BOUNDEDNESS OF THE FIRST EIGENVALUE OF THE p -LAPLACIAN

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ABSTRACT. We prove that for any $p > 1$, any compact manifold of three or more dimensions carries Riemannian metrics of volume one with the first eigenvalue of the p -Laplacian arbitrarily large.

INTRODUCTION

The p -Laplace operator has been extensively studied in recent years, especially in the context of a bounded domain in $\mathbb{R}^n$. Recently, there has been an increasing interest in the study of this operator – and in particular of its first eigenvalue – in the more general setting of Riemannian manifolds.

Let M be a compact connected manifold. The p -Laplace operator ($p > 1$) associated to a Riemannian metric g on M is given by

$$\Delta_p f := \delta(|df|^{p-2} df),$$

where $\delta = -\operatorname{div}_g$ is the adjoint of d for the L^2 -norm induced by g on the space of differential forms. This operator can be viewed as a natural generalization of the well-known Laplace-Beltrami operator which corresponds to $p = 2$. When $\partial M = \emptyset$, the nonlinear partial differential equation

$$(0.1) \quad \Delta_p f = \lambda |f|^{p-2} f$$

characterizes the critical points of the p -energy functional $E_p(f) = \int_M |df|^p \nu_g$ under the constraint $\int_M |f|^p \nu_g = 1$ (ν_g denotes the Riemannian volume element induced by g).

The real numbers λ for which this equation has nontrivial solutions are called *eigenvalues* of Δ_p . They represent the critical energy levels. The associated solutions, i.e. the critical functions, are called *eigenfunctions*.

Obviously, zero is an eigenvalue of Δ_p , the associated eigenfunctions being the constant functions. The set $\sigma_p(M, g)$ of the nonzero eigenvalues is a nonempty, unbounded subset of $(0, \infty)$ [6]. Its infimum $\lambda_{1,p}(M, g) = \inf \sigma_p(M, g)$ is itself a positive eigenvalue called the *first eigenvalue* of Δ_p and has the following variational characterization [14] :

$$(0.2) \quad \lambda_{1,p}(M, g) = \inf \left\{ \frac{\int_M |df|^p \nu_g}{\int_M |f|^p \nu_g} \mid f \in W^{1,p}(M) \setminus \{0\}, \int_M |f|^{p-2} f \nu_g = 0 \right\}.$$

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In the present work we consider $\lambda_{1,p}$ as a functional on the space of Riemannian metrics on M . Since $\lambda_{1,p}$ is noninvariant under dilatations, we will study the restriction of $\lambda_{1,p}$ to $\mathcal{M}(M)$, the space of Riemannian metrics of volume one on M . The question we ask is whether $\lambda_{1,p}$ is bounded on $\mathcal{M}(M)$?

A classical elementary argument [3] shows that the infimum of $\lambda_{1,2}$ on this space is zero. The same argument shows that in fact for any $p > 1$ the infimum of $\lambda_{1,p}$ on $\mathcal{M}(M)$ is equally zero.

A series of authors (Bérard Bergery and Bourguignon [2], Muto [10], Tanno [11], Urakawa [13]) have showed in various degrees of generality that if M is a sphere of dimension $m \geq 3$, then there is no upper bound for $\lambda_{1,2}$ on $\mathcal{M}(S^m)$. Colbois and Dodziuk [4] (and independently Xu [15]) have extended this result to any compact manifold of dimension $m \geq 3$.

In this paper we prove that this phenomenon is still true for the functional $\lambda_{1,p}$, for any $p > 1$.

1. PRELIMINARIES

We review here some basic properties of the first eigenvalue for the Dirichlet and the Neumann problems for Δ_p .

Let Ω be a domain in M and consider the Dirichlet problem:

$$\begin{cases} \Delta_p f &= \mu |f|^{p-2} f & \text{in } \Omega, \\ f &= 0 & \text{on } \partial\Omega. \end{cases}$$

Denote by $\mu_{1,p}(\Omega, g)$ the infimum of the set of eigenvalues for this problem. Here too, $\mu_{1,p}(\Omega, g)$ is a positive eigenvalue with the variational characterization

$$\mu_{1,p}(\Omega, g) = \inf \left\{ \frac{\int_{\Omega} |df|^p \nu_g}{\int_{\Omega} |f|^p \nu_g} \mid f \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}.$$

The eigenfunctions associated to $\mu_{1,p}(\Omega, g)$ are essentially unique; they form a one-dimensional space. Moreover, they have no zeros in Ω (see for instance [7]).

In [14], Veron proved the following result:

$$(1.1) \quad \lambda_{1,p}(M, g) = \min_{(\omega_1, \omega_2)} \max \{ \mu_{1,p}(\omega_1, g), \mu_{1,p}(\omega_2, g) \},$$

where (ω_1, ω_2) runs over the set of couples of nonempty disjoint open subsets of M .

