THE LINEAR SPACE OF GENERALIZED BROWNIAN MOTIONS WITH APPLICATIONS

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(Communicated by Richard C. Bradley)

ABSTRACT. In this paper, we define, motivated by recent works of Chang and Skoug, stochastic integrals for a generalized Brownian motion (gBm) X and then use it to study the representation problem on the linear space $H(X)$ spanned by X . We next establish a translation theorem for L^p -functionals of X , $p \geq 1$, and then use this translation to establish an integration by parts formula for L^p -functionals of X .

1. INTRODUCTION

Let $(C_0[0, T], \mathcal{B}(C_0[0, T]), m_w)$ be the Wiener measure space. Let $a \in C_0[0, T]$ be of bounded variation on $[0, T]$ and let $b \in C_0[0, T]$ be strictly increasing and of bounded variation on $[0, T]$. Then by Theorem 14.2 ([12], p. 187) there exists a Gaussian measure μ on $(C_0[0, T], \mathcal{B}(C_0[0, T]))$ such that the coordinate process defined as $X(t, x) = x(t)$ is a continuous additive process on $(C_0[0, T], \mathcal{B}(C_0[0, T]), \mu)$ and $[0, T]$ on which the probability distribution of $X(t_2, \cdot) - X(t_1, \cdot)$, $t_1 < t_2$ is normally distributed with mean $a(t_2) - a(t_1)$ and variance $b(t_2) - b(t_1)$.

Such a process $X = (X_t, t \in [0, T])$ will be referred to as *the generalized Brownian motion* (gBm) determined by the mean function $a(t)$ and the variance function $b(t)$. We will write the space $(C_0[0, T], \mathcal{B}(C_0[0, T]), \mu)$ as $(C_{a,b}[0, T], \mathcal{B}(C_{a,b}[0, T]), \mu)$. We note that the Wiener process $W(t, x) = x(t)$ on $(C_0[0, T], \mathcal{B}(C_0[0, T]), m_w)$ and $[0, T]$ is free of drift and is stationary in time, while the process $X = (X_t, t \in [0, T])$ on $(C_{a,b}[0, T], \mathcal{B}(C_{a,b}[0, T]), \mu)$ is subject to the drift $a(t)$ and is nonstationary in time, and can be used to describe the Black-Scholes model with time-dependent drift and diffusion coefficient in the financial market, as well as the model of interest rates and bond prices. For more details about the Black-Scholes model with time-dependent parameters, see e.g. ([9], p. 80, p. 87). For the model of bond prices and interest rates, see e.g. ([9], p. 125, p. 127).

Let $X = (X_t, t \in [0, T])$ be a gBm determined by $a(t)$ and $b(t)$. Then we note that X is an L^2 -process and $\sigma(X) = \mathcal{B}(C_{a,b}[0, T])$. There are two Hilbert spaces associated to the gBm X . The one is the *nonlinear space* of X , $L^2(X) = L^2(C_{a,b}[0, T], \sigma(X), \mu)$ (consisting of all $\sigma(X)$ -measurable random variables with

Received by the editors January 20, 2004 and, in revised form, March 19, 2004.

2000 *Mathematics Subject Classification.* Primary 60J65, 28C20.

Key words and phrases. Generalized Brownian motion, translation theorem, directional derivative, integration by parts formula.

This work was supported by Korea Research Foundation Grant (KRF-2003-015-C00065).

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finite second moment). Elements of $L^2(X)$ are called (*nonlinear*) L^2 -functionals of X . The second Hilbert space is the *linear space* $H(X)$ of X , which is the closed subspace of $L^2(X)$ spanned by X . Elements of $H(X)$ are called *linear* L^2 -functionals of X .

Let $\tilde{X} = (\tilde{X}_t \equiv X_t - a(t), t \in [0, T])$. Let $L^2(\tilde{X})$ ($H(\tilde{X})$, resp.) denote the *nonlinear* (*linear*, resp.) space of $\tilde{X}$.

The authors in [4], [5] used a gBm to define a generalized analytic Feynman integral and a generalized analytic Fourier-Feynman transform and then studied the first variations and the integration by parts formula involving these integrals and transforms. Motivated by these recent works ([4], [5]), in this paper we introduce a stochastic integral for the gBm X for which the space of the integrands is wider than the one for the stochastic integrals defined in [4], and then use it to study the representation problem on the linear space $H(X)$. We next establish a translation theorem for L^p -functionals, $p \geq 1$, which is proved by using a different method from the one given in [3], [5] and then use this translation theorem to establish an integration by parts formula for L^p -functionals of X with $p \geq 1$.

