\frac{1}{|1 - ae^{i\theta}|}]^2 + [\arg\frac{1}{1 - ae^{i\theta}}]^2\}^{1/2}.$$

Since $\max|1 - ae^{i\theta}| = |1 - ae^{i\pi}| = 1 + a$ and $[\arg\frac{1}{1 - ae^{i\pi}}]^2 = 0$, it is easy to see that

$$-\frac{1}{A}\log\frac{1}{1 + a} \le |B_a(e^{i\theta})| \le \frac{1}{A}\log\frac{1}{1 - a}$$

for $0 < a < 1$. Thus $d_a = -\frac{1}{A}\log\frac{1}{1+a}$. We want to show that d_a is a decreasing function of a for $0 < a < 1$. It is clear that $d_a \to 1$ as $a \to 0$. Let

$$f(a) = \frac{\log(1 + a)}{A}.$$

Then

$$f'(a) = -\frac{[(1 - a)\log(1 - a) + \log(1 + a)]}{(1 - a^2)A^3}.$$

Let

$$f_1(a) = (1-a)\log(1-a) + \log(1+a).$$

An easy computation shows that $f_1''(a) > 0$ for $0 < a < 1$. Thus f_1' is an increasing function of a, and it follows that $f_1'(a) > 0$ for $0 < a < 1$ since $f_1'(0) = 0$. Therefore f_1 is an increasing function of a for $0 < a < 1$ and $f_1(a) > 0$ since $f_1(0) = 0$. Finally, this implies that $f'(a) < 0$ for $0 < a < 1$, and thus f is a decreasing function of a. Therefore $d_{a_0} > d_a$ for all a, $a_0 < a < 1$.

In the disk $|\zeta| < d_a$, $u_a(\zeta)$ has the form (2.8), where u_{1a} is harmonic in $|\zeta| < d_a$ and $u_{1a}(0) = 0$. Thus

$$u_a^*(\rho e^{i\pi}) = -2\pi \log\frac{1}{\rho} - 2\pi\log\frac{A}{a}$$

and, similarly,

$$u_{a_0}^*(\rho e^{i\pi}) = -2\pi \log\frac{1}{\rho} - 2\pi\log\frac{A_0}{a_0}$$

for $0 < \rho < d_a$. Hence $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ for $-d_a \le \zeta < 0$.

(c) Since $u_{1a}(\zeta)$ and $u_{1a_0}(\zeta)$ are harmonic in $|\zeta| < d_a$ and $u_{1a}(0) = u_{1a_0}(0) = 0$, then for every $\epsilon > 0$ there is a ρ_0, $\rho_0 = |\zeta_0| < d_a$, such that $|u_{1a}(\zeta)| < \epsilon/2$ and $|u_{1a_0}(\zeta)| < \epsilon/2$ for all ζ, $|\zeta| \le \rho_0$. Thus

$$\begin{aligned} u_a^*(\rho e^{i\phi}) &= \sup_{|E|=2\phi}\int_E u_a(\rho e^{it})dt \\ &= -2\phi\log\rho - 2\phi\log\frac{A}{a} + \sup_{|E|=2\phi}\int_E u_{1a}(\rho e^{it})dt \\ &\le -2\phi\log\rho - 2\phi\log\frac{A}{a} + \phi\epsilon \end{aligned}$$

and

$$\begin{aligned} u_{a_0}^*(\rho e^{i\phi}) &= \sup_{|E|=2\phi}\int_E u_{a_0}(\rho e^{it})dt \\ &= -2\phi\log\rho - 2\phi\log\frac{A_0}{a_0} + \sup_{|E|=2\phi}\int_E u_{1a_0}(\rho e^{it})dt \\ &\ge -2\phi\log\rho - 2\phi\log\frac{A_0}{a_0} - \phi\epsilon \end{aligned}$$

for $0 < \rho \le \rho_0$ and $0 < \phi < \pi$. Now choose ϵ such that $\epsilon < \log(Aa_0/aA_0)$. Then

$$u_a^*(\rho e^{i\phi}) - u_{a_0}^*(\rho e^{i\phi}) \le -2\phi\log\frac{Aa_0}{aA_0} + 2\phi\epsilon < 0$$

for all $0 < \rho \le \rho_0$ and $0 < \phi < \pi$. Hence $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ for $|\zeta| \le \rho_0 < d_a$ and $0 < \phi < \pi$.