Consider now the Neumann problem for Δ_p on Ω :

$$\begin{cases} \Delta_p f &= \lambda |f|^{p-2} f & \text{in } \Omega, \\ df(\eta) &= 0 & \text{on } \partial\Omega, \end{cases}$$

where η denotes the exterior normal unit vector field to $\partial\Omega$.

Denote by $\lambda_{1,p}^N(\Omega, g)$ the infimum of the set of nonzero eigenvalues for this problem. By mimicking the proof of the closed case [14] we obtain that $\lambda_{1,p}^N(\Omega, g)$ is itself a positive eigenvalue with the variational characterization

$$\lambda_{1,p}^N(\Omega, g) = \inf \left\{ \frac{\int_{\Omega} |df|^p \nu_g}{\int_{\Omega} |f|^p \nu_g} \mid f \in W^{1,p}(\Omega) \setminus \{0\}, \int_{\Omega} |f|^{p-2} f \nu_g = 0 \right\}.$$

A more general regularity result [12] says that the eigenfunctions for these problems are C^1 , with gradient locally Hölder continuous.

2. RIEMANNIAN METRICS WITH ARBITRARILY LARGE $\lambda_{1,p}$

Let M be a compact connected manifold and denote by $\mathcal{M}(M)$ the space of Riemannian metrics of volume one on M .

It is known that we may diffeomorphically transform a local disc of M into a “mushroom” attached to M by a cylinder C . We can choose a Riemannian metric $g_s \in \mathcal{M}(M)$ such that some subset $C_0 \subset C$ is the Riemannian product $[0, l] \times S^{m-1}(s)$ (l, s are positive constants and $S^{m-1}(s)$ denotes the $(m-1)$ -dimensional Euclidean sphere of radius s). Let M_1 and M_2 be the connected components of $M \setminus C_0$. We may choose g_s such that $\text{Vol}(M_1, g_s) = \text{Vol}(M_2, g_s)$.

Let us consider a C^∞ function f equal to 1 on M_1 , to -1 on M_2 and linear along C_0 (constant on the cross sections of C_0). Then f satisfies the orthogonality condition $\int_M |f|^{p-2} f \nu_{g_s} = 0$ and

$$\lim_{s \rightarrow 0} \frac{\int_M |df|^p \nu_{g_s}}{\int_M |f|^p \nu_{g_s}} = 0.$$

The variational characterization for $\lambda_{1,p}(M, g_s)$ implies $\lim_{s \rightarrow 0} \lambda_{1,p}(M, g_s) = 0$, i.e.:

Proposition 2.1. *For any $p > 1$,*

$$\inf_{g \in \mathcal{M}(M)} \lambda_{1,p}(M, g) = 0.$$

Our main result is addressing the nonexistence of universal upper bounds for $\lambda_{1,p}$ on $\mathcal{M}(M)$:

Theorem 2.2. *For any $p > 1$, every compact manifold M of dimension $m \geq 3$ carries Riemannian metrics of volume one with $\lambda_{1,p}$ arbitrarily large.*

In the case $p = 2$ this Theorem is due to Colbois and Dodziuk [4].

The proof of Theorem 2.2 is divided into two parts: for $1 < p < 2$ the path we follow is reminiscent of the case $p = 2$; however, new ideas are required since the arguments from the linear case do not apply when $p \neq 2$. In the case $p > 2$, the proof follows directly from a monotonicity result.

Proof. I. The case $1 < p \leq 2$

The main part of the proof consists in constructing a domain $D \subset S^m$ (S^m denotes the m -dimensional unit Euclidean sphere) and a family of Riemannian metrics $(g_n)_{n \in \mathbb{N}}$ on S^m such that

$$(2.1) \quad \lim_{n \rightarrow \infty} \lambda_{1,p}^N(D, g_n) > 0, \quad \lim_{n \rightarrow \infty} \lambda_{1,p}(S^m, g_n) > 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \text{Vol}(D, g_n) = \infty.$$

We do this first on S^3 and then by following a path similar to [5] we show how to construct such domains and Riemannian metrics inductively on S^m , $m > 3$ (can and $\widehat{can}$ denote the canonical metrics on S^3 , respectively $\mathbb{CP}^1$). For any $x \in S^3$, denote by $V_x = \text{Ker}(d\pi_x)$ and by H_x the space orthogonal to V_x in $T_x S^3$ w.r.t. can_x . Let $\{g_t\}_{t>0}$ be the *Berger metrics* defined by

$$(2.2) \quad g_t|_{V_x \times H_x} = 0, \quad g_t|_{V_x \times V_x} = t^2 can|_{V_x \times V_x}, \quad g_t|_{H_x \times H_x} = can|_{H_x \times H_x}.$$

The map π is a Riemannian submersion from (S^3, g_t) to $(\mathbb{CP}^1, \widehat{can})$ with totally geodesic fibers $F_x = \pi^{-1}(\pi(x))$ isometric to $(S^1, t^2 can)$.