The organization of this paper is as follows. In Section 2, we introduce the function Hilbert space $L_{a,b}^{1,2}[0, T]$, which is our choice of function Hilbert spaces to represent the linear space $H(X)$. In Section 3, we establish a translation theorem for L^p -functionals, $p \geq 1$. In Section 4, we use the translation theorem obtained in Section 3 to establish an integration by parts formula for the functionals in $L^p(X)$, $p \geq 1$.

2. STOCHASTIC INTEGRATION FOR gBM

In this section we introduce the function Hilbert space $L_{a,b}^{1,2}[0, T]$, which will be our choice of function Hilbert spaces for the linear representation problem on the gBm X .

We will assume throughout this paper that $a(t)$ is an absolutely continuous function on $[0, T]$ with $a(0) = 0$ and $a'(t) \in L^2[0, T]$, and $b(t)$ is a differentiable function with $b(0) = 0$ and $b_1 \leq b'(t) \leq b_2$ ($b_1, b_2 > 0$) for all $t \in [0, T]$. This assumption is slightly milder than the one given in [4].

Let $L_b^2[0, T]$ be the Hilbert space of functions on $[0, T]$ given by

$$L_b^2[0, T] = \{f \in L^2[0, T] \mid \int_0^T f^2(s)db(s) < \infty\}$$

equipped with an inner product

$$\langle f, g \rangle_b = \int_0^T f(s)g(s)db(s).$$

Let $L_{a,b}^{1,2}[0, T]$ be the Hilbert space of functions on $[0, T]$ defined by

$$L_{a,b}^{1,2}[0, T] = \{f \in L_b^2[0, T] \mid \int_0^T |f(s)|d|a|(s) < \infty\}$$

equipped with an inner product

$$\langle f, g \rangle_{a,b} = \int_0^T f(s)g(s)db(s) + \int_0^T f(s)da(s) \int_0^T g(s)da(s)$$

where $|a|(t) = \int_0^t |da|$, $t \in [0, T]$. It is easy to see that $\|f\|_{a,b} = 0$ if and only if $f = 0$ a.e. on $[0, T]$ and that the three norms $\|\cdot\|$, $\|\cdot\|_b$ and $\|\cdot\|_{a,b}$ are equivalent. Hence $L^2[0, T]$, $L_b^2[0, T]$ and $L_{a,b}^{1,2}[0, T]$ coincide as sets, and furthermore, $L_{a,b}^{1,2}[0, T]$ is a separable Hilbert space.

Let $\mathcal{S}$ be the set of all step functions on $[0, T]$, $f = \sum_{j=0}^{n-1} c_j 1_{(t_j, t_{j+1}]}$, where $0 = t_0 < t_1 < t_2 < \dots < t_n = T$ and $c_j \in \mathbb{R}$. We define $\theta : \mathcal{S} \rightarrow H(X)$ by

$$\theta(f) = \sum_{j=0}^{n-1} c_j (X_{t_{j+1}} - X_{t_j}) = \sum_{j=0}^{n-1} c_j [(\tilde{X}_{t_{j+1}} - \tilde{X}_{t_j}) + (a(t_{j+1}) - a(t_j))].$$

Clearly $\mathcal{S}$ is a linear dense subspace of $L_{a,b}^{1,2}[0, T]$ and for all $f, g \in \mathcal{S}$, we have the following identity:

$$E[\theta(f)] = \int_0^T f da \quad \text{and} \quad E[\theta(f)\theta(g)] = \langle f, g \rangle_{a,b},$$

where $\int_0^T f da$ denotes the Stieltjes integral of f with respect to the absolutely continuous function a . Thus the correspondence $f \mapsto \theta(f)$ is an inner product preserving mapping from $\mathcal{S}$ to $H(X)$. Hence θ is uniquely extended to $L_{a,b}^{1,2}[0, T]$. For $f \in L_{a,b}^{1,2}[0, T]$, we call $\theta(f)$ the *stochastic integral* of f against X and write it as

$$(2.1) \quad \theta(f) = \int_0^T f dX \quad \text{or} \quad \int_0^T f d\tilde{X} + \int_0^T f da.$$