(d) To establish the inequality on $-\infty < \zeta < -d_a$, we fix $\epsilon > 0$ and consider the function

$$Q(\zeta) = u_a^*(\zeta) - u_{a_0}^*(\zeta) - \epsilon\phi,$$

$\zeta = \rho e^{i\phi}$, which is subharmonic in $\mathcal{A} = \{\zeta : \rho_0 < |\zeta|, 0 \le \Im\zeta\}$ and continuous in the closure of $\mathcal{A}$. Let M be the maximum of $Q(\zeta)$ in $\overline{\mathcal{A}}$. Then $M \ge 0$ and, according to the maximum principle for subharmonic functions, the maximum is attained somewhere on the boundary of $\mathcal{A}$. Suppose $M > 0$. Since $u_a^*(\zeta) \le u_{a_0}^*(\zeta)$ on the set $\{\zeta : -d_a \le \zeta \le \rho_0\} \cup \{\zeta : |\zeta| = \rho_0, \Im\zeta > 0\} \cup \{\zeta : \rho_0 \le \zeta < \infty\}$, there is some point $-\zeta_1 = -\rho_1$ for which $-\infty < \zeta_1 < -d_a$ and $Q(\zeta_1) = M$. Let $G_a(\phi)$ denote the symmetric decreasing rearrangement of $u_a(\rho_1 e^{i\phi})$. Then

$$\frac{\partial u_a^*}{\partial \phi}(\rho_1 e^{i\phi}) = 2G_a(\phi)$$

for $0 \le \phi \le \pi$ by [2, Proposition 2]. But because $\rho_1 > d_a$, there is some point on the circle $|\zeta| = \rho_1$ that lies outside D_a, so

$$G_a(\pi) = \inf_{0\le\phi\le\pi} u_a(\rho_1 e^{i\phi}) = 0.$$

Applying the same argument to $u_{a_0}^*$ we obtain

$$\frac{\partial u_{a_0}^*}{\partial \phi}(\rho_1 e^{i\phi}) = 2G_{a_0}(\phi)$$

for $0 \le \phi \le \pi$. If $d_a < \rho_1 \le d_{a_0}$, then

$$G_{a_0}(\phi) = \inf_{0\le\phi<\pi} \{t : \lambda(t) \le 2\phi\},$$

where λ is the distribution function of u_{a_0}, $\lambda(t) = |\{\phi : u_{a_0}(\rho_0 e^{i\phi}) > t\}|$, and

$$G_{a_0}(\pi) = \lim_{\phi\to\pi^-} G_{a_0}(\phi).$$

Hence $G_{a_0}(\pi) \ge 0$ if $d_a < \rho_1 \le d_{a_0}$. If $d_{a_0} < \rho_1$, there is some point on the circle $|\zeta| = \rho_1$ that lies outside D_{a_0}, so

$$G_{a_0}(\pi) = \inf_{0\le\phi\le\pi} u_{a_0}(\rho_1 e^{i\phi}) = 0.$$

Therefore

$$\frac{\partial Q}{\partial \phi}(\zeta_1) \le -\epsilon < 0,$$

which contradicts the assumption that $Q(\zeta)$ has a relative maximum at ζ_1. Hence $M = 0$ and

$$u_a^*(\zeta) \le u_{a_0}^*(\zeta) + \epsilon\phi \le u_{a_0}^*(\zeta) + \epsilon\pi$$

for $\zeta \in \overline{\mathcal{A}}$. Letting $\epsilon \to 0$ we obtain that

$$u_a^*(\rho e^{i\phi}) \le u_{a_0}^*(\rho e^{i\phi})$$

for $\zeta \in \overline{\mathcal{A}}$.