Lemma 2.3. *Let $A = \{(x_1, x_2, x_3, x_4) \in S^3 \mid x_3^2 + x_4^2 < \frac{1}{4}\}$. Then for any $1 < p \leq 2$,*

$$\lambda_{1,p}^N(A, g_t) \geq \lambda_{1,p}(S^3, g_t) \geq 2^{-p} \mu_{1,p}(A, g_t).$$

Proof of Lemma 2.3. Fix t and let $a = \{x \in S^3 \mid x = (x_1, x_2, 0, 0)\}$, $b = \{x \in S^3 \mid x = (0, 0, x_3, x_4)\}$ and $\mathcal{S} = S^3 \setminus \{a \cup b\}$. Since a and b have codimension 2 in S^3 , [1, Lemma 1] implies that $C_0^\infty(\mathcal{S})$ is dense in $W^{1,2}(S^3)$ and therefore in $W^{1,p}(S^3)$ for $1 < p \leq 2$. We can then replace $W^{1,p}(S^3)$ by $C_0^\infty(\mathcal{S})$ in the variational characterization of $\lambda_{1,p}(S^3, g_t)$.

Parameterize $\mathcal{S}$ by

$$\Phi : (0, 1) \times [0, 2\pi) \times [0, 2\pi) \rightarrow \mathcal{S},$$

$$\Phi(r, \theta_1, \theta_2) = (\sqrt{1-r^2} \cos \theta_1, \sqrt{1-r^2} \sin \theta_1, r \cos \theta_2, r \sin \theta_2).$$

It is immediate that $\Phi(r, \theta_1, \theta_2) \in A \setminus a \Leftrightarrow r \in (0, \frac{1}{2})$. Consider the diffeomorphism $\tau : A \setminus a \rightarrow \mathcal{S}$ defined by

$$(2.3) \quad \tau(\Phi(r, \theta_1, \theta_2)) = \Phi(2r, \theta_1, \theta_2).$$

Let $f \in C_0^\infty(\mathcal{S}) \setminus \{0\}$ with $\int_S |f|^{p-2} f \nu_{g_t} = 0$. The function $f \circ \tau \in W_0^{1,p}(A \setminus a) \subset W_0^{1,p}(A)$ and the variational characterization of $\mu_{1,p}(A, g_t)$ yields

$$(2.4) \quad \mu_{1,p}(A, g_t) \leq \frac{\int_{A \setminus a} |d(f \circ \tau)|_{g_t}^p \nu_{g_t}}{\int_{A \setminus a} |f \circ \tau|^p \nu_{g_t}},$$

where $|\cdot|_{g_t}$ denotes the Hilbert-Schmidt norm w.r.t. g_t . To evaluate the right side of inequality (2.4) note that for any $x \in \mathcal{S}$, $x = \Phi(r, \theta_1, \theta_2)$, the vectors

$$v = d\Phi(0, 1, 1) = (-\sqrt{1-r^2} \sin \theta_1, \sqrt{1-r^2} \cos \theta_1, -r \sin \theta_2, r \cos \theta_2),$$

$$u_1 = d\Phi(0, -\frac{r}{\sqrt{1-r^2}}, \frac{\sqrt{1-r^2}}{r}) = (r \sin \theta_1, -r \cos \theta_1, -\sqrt{1-r^2} \sin \theta_2, \sqrt{1-r^2} \cos \theta_2),$$

$$u_2 = d\Phi(\sqrt{1-r^2}, 0, 0) = (-r \cos \theta_1, -r \sin \theta_1, \sqrt{1-r^2} \cos \theta_2, \sqrt{1-r^2} \sin \theta_2)$$

form an orthonormal basis of $T_x S^3$ w.r.t. can_x . Moreover $v = \frac{d\mathbf{c}}{ds} \big|_{s=0}$ where $\mathbf{c} : [0, 2\pi) \rightarrow F_x$, $\mathbf{c}(s) = \Phi(r, \theta_1 + s, \theta_2 + s)$. It is easy to see that $\pi \circ \mathbf{c}(s) = \pi(x)$ for all s and therefore $v \in \text{Ker}(d\pi_x) = V_x$. Hence, $\{t^{-1}v, u_1, u_2\}$ is an orthonormal basis of $T_x S^3$ w.r.t. g_t , for any $t > 0$. On the other hand, (2.3) implies

$$(2.5) \quad d\tau(v) = v, \quad d\tau(u_1) = u_1, \quad d\tau(u_2) = 2u_2.$$

We have

$$\begin{aligned} |d(f \circ \tau)|_{g_t}^p &= (|d(f \circ \tau)(t^{-1}v)|^2 + |d(f \circ \tau)(u_1)|^2 + |d(f \circ \tau)(u_2)|^2)^{\frac{p}{2}} \\ &= (|df(t^{-1}v)|^2 + |df(u_1)|^2 + |df(2u_2)|^2)^{\frac{p}{2}} \circ \tau \\ &\leq 2^p |df|_{g_t}^p \circ \tau. \end{aligned}$$