Definition 2.1. The *stochastic integral* $\int_0^T f dX_t$ for $f \in L_{a,b}^{1,2}[0, T]$ is defined by

$$\int_0^T f dX_t = \int_0^T f d\tilde{X}_t + \int_0^T f da.$$

Proposition 2.2. Let X be a gBm determined by $a(t)$ and $b(t)$. Then

$$L_{a,b}^{1,2}[0, T] \cong H(X).$$

Proof. Let $\theta : \mathcal{S} \rightarrow H(X)$ be the mapping defined as above. Then $\theta(1_{(0,t]}) = \int_0^T 1_{(0,t]}(u) dX(u) = X_t$ for any $t \in [0, T]$. But the set $\{X_t | t \in [0, T]\}$ generates $H(X)$. Hence the mapping θ is uniquely extended to a unitary isomorphism from $L_{a,b}^{1,2}[0, T]$ onto $H(X)$. This completes the proof. $\square$

Theorem 2.3. Let X be a gBm determined by $a(t)$ and $b(t)$. For $f, g \in L_{a,b}^{1,2}[0, T]$ and $\alpha, \beta \in \mathbb{R}$, the stochastic integral θ satisfies the following:

- (1) $\theta(\alpha f + \beta g) = \alpha \theta(f) + \beta \theta(g)$.
- (2) $E[\theta(f)] = \int_0^T f da$.
- (3) $\langle \theta(f), \theta(g) \rangle_{L^2(X)} = \langle f, g \rangle_{a,b}$.
- (4) $\|\theta(f)\|_{L^2(X)}^2 = \|f\|_{a,b}^2$.
- (5) $\theta(f)$ is normally distributed with $N(\int_0^T f da, \|f\|_b^2)$.
- (6) $\{\theta(f), f \in L_{a,b}^{1,2}[0, T]\}$ is a Gaussian system of random variables.

Proof. The proof of (1)–(4) has been done in Proposition 2.2.

(5) For any $f \in L_{a,b}^{1,2}[0, T]$, there exists a sequence $f_n \in \mathcal{S}$, $n \geq 1$ such that $f_n \rightarrow f$. Then $\theta(f_n) \rightarrow \theta(f)$ in $L^2(X)$ and clearly the $\theta(f_n)$'s are Gaussian variables. Hence the L^2 -limit $\theta(f)$ of a sequence $\theta(f_n)$ is normally distributed with $N(\int_0^T f da, \|f\|_b^2)$.

(6) The proof follows from the fact that for any finite linear combination of $\theta(f)$'s, $\sum_{i=1}^n \alpha_i \theta(f_i)$ is normally distributed with $N(\sum_{i=1}^n \alpha_i \int_0^T f_i da, \sum_{i,j=1}^n \alpha_i \alpha_j \langle f_i, f_j \rangle_b)$. $\square$

3. TRANSLATION THEOREM FOR L^p -FUNCTIONALS

Let D be any set and C be a real-valued function on $D \times D$. Then C is called a *covariance function* on $D \times D$ if

- (1) $C(s, t) = C(t, s)$,
- (2) $\sum_{t,s \in I} a_t a_s C(t, s) \geq 0$ for all finite subsets I of D and $\{a_s, s \in I\} \subset \mathbb{R}$.

Let C be a real-valued covariance function on $D \times D$. Then according to the following theorem due to Aronszajn [1], there exists a unique Hilbert space $K(C)$ of functions on D satisfying the conditions in the following theorem.

Theorem 3.1. *Let D be any separable metric space and C be a real-valued covariance function on $D \times D$. Then there exists a unique Hilbert space $K(C)$ of real-valued functions on D , with the inner product denoted by $\langle \cdot, \cdot \rangle_{K(C)}$ such that*

- (1) $C(t, \cdot) \in K(C)$, for each $t \in D$,
- (2) $\langle f(\cdot), C(t, \cdot) \rangle_{K(C)} = f(t)$, for each $t \in D$ and $f \in K(C)$.

The space $K(C)$ in Theorem 3.1, is called the *reproducing kernel Hilbert space* (RKHS) of C . Indeed, $K(C)$ is defined as the closure of the linear span of the functions $\{C(t, \cdot) \mid t \in D\}$ with respect to the inner product $\langle C(t, \cdot), C(s, \cdot) \rangle_{K(C)} = C(t, s)$. It is well known that if C is continuous on $D \times D$, then $K(C)$ is a separable Hilbert space.