We are in a position now to prove that $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ in the open upper half-plane. Combining (a)–(d) we obtain (2.7). Furthermore, $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ on the set $\{\zeta : -d_a \le \zeta \le \rho_0\} \cup \{\zeta : |\zeta| = \rho_0, \Im\zeta > 0\}$ by (b) and (c). Hence $u_a^* - u_{a0}^*$ is a subharmonic function on $\mathcal{A}$ that is not identically equal to zero there and, by the maximum principle, this implies that $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ everywhere in $\mathcal{A}$. Also, $u_a^*(\zeta) < u_{a_0}^*(\zeta)$ for $\{\zeta : 0 < |\zeta| \le \rho_0 < d_a, 0 < \Im\zeta\}$ by (c). Therefore,

$$u_a^*(\zeta) < u_{a_0}^*(\zeta)$$

in the open upper half-plane.

It follows from Proposition 2.2 that

$$\int_0^{2\pi} \Phi(\log|B_a(re^{i\theta})|)d\theta \le \int_0^{2\pi} \Phi(\log|B_{a_0}(re^{i\theta)}|)d\theta \tag{2.9}$$

for all $0 \le a_0 < a < 0$ and $0 < r < 1$. The proof of strict inequality in (2.9) is identical to the proof of strict inequality in Theorem 1 in [2, pp. 157-158] and will be omitted. This completes the proof of Theorem 2.1. $\square$

Proof of Theorem 1.1. The choice $\Phi(x) = e^{e^{2x}}$ in (2.1) allows us to conclude that

$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) < \Lambda_{\Phi_1}(B_{a_0}(re^{i\theta}))$$

for all $0 \leq a_0 < a < 0$ and $0 < r < 1$. Let

$$\|B_a(re^{i\theta})\|_p^p = \frac{1}{2\pi}\int_0^{2\pi} |B_a(re^{i\theta})|^p d\theta.$$

Since

$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) = 1 + \sum_{n=1}^{\infty} \frac{\|B_a(re^{i\theta})\|_{2n}^{2n}}{n!},$$

and, by Lemma 1 of [1], $B_a \in H^p$ for $0 < p < \infty$, we can choose a sequence $r_n \to 1$ as $n \to \infty$ for which the inequalities $\Lambda_{\Phi_1}(B_a(r_n e^{i\theta})) < \Lambda_{\Phi_1}(B_{a_0}(r_n e^{i\theta}))$ hold. Hence

$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) \leq \Lambda_{\Phi_1}(B_{a_0}(re^{i\theta}))$$

for all $0 < r \leq 1$ by Hardy's convexity theorem for integral means (see, e.g., [6, Theorem 1.5]).

It now remains to demonstrate that strict inequality holds true in Theorem 1.1. According to Theorem 2 of [4], B_0 is a local maximum on the set of Beurling functions. Thus there is an a_0, $0 < a_0$, such that

$$\Lambda_{\Phi_1}(B_a(e^{i\theta})) < \Lambda_{\Phi_1}(B_0(e^{i\theta}))$$

for $0 < a \leq a_0$. (James and Matheson [8] have informed the author that, using a numerical method, they have proved the last inequality for $0 < a < 1/2$.)

Finally, combine the last inequality with the fact that Λ_{Φ_1} is log-convex [4, p. 387] to complete the proof of Theorem 1.1. □

It was pointed out in [1] that B_0 does not maximize the integral means over $\mathcal{B}$. If we choose $\Phi(x) = e^{px}$, $0 < p < \infty$, in Theorem 2.1, we obtain that B_0 maximizes the integral means over $\mathcal{B}_0$.

Corollary 2.3. *The inequality*

$$\frac{1}{2\pi}\int_0^{2\pi} |B_a(re^{i\theta})|^p d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |B_{a_0}(re^{i\theta})|^p d\theta$$

holds true for all $0 \leq |a_0| < |a| < 0$, $0 < r \leq 1$, *and all* $0 < p < \infty$.

It will be interesting to see if the approach in Theorem 2.1 can be extended to the univalent functions in $\mathcal{D}$. The result of this paper provides further evidence in favor of a conjecture made in [1]:

Conjecture 1. Λ_{Φ_1} attains its maximum on $\mathcal{B}$ at B_0.

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