Integrating over $A \setminus a$ we obtain

$$\begin{aligned}
 \int_{A \setminus a} |d(f \circ \tau)|_{g_t}^p \nu_{g_t} &\leq 2^p \int_{A \setminus a} |df|_{g_t}^p \circ \tau \nu_{g_t} \\
 (2.6) \qquad \qquad \qquad &= 2^{p-1} \int_{A \setminus a} |df|_{g_t}^p \circ \tau \nu_{\tau^* g_t} \\
 &= 2^{p-1} \int_{\mathcal{S}} |df|_{g_t}^p \nu_{g_t}
 \end{aligned}$$

(where in the first equality we have used $\nu_{\tau^* g_t} = 2\nu_{g_t}$, which follows immediately from (2.5)). Similarly

$$(2.7) \qquad \qquad \int_{A \setminus a} |f \circ \tau|^p \nu_{g_t} = \frac{1}{2} \int_{\mathcal{S}} |f|^p \nu_{g_t}.$$

Equations (2.4), (2.6) and (2.7) imply that

$$\mu_{1,p}(A, g_t) \leq 2^p \frac{\int_{\mathcal{S}} |df|^p \nu_{g_t}}{\int_{\mathcal{S}} |f|^p \nu_{g_t}}.$$

Taking now the infimum over all $f \in C_0^\infty(\mathcal{S}) \setminus \{0\}$ with $\int_{\mathcal{S}} |f|^{p-2} f \nu_{g_t} = 0$ we obtain

$$\mu_{1,p}(A, g_t) \leq 2^p \lambda_{1,p}(S^3, g_t).$$

To prove the left inequality in Lemma 2.3 note that since $C^\infty(\overline{A})$ is dense in $W^{1,p}(A)$, by the same argument as above we obtain that $C_0^\infty(\overline{A} \setminus a)$ is dense in $W^{1,p}(A)$. We can thus replace $W^{1,p}(A)$ by $C_0^\infty(\overline{A} \setminus a)$ in the variational characterization for $\lambda_{1,p}^N(A, g_t)$. Let $f \in C_0^\infty(\overline{A} \setminus a) \setminus \{0\}$ with $\int_{A \setminus a} |f|^{p-2} f \nu_{g_t} = 0$. Let $B = \{(x_1, x_2, x_3, x_4) \in S^3 \mid x_3^2 + x_4^2 > \frac{1}{4}\}$ and consider the diffeomorphism $\Psi : \mathcal{S} \rightarrow \mathcal{S}$ given by

$$(2.8) \qquad \qquad \Psi(\Phi(r, \theta_1, \theta_2)) = \Phi(1-r, \theta_1, \theta_2).$$

Extend f into a function $F \in W_0^{1,p}(\mathcal{S})$ defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in \overline{A} \setminus a, \\ f \circ \Psi(x) & \text{if } x \in B \setminus b. \end{cases}$$

Equation (2.8) implies $d\Psi(v) = v$, $d\Psi(u_1) = u_1$ and $d\Psi(u_2) = -u_2$. Hence, Ψ is an isometry of $(\mathcal{S}, \text{can})$; in fact, since v is invariant by $d\Psi$, Ψ is an isometry of $(\mathcal{S}, g_t)$, for any $t > 0$. In particular, given that $\Psi(A \setminus a) = B \setminus b$, we have

$$\begin{aligned}
 \int_{\mathcal{S}} |dF|_{g_t}^p \nu_{g_t} &= 2 \int_{A \setminus a} |df|_{g_t}^p \nu_{g_t}, \quad \int_{\mathcal{S}} |F|^p \nu_{g_t} = 2 \int_{A \setminus a} |f|^p \nu_{g_t}, \\
 \int_{\mathcal{S}} |F|^{p-2} F \nu_{g_t} &= 2 \int_{A \setminus a} |f|^{p-2} f \nu_{g_t} = 0.
 \end{aligned}$$

The variational characterization for $\lambda_{1,p}(S^3, g_t)$ yields

$$\lambda_{1,p}(S^3, g_t) \leq \frac{\int_{\mathcal{S}} |dF|_{g_t}^p \nu_{g_t}}{\int_{\mathcal{S}} |F|^p \nu_{g_t}} = \frac{\int_{A \setminus a} |df|_{g_t}^p \nu_{g_t}}{\int_{A \setminus a} |f|^p \nu_{g_t}}.$$

Taking now the infimum over all $f \in C_0^\infty(\overline{A} \setminus a) \setminus \{0\}$ with $\int_A |f|^{p-2} f \nu_{g_t} = 0$ we obtain $\lambda_{1,p}(S^3, g_t) \leq \lambda_{1,p}^N(A, g_t)$. This ends the proof of Lemma 2.3. $\square$

Lemma 2.4. *Let A be as in Lemma 2.3. Then for any $1 < p \leq 2$,*

$$\mu_{1,p}(A, g_t) \geq 2^{\frac{p}{2}-1} \lambda_{1,p}(C\mathbb{P}^1, \widehat{can}).$$