Proposition 3.2. *Let $D = [0, T]$ and $C(t, s) = \min\{b(t), b(s)\}$, $s, t \in [0, T]$. Let $C'_b[0, T]$ be the Hilbert space given by*

$$C'_b[0, T] = \left\{ \gamma \mid \gamma(t) = \int_0^t \frac{d\gamma}{db} db, \frac{d\gamma}{db} \in L_b^2[0, T] \right\}$$

equipped with the inner product defined by

$$\langle \gamma_1, \gamma_2 \rangle_{C'_b[0, T]} = \left\langle \frac{d\gamma_1}{db}, \frac{d\gamma_2}{db} \right\rangle_b,$$

where $\frac{d\gamma}{db} \equiv \frac{d\nu}{d\mu}$ with μ = the Borel-Stieltjes measure induced by b and ν = the signed measure induced by $\gamma(t) = \int_0^t g db$, $g \in L_b^2[0, T]$. Then $K(C) = C'_b[0, T]$.

Proof. We first note that for all $s, t \in [0, T]$, $C(t, s) = \min\{b(t), b(s)\} = \int_0^T 1_{[0, t \wedge s]} db$ and $\frac{dC(t, \cdot)}{db} = 1_{[0, t]}(\cdot)$. Since $1_{[0, t]} \in L_b^2[0, T]$ for all $t \in [0, T]$, we have

$$C(t, \cdot) \in C'_b[0, T], \quad \text{for each } t \in [0, T].$$

For any $f \in C'_b[0, T]$ and for each $t \in [0, T]$, we have

$$\begin{aligned}\langle C(t, \cdot), f \rangle_{C'_b[0, T]} &= \langle 1_{[0, t]}, \frac{df}{db} \rangle_b \\ &= \int_0^t \frac{df}{db} db = f(t).\end{aligned}$$

Hence by Theorem 3.1, $K(C) = C'_b[0, T]$. $\square$

Proposition 3.3. *Let $D = [0, T]$ and $C(t, s) = \min\{b(t), b(s)\}$, $s, t \in [0, T]$. Then we have $K(C) \cong L_b^2[0, T]$.*

Proof. We define an operator U from $L_b^2[0, T]$ to $C'_b[0, T]$ by

$$Uf(t) = \int_0^t f(s)db(s), \quad f \in L_b^2[0, T].$$

Then we see that $\langle f, g \rangle_b = \langle \int_0^t f(s)db(s), \int_0^t g(s)db(s) \rangle_{C'_b[0, T]}$. Thus U is a unitary operator from $L_b^2[0, T]$ onto $C'_b[0, T]$. Hence $K(C) = C'_b[0, T] \cong L_b^2[0, T]$. $\square$

For $p \geq 1$, let $L^p(X) \equiv L^p(C_{a,b}[0, T], \mathcal{B}(C_{a,b}[0, T]), \mu)$ be the space of all random variables $F : C_{a,b}[0, T] \rightarrow \mathbb{R}$ such that

$$\|F\|_p = E(|F|^p)^{\frac{1}{p}} < \infty.$$

Lemma 3.4. *For $\phi \in L^p(X)$, $p \geq 1$, there exists a sequence $\{\xi_n\}_{n=1}^\infty$ in $H(X)$ such that ϕ is $\sigma(\xi_n, n \geq 1)$ -measurable.*

Proof. Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal basis for $L_{a,b}^{1,2}[0, T]$ and let $\xi_n = \int_0^T e_n(t)dX_t$. Since $1_{[0, t]} \in L_{a,b}^{1,2}[0, T]$ for each $t \in [0, T]$, we have

$$X_t = \int_0^T 1_{[0, t]}(s)dX_s = \sum_{n=1}^\infty \langle 1_{[0, t]}, e_n \rangle_{a,b} \int_0^T e_n(s)dX_s = \sum_{n=1}^\infty \langle 1_{[0, t]}, e_n \rangle_{a,b} \xi_n.$$

Hence X_t is $\sigma(\xi_n, n \geq 1)$ -measurable for each $t \in [0, T]$. Therefore, ϕ is $\sigma(\xi_n, n \geq 1)$ -measurable since ϕ is $\sigma(X_t, t \in [0, T])$ -measurable. $\square$

Theorem 3.5. *The set $A = \{e^\xi | \xi \in H(X)\}$ is total in $L^p(X)$, where $e^\xi = L^p$ -limit of $\sum_{k=0}^n \frac{\xi^k}{k!}$.*

Proof. We first note that each e^ξ is an element of $L^p(X)$ for each $p \geq 1$. Let V be the L^p -closure of the linear span of A . Suppose $V \neq L^p(X)$. Then there exists $\phi \in L^q(X)$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that $\phi \neq 0$ and $E[\phi \cdot e^\xi] = 0$ for all $\xi \in H(X)$. Let $\{\xi_n\}$ be as in Lemma 3.4. For any $E \in \sigma(\xi_1, \xi_2, \dots, \xi_n) \equiv \mathcal{B}_n$, $\xi_i \in H(X)$, $i = 1, 2, \dots, n$, we have

$$\int_E \phi \cdot e^\xi d\mu = \int_E E[\phi \cdot e^\xi | \mathcal{B}_n] d\mu,$$

for all $\xi \in H(X)$. Hence we have

$$\int_{C_{a,b}[0, T]} \phi \cdot e^{t_1 \xi_1 + \dots + t_n \xi_n} d\mu = \int_{C_{a,b}[0, T]} e^{t_1 \xi_1 + \dots + t_n \xi_n} E[\phi | \mathcal{B}_n] d\mu = 0.$$

By using the Stone-Weierstrass Theorem, one can show that the set $\{e^{t_1 \xi_1 + \dots + t_n \xi_n} | t_i \in \mathbb{R}, \xi_i \in H(X), i = 1, 2, \dots, n\}$ is total in $L^p(\xi_1, \dots, \xi_n)$. This follows because

$E[\phi|\mathcal{B}_n] = 0$ a.e. for all n . Since ϕ is integrable, we have by a well-known martingale convergence theorem and Lemma 3.4,

$$\phi = \lim_{n \rightarrow \infty} E[\phi|\mathcal{B}_n] = 0.$$

This contradiction completes the proof. $\square$

Theorem 3.6 (Translation Theorem). *Let $F : C_{a,b}[0, T] \rightarrow \mathbb{R}$ be a random variable such that $F \in L^p(X)$, and $\gamma(t) = \int_0^t g(s)db(s)$ for $g \in L_b^2[0, T]$. Then*

$$\begin{aligned} E[F(\cdot + \gamma)] &= E[F(\cdot) e^{\int_0^T g dX(\cdot) - \int_0^T g da - \frac{1}{2} \int_0^T g^2 db}] \\ &= E[F(\cdot) e^{\int_0^T g d\tilde{X}(\cdot) - \frac{1}{2} \int_0^T g^2 db}]. \end{aligned}$$

Proof. For $f \in L_{a,b}^{1,2}[0, T]$, let $F \in L^p(C_{a,b}[0, T])$ be given by $F(x) = e^{\int_0^T f(t)dX_t(x)}$. Then we have

$$F(x + \gamma) = e^{\int_0^T f(t)d(X_t(x) + \gamma(t))} = e^{\int_0^T f(t)dX_t(x) + \int_0^T f(t)g(t)db(t)}.$$

Hence it follows that

$$E[F(\cdot + \gamma)] = e^{\int_0^T f da + \frac{1}{2} \int_0^T f^2 db + \int_0^T f g db}.$$

On the other hand, we have

$$\begin{aligned} E[F(\cdot) e^{\int_0^T g dX(\cdot) - \int_0^T g da - \frac{1}{2} \int_0^T g^2 db}] &= E[e^{\int_0^T (f+g)dX(\cdot) - \int_0^T g da - \frac{1}{2} \int_0^T g^2 db}] \\ &= e^{\int_0^T f da + \frac{1}{2} \int_0^T f^2 db + \int_0^T f g db}. \end{aligned}$$

Hence the theorem is true for all functionals of the form $e^{\int_0^T f dX}$, $f \in L_{a,b}^{1,2}[0, T]$. By Theorem 3.5, the linear span of $\{e^{\int_0^T f dX} | f \in L_{a,b}^{1,2}[0, T]\}$ is dense in $L^p(X)$. Hence the theorem is proved. $\square$

4. INTEGRATION BY PARTS FORMULA FOR L^p -FUNCTIONALS

In this section, we define the directional derivative of L^p -functionals and then use the translation theorem obtained in Section 3 to establish an integration by parts formula for L^p -functionals.