Proof of Lemma 2.4. Fix t and let $f \in W_0^{1,p}(A)$ be an eigenfunction for $\mu_{1,p}(A, g_t)$. Then f has no zeros in A , and we may assume that $f > 0$ in A . It is easy to see that for any $x \in A$, $F_x \subset A$. We can construct therefore a positive function $\tilde{f} : \pi(A) \subset C\mathbb{P}^1 \rightarrow \mathbb{R}$ by setting

$$(2.9) \quad \tilde{f}(\pi(x))^p = \int_{F_x} f^p \nu_{g_t}.$$

Let u_1, u_2 be two vector fields such that for all $x \in A$, $\{u_1, u_2\}$ is an orthonormal basis of H_x w.r.t. g_t . Then $\{\tilde{u}_1 = d\pi(u_1), \tilde{u}_2 = d\pi(u_2)\}$ is an orthonormal basis of $T_{\pi(x)}C\mathbb{P}^1$ w.r.t. $\widehat{can}$. Given that the fibers F_x are totally geodesic, we can apply Lemma 3.14 in [2] to the function f^p and the vector field u_1 and obtain

$$p \tilde{f}^{p-1} d\tilde{f}(\tilde{u}_1) = p \int_{F_x} f^{p-1} df(u_1) \nu_{g_t}.$$

The Hölder inequality yields

$$\begin{aligned} \tilde{f}^{p-1} |d\tilde{f}(\tilde{u}_1)| &\leq \int_{F_x} f^{p-1} |df(u_1)| \nu_{g_t} \\ &\leq \left(\int_{F_x} f^p \nu_{g_t} \right)^{\frac{p-1}{p}} \left(\int_{F_x} |df(u_1)|^p \nu_{g_t} \right)^{\frac{1}{p}} \\ &= \tilde{f}^{p-1} \left(\int_{F_x} |df(u_1)|^p \nu_{g_t} \right)^{\frac{1}{p}}. \end{aligned}$$

Since $\tilde{f} > 0$, it follows that $|d\tilde{f}(\tilde{u}_1)|^p \leq \int_{F_x} |df(u_1)|^p \nu_{g_t}$. In the same way we obtain $|d\tilde{f}(\tilde{u}_2)|^p \leq \int_{F_x} |df(u_2)|^p \nu_{g_t}$. Hence

$$\begin{aligned} |d\tilde{f}|_{\widehat{can}}^p &= \left(|d\tilde{f}(\tilde{u}_1)|^2 + |d\tilde{f}(\tilde{u}_2)|^2 \right)^{\frac{p}{2}} \leq |d\tilde{f}(\tilde{u}_1)|^p + |d\tilde{f}(\tilde{u}_2)|^p \\ (2.10) \quad &\leq \int_{F_x} |df(u_1)|^p + |df(u_2)|^p \nu_{can} \\ &\leq 2^{1-\frac{p}{2}} \int_{F_x} (|df(u_1)|^2 + |df(u_2)|^2)^{\frac{p}{2}} \nu_{can} \\ &\leq 2^{1-\frac{p}{2}} \int_{F_x} |df|_{g_t}^p \nu_{can} \end{aligned}$$

(where we have used the inequality: $(a+b)^{\frac{p}{2}} \leq a^{\frac{p}{2}} + b^{\frac{p}{2}} \leq 2^{1-\frac{p}{2}}(a+b)^{\frac{p}{2}}$ for any $a, b > 0$ and $1 < p \leq 2$). Integrating (2.9) and (2.10) over $\pi(A)$ we obtain

$$(2.11) \quad \int_{\pi(A)} |\tilde{f}|^p \nu_{\widehat{can}} = \int_A |f|^p \nu_{g_t} \quad \text{and} \quad \int_{\pi(A)} |d\tilde{f}|^p \nu_{\widehat{can}} \leq 2^{1-\frac{p}{2}} \int_A |df|_{g_t}^p \nu_{g_t}.$$

In particular, (2.11) implies that $\tilde{f} \in W^{1,p}(\pi(A))$. In fact, since $f \in W_0^{1,p}(A)$ and $F_x \subset \partial A$ for any $x \in \partial A$, it is immediate from (2.9) that $\tilde{f} \in W_0^{1,p}(\pi(A))$. The variational characterization for $\mu_{1,p}(\pi(A), \widehat{can})$ and (2.11) imply then

$$(2.12) \quad \mu_{1,p}(\pi(A), \widehat{can}) \leq \frac{\int_{\pi(A)} |d\tilde{f}|^p \nu_{\widehat{can}}}{\int_{\pi(A)} |\tilde{f}|^p \nu_{\widehat{can}}} \leq 2^{1-\frac{p}{2}} \frac{\int_A |df|_{g_t}^p \nu_{g_t}}{\int_A |f|^p \nu_{g_t}} = 2^{1-\frac{p}{2}} \mu_{1,p}(A, g_t).$$