Definition 4.1. Let $p \geq 1$. The directional derivative of a random variable $F \in L^p(X)$ in the direction γ , $\gamma(t) = \int_0^t g(s)db(s)$ where $g \in L_b^2[0, T]$, is defined as

$$\mathcal{D}_\gamma F(\cdot) = \lim_{t \rightarrow 0} \frac{1}{t} \{F(\cdot + t\gamma) - F(\cdot)\},$$

where the limit is taken in the $L^p(X)$ -sense. Let

$$\mathcal{D}^p(X) = \{F \in L^p(X) | \mathcal{D}_\gamma F \text{ exists in } L^p(X) \text{ for all } \gamma \in C'_{a,b}[0, T]\}.$$

Lemma 4.2. Let $\epsilon(h)(x) = e^{\int_0^T h d\tilde{X}(x) - \frac{1}{2} \int_0^T h^2 db}$, $h \in L_b^2[0, T]$, be a functional on $C_b[0, T]$. Then $\epsilon(h) \in L^p(X)$ for all $p \geq 1$. Moreover, for any $p \geq 1$,

$$(4.1) \quad \lim_{t \rightarrow 0} \frac{1}{t} \{\epsilon(th) - 1\} = \int_0^T h d\tilde{X}$$

where the limit is taken in the $L^p(X)$ -sense.

Proof. Let $h \in L_b^2[0, T]$. We first note that $e^{|\int_0^T h d\tilde{X}|} \in L^p(X)$ for all $p \geq 1$. So we see that $\|\epsilon(h)\|_p < \infty$. Let $A > 0$ be fixed. Define a function f on $C_{a,b}[0, T] \times [-A, A]$ by

$$f(x, t) = \epsilon(th)(x) = e^{t \int_0^T h d\tilde{X}(x) - \frac{t^2}{2} \int_0^T h^2 db}.$$

Then for almost all $x \in C_{a,b}[0, T]$, $f(x, \cdot)$ is differentiable on $(-A, A)$ and

$$\frac{\partial f}{\partial t}(x, t) = \left(\int_0^T h d\tilde{X} - t \int_0^T h^2 db \right) f(x, t).$$

If $-A < t < A$, then by the mean value theorem, for each $x \in C_{a,b}[0, T]$ there exists $\theta \equiv \theta(x)$, $0 < \theta < 1$, such that

$$\frac{1}{t}(f(x, t) - f(x, 0)) = \frac{\partial f}{\partial t}(x, \theta t).$$

From this fact and the inequalities $|\int_0^T h d\tilde{X}| \leq e^{|\int_0^T h d\tilde{X}|}$ and $|e^u| \leq e^{|u|}$ for $u \in \mathbb{R}$, we have the following inequalities:

$$\begin{aligned} & \left| \frac{1}{t} \{ \epsilon(th)(x) - 1 \} - \int_0^T h d\tilde{X}(x) \right| \\ &= \left| \left(\int_0^T h d\tilde{X}(x) - \theta t \int_0^T h^2 db \right) \cdot \epsilon(\theta th)(x) \right| - \int_0^T h d\tilde{X}(x) \\ &\leq \alpha e^{\beta |\int_0^T h d\tilde{X}(x)|} \end{aligned}$$

for some constants $\alpha > 0$ and $\beta > 0$. Since $\alpha e^{\beta |\int_0^T h d\tilde{X}|} \in L^p(X)$ for all $p \geq 1$, by the dominated convergence theorem,

$$\lim_{t \rightarrow 0} E \left[\left| \frac{1}{t} (\epsilon(th) - 1) - \int_0^T h d\tilde{X} \right|^p \right] = 0$$

and the lemma is proved. $\square$

Theorem 4.3. Let $F \in \mathcal{D}^p(X)$ and $\gamma(t) = \int_0^t g(u) db(u)$ with $g \in L_{a,b}^{1,2}[0, T]$. Then

$$(4.2) \quad E[\mathcal{D}_\gamma F] = E[F \cdot \int_0^T g dX] - \int_0^T g da \cdot E[F].$$

Proof. Since $F \in \mathcal{D}^p$, $\frac{1}{t} \{F(\cdot + t\gamma) - F(\cdot)\}$ converges to $\mathcal{D}_\gamma F$ in $L^p(X)$. Hence it converges to $\mathcal{D}_\gamma F$ in $L^1(X)$. From Theorem 3.6 and Lemma 4.2, it follows that

$$\begin{aligned} E[\mathcal{D}_\gamma F] &= \lim_{t \rightarrow 0} E \left[\frac{1}{t} \{F(\cdot + t\gamma) - F(\cdot)\} \right] \\ &= \lim_{t \rightarrow 0} E \left[F(\cdot) \left\{ \frac{1}{t} (\epsilon(tg) - 1) \right\} \right] \\ &= E \left[F \cdot \left(\int_0^T g dX - \int_0^T g da \right) \right] \end{aligned}$$

and the theorem is proved. $\square$

Theorem 4.4 (Integration by parts formula). Let $F \in \mathcal{D}^p(X)$, $G \in \mathcal{D}^q(X)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have, for $\gamma(t) = \int_0^t g(u) db(u)$ with $g \in L_{a,b}^{1,2}[0, T]$,

$$(4.3) \quad E[G \cdot \mathcal{D}_\gamma F] = E[F \cdot G \cdot \int_0^T g dX] - \int_0^T g da \cdot E[F \cdot G] - E[F \cdot \mathcal{D}_\gamma G].$$

Proof. We first note that for $F \in L^p(X)$ and $G \in L^q(X)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have $F \cdot G \in L^1(X)$. Now we shall show that the limit

$$\lim_{t \rightarrow 0} \frac{1}{t} \{F(\cdot + t\gamma) \cdot G(\cdot + t\gamma) - F(\cdot) \cdot G(\cdot)\}$$

exists in $L^1(X)$ and the limit is equal to $G \cdot \mathcal{D}_\gamma F + F \cdot \mathcal{D}_\gamma G$. This assertion follows from the following inequality:

$$\begin{aligned} & E\left[\left|\frac{1}{t}\{F(\cdot + t\gamma) \cdot G(\cdot + t\gamma) - F(\cdot) \cdot G(\cdot)\} - \{G \cdot \mathcal{D}_\gamma F + F \cdot \mathcal{D}_\gamma G\}\right|\right] \\ & \leq \left\|\frac{1}{t}\{F(\cdot + t\gamma) - F(\cdot)\} - \mathcal{D}_\gamma F\right\|_1 \|G(\cdot + t\gamma)\|_1 + \|\mathcal{D}_\gamma F\|_1 \|G(\cdot + t\gamma) - G(\cdot)\|_1 \\ & \quad + \|F\|_1 \left\|\frac{1}{t}\{G(\cdot + t\gamma) - G(\cdot)\} - \mathcal{D}_\gamma G\right\|_1 \\ & \leq \left\|\frac{1}{t}\{F(\cdot + t\gamma) - F(\cdot)\} - \mathcal{D}_\gamma F\right\|_p \|G(\cdot + t\gamma)\|_q + \|\mathcal{D}_\gamma F\|_p \|G(\cdot + t\gamma) - G(\cdot)\|_q \\ & \quad + \|F\|_p \left\|\frac{1}{t}\{G(\cdot + t\gamma) - G(\cdot)\} - \mathcal{D}_\gamma G\right\|_q \end{aligned}$$

since all three terms on the right-hand side above go to zero as t goes to zero. It then follows that

$$(4.4) \quad E[\mathcal{D}_\gamma(F \cdot G)] = E[G \cdot \mathcal{D}_\gamma F] + E[F \cdot \mathcal{D}_\gamma G].$$

But this implies that $F \cdot G \in \mathcal{D}^1(X)$, and hence by Theorem 4.3,

$$E[\mathcal{D}_\gamma(F \cdot G)] = E[(F \cdot G) \cdot \int_0^T g dX] - \int_0^T g da E[F \cdot G].$$

Therefore, we conclude that

$$E[G \cdot \mathcal{D}_\gamma F] = E[(F \cdot G) \cdot \int_0^T g dX] - \int_0^T g da E[F \cdot G] - E[F \cdot \mathcal{D}_\gamma G]$$

and the theorem is proved. $\square$

ACKNOWLEDGEMENT

The author thanks Professor Dong Myung Chung for his valuable discussion on this problem.

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