In the same way we can prove that $\mu_{1,p}(\pi(B), \widehat{can}) \leq 2^{1-\frac{p}{2}} \mu_{1,p}(B, g_t)$, where B is as in Lemma 2.3. The map Ψ is an isometry between $A \setminus a$ and $B \setminus b$ and therefore it is straightforward that $\mu_{1,p}(A, g_t) = \mu_{1,p}(B, g_t)$. Since the open sets $\pi(A)$ and $\pi(B)$ are disjoint, (1.1) and (2.12) imply

$$\lambda_{1,p}(C\mathbb{P}^1, \widehat{can}) \leq \max\{\mu_{1,p}(\pi(A), \widehat{can}), \mu_{1,p}(\pi(B), \widehat{can})\} \leq 2^{1-\frac{p}{2}} \mu_{1,p}(A, g_t).$$

□

Combining Lemma 2.3 and Lemma 2.4 we obtain

$$\lambda_{1,p}^N(A, g_t) \geq \lambda_{1,p}(S^3, g_t) \geq 2^{-\frac{p}{2}-1} \lambda_{1,p}(C\mathbb{P}^1, \widehat{can}).$$

Finally, since $\nu_{g_t} = t\nu_g$ we have $\text{Vol}(A, g_t) \rightarrow \infty$ as $t \rightarrow \infty$. In particular, we can find a sequence $t_n \rightarrow \infty$ such that the metrics $g_n := g_{t_n}$ and the domain A satisfy (2.1) on S^3 .

We will now construct a domain $D \subset S^{m+1}$ and a family of metrics on S^{m+1} with the property (2.1), given such domain and metrics on S^m . To do this we follow the general idea from [5].

Let $D \subset S^m$ and $(g_n)_{n \in \mathbb{N}}$ with property (2.1). (In fact we need only the last two assumptions in (2.1) for the construction on S^{m+1} .) Let $C = S^m \times (0, 1)$ endowed with the metric $h_n = g_n \times dr^2$ where dr^2 denotes the Euclidean metric on $(0, 1)$. Obviously $\text{Vol}(C, h_n) = \text{Vol}(S^m, g_n) > \text{Vol}(D, g_n)$. Let f_n be an eigenfunction for $\lambda_{1,p}^N(C, h_n)$ and denote by f_n^+ , f_n^- its positive, respectively, its negative part. In the same way as in [8, Lemma 3], it is straightforward that

$$\lambda_{1,p}^N(C, h_n) = \frac{\int_C |df_n|_{h_n}^p \nu_{h_n}}{\int_C |f_n|^p \nu_{h_n}} = \frac{\int_C |df_n^+|_{h_n}^p \nu_{h_n}}{\int_C |f_n^+|^p \nu_{h_n}} = \frac{\int_C |df_n^-|_{h_n}^p \nu_{h_n}}{\int_C |f_n^-|^p \nu_{h_n}}.$$

After multiplying f_n^+ and f_n^- by constants we may assume that $\int_C |f_n^+|^p \nu_{h_n} = \int_C |f_n^-|^p \nu_{h_n} = 1$. Then

$$(2.13) \quad \lambda_{1,p}^N(C, h_n) = \int_C |df_n^+|_{h_n}^p \nu_{h_n} = \int_C |df_n^-|_{h_n}^p \nu_{h_n}.$$

Consider the positive functions $\tilde{f}_n^+(r)$ and $\tilde{f}_n^-(r)$ defined by

$$(2.14) \quad \tilde{f}_n^+(r)^p = \int_{S^m} |f_n^+|^p(\cdot, r) \nu_{g_n} \quad \text{and} \quad \tilde{f}_n^-(r)^p = \int_{S^m} |f_n^-|^p(\cdot, r) \nu_{g_n}.$$

As in the proof of Lemma 2.4, we can apply Lemma 3.14 of [2] to the function $(\tilde{f}_n^+)^p$ and the vector field $\frac{\partial}{\partial r}$ and obtain

$$(2.15) \quad |(\tilde{f}_n^+)'|^p \leq \int_{S^m} |df_n^+|_{h_n}^p \left(\frac{\partial}{\partial r} \right)^p \nu_{g_n}.$$

Integrating (2.14) and (2.15) we get $\int_0^1 |\tilde{f}_n^+|^p dr = \int_C |f_n^+|^p \nu_{h_n} = 1$ and

$$(2.16) \quad \int_0^1 |(\tilde{f}_n^+)'|^p dr \leq \int_C |df_n^+|_{h_n}^p \left(\frac{\partial}{\partial r} \right)^p \nu_{h_n} \leq \int_C |df_n^+|_{h_n}^p \nu_{h_n} = \lambda_{1,p}^N(C, h_n).$$

Suppose now that $\lim_{n \rightarrow \infty} \lambda_{1,p}^N(C, h_n) = 0$. Then (2.16) implies that the sequence $(\tilde{f}_n^+)_{n \in \mathbb{N}}$ is bounded in $W^{1,p}(0, 1)$. After extracting a subsequence, we may assume

that there exists a function $\tilde{f}_0^+ \in W^{1,p}(0,1)$ such that $\tilde{f}_n^+$ converges to $\tilde{f}_0^+$ weakly in $W^{1,p}(0,1)$ and strongly in $L^p(0,1)$. The weak convergence and (2.16) yield

$$\int_0^1 |(\tilde{f}_0^+)'|^p dr \leq \liminf_{n \rightarrow \infty} \int_0^1 |(\tilde{f}_n^+)'|^p dr \leq \lim_{n \rightarrow \infty} \lambda_{1,p}^N(C, h_n) = 0 \Rightarrow \tilde{f}_0^+ \text{ is constant.}$$

The strong convergence implies that $\int_0^1 |\tilde{f}_0^+|^p dr = \lim_{n \rightarrow \infty} \int_0^1 |\tilde{f}_n^+|^p dr = 1$. Since $\tilde{f}_0^+$ is constant, we have $\tilde{f}_0^+ \equiv 1$, i.e. $\tilde{f}_n^+ \rightarrow 1$ in $L^p(0,1)$. Similarly, after extracting a subsequence again, we obtain $\tilde{f}_n^- \rightarrow 1$ in $L^p(0,1)$. In particular,

$$(2.17) \quad \lim_{n \rightarrow \infty} \int_{S^m} |f_n^+|^p(\cdot, r) \nu_{g_n} = \lim_{n \rightarrow \infty} \int_{S^m} |f_n^-|^p(\cdot, r) \nu_{g_n} = 1 \text{ for almost all } r.$$

On the other hand, (2.13) implies

$$\lambda_{1,p}^N(C, h_n) = \int_0^1 \int_{S^m} |df_n^+|_{h_n}^p \nu_{g_n} dr \geq \int_0^1 \int_{S^m} |df_n^+|_{g_n}^p \nu_{g_n} dr,$$

and a similar inequality for f_n^- . Hence

$$(2.18) \quad \lim_{n \rightarrow \infty} \int_{S^m} |df_n^+|_{g_n}^p(\cdot, r) \nu_{g_n} = \lim_{n \rightarrow \infty} \int_{S^m} |df_n^-|_{g_n}^p(\cdot, r) \nu_{g_n} = 0 \text{ for almost all } r.$$

Combining (2.17) and (2.18) we obtain

$$(2.19) \quad \lim_{n \rightarrow \infty} \frac{\int_{S^m} |df_n^+|_{g_n}^p(\cdot, r) \nu_{g_n}}{\int_{S^m} |f_n^+|^p(\cdot, r) \nu_{g_n}} = \lim_{n \rightarrow \infty} \frac{\int_{S^m} |df_n^-|_{g_n}^p(\cdot, r) \nu_{g_n}}{\int_{S^m} |f_n^-|^p(\cdot, r) \nu_{g_n}} = 0 \text{ for almost all } r.$$

Fix r and denote by ω_n^+ , ω_n^- the (disjoint) supports in S^m of $f_n^+(\cdot, r)$ and $f_n^-(\cdot, r)$ respectively. The variational characterization for $\mu_{1,p}(\omega_r^+, g_n)$ and $\mu_{1,p}(\omega_r^-, g_n)$ together with (1.1) implies

$$\begin{aligned} & \max \left\{ \frac{\int_{S^m} |df_n^+|_{g_n}^p(\cdot, r) \nu_{g_n}}{\int_{S^m} |f_n^+|^p(\cdot, r) \nu_{g_n}}, \frac{\int_{S^m} |df_n^-|_{g_n}^p(\cdot, r) \nu_{g_n}}{\int_{S^m} |f_n^-|^p(\cdot, r) \nu_{g_n}} \right\} \\ & \geq \max \{ \mu_{1,p}(\omega_n^+, g_n), \mu_{1,p}(\omega_n^-, g_n) \} \\ & \geq \lambda_{1,p}(S^m, g_n). \end{aligned}$$

Equation (2.19) implies then that $\lim_{n \rightarrow \infty} \lambda_{1,p}(S^m, g_n) = 0$. This contradicts our choice of g_n . It follows that $\lim_{n \rightarrow \infty} \lambda_{1,p}^N(C, h_n) \neq 0$ and therefore (after extracting a subsequence) we may assume $\lim_{n \rightarrow \infty} \lambda_{1,p}^N(C, h_n) > 0$.

To conclude our construction on S^{m+1} we need the first two assertions of the following technical result [9]:

Lemma 2.5. *Let (M, g) be a compact Riemannian manifold, $p > 1$ and $\Omega \subset M$ a domain with Lipschitzian boundary. Then for any $\delta > 0$, there exists a metric $\tilde{g}$ on M such that:*

- $g|_{\Omega} = \tilde{g}|_{\Omega}$,
- $\lambda_{1,p}(M, \tilde{g}) > \lambda_{1,p}^N(\Omega, g) - \delta$,
- $|\text{Vol}(M, \tilde{g}) - \text{Vol}(\Omega, g)| < \delta$.

We can now glue two caps to C to obtain a manifold diffeomorphic to S^{m+1} . Let $2\delta = \lim_{n \rightarrow \infty} \lambda_{1,p}^N(C, h_n) > 0$. Lemma 2.5 implies that we can extend h_n into a metric $\tilde{h}_n$ on the whole of S^{m+1} such that $\lambda_{1,p}(S^{m+1}, \tilde{h}_n) > \lambda_{1,p}^N(C, h_n) - \delta$. Then

C and (after extracting a subsequence) the family $(\tilde{h}_n)_{n \in \mathbb{N}}$ will satisfy the property (2.1) on S^{m+1} .

Now let M be an arbitrarily compact manifold of dimension $m \geq 3$. Consider a domain $D \subset S^m$ and a family of Riemannian metrics $(g_n)_{n \in \mathbb{N}}$ on S^m with the property (2.1). Let $x_0 \in S^m$ such that $x_0 \notin D$ and let U be a small open set in M diffeomorphic to $S^m \setminus \{x_0\}$ (since $S^m \setminus \{x_0\}$ is diffeomorphic to a ball in $\mathbb{R}^m$, it suffices to take U diffeomorphic to a ball). We can pull back the metric g_n via this diffeomorphism into a metric $\bar{g}_n$ on U . Then (D, g_n) is isometric to $(\Omega, \bar{g}_n)$, where Ω is some subset of U . We have

$$\lambda_{1,p}^N(D, g_n) = \lambda_{1,p}^N(\Omega, \bar{g}_n) \quad \text{and} \quad \text{Vol}(\Omega, \bar{g}_n) = \text{Vol}(D, g_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Now let $2\delta = \lim_{n \rightarrow \infty} \lambda_{1,p}^N(\Omega, \bar{g}_n) > 0$. We can perturb each $\bar{g}_n$ outside Ω as in Lemma 2.5 and obtain a family of metrics $(\tilde{g}_n)_{n \in \mathbb{N}}$ on M such that $\lambda_{1,p}(M, \tilde{g}_n) > \lambda_{1,p}^N(\Omega, \bar{g}_n) - \delta$. Since $\text{Vol}(M, \tilde{g}_n) > \text{Vol}(\Omega, \bar{g}_n)$ we have $\lim_{n \rightarrow \infty} \text{Vol}(M, \tilde{g}_n) = \infty$. Now let $\hat{g}_n = \text{Vol}(M, \tilde{g}_n)^{-\frac{2}{m}} \tilde{g}_n$. Then $\hat{g}_n \in \mathcal{M}(M)$ and

$$\lambda_{1,p}(M, \hat{g}_n) = \text{Vol}(M, \tilde{g}_n)^{\frac{2}{m}} \lambda_{1,p}(M, \tilde{g}_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

(If $M = S^m$, it suffices to construct the metrics $\hat{g}_n = \text{Vol}(S^m, g_n)^{-\frac{2}{m}} g_n \in \mathcal{M}(S^m)$ to obtain $\lim_{n \rightarrow \infty} \lambda_{1,p}(S^m, \hat{g}_n) = \infty$.) This ends the proof of the case $1 < p \leq 2$.

II. The case $p > 2$

In this case the proof follows directly from the following monotonicity result:

Proposition 2.6. *Let (M, g) be a compact Riemannian manifold. Then the map*

$$p \mapsto p \lambda_{1,p}(M, g)^{\frac{1}{p}}$$

is strictly increasing on $(1, \infty)$.

Proof of Proposition 2.6. Let $1 < p < q$ and A_1, A_2 be the nodal domains of an eigenfunction for $\lambda_{1,q}(M, g)$. Then $\lambda_{1,q}(M, g) = \mu_{1,q}(A_1, g) = \mu_{1,q}(A_2, g)$ [8].

For any bounded domain Ω of M , Lindqvist [7] proved that the map $p \mapsto p \mu_{1,p}(\Omega)^{\frac{1}{p}}$ is strictly increasing on $]1, \infty[$. We then have

$$p \mu_{1,p}(A_1, g)^{\frac{1}{p}} < q \mu_{1,q}(A_1, g)^{\frac{1}{q}} \quad \text{and} \quad p \mu_{1,p}(A_2, g)^{\frac{1}{p}} < q \mu_{1,q}(A_2, g)^{\frac{1}{q}}.$$

On the other hand, (1.1) implies $\lambda_{1,p}(M, g) \leq \max\{\mu_{1,p}(A_1, g), \mu_{1,p}(A_2, g)\}$. Hence

$$\begin{aligned} p \lambda_{1,p}(M, g)^{\frac{1}{p}} &\leq \max \left\{ p \mu_{1,p}(A_1, g)^{\frac{1}{p}}, p \mu_{1,p}(A_2, g)^{\frac{1}{p}} \right\} \\ &< \max \left\{ q \mu_{1,q}(A_1, g)^{\frac{1}{q}}, q \mu_{1,q}(A_2, g)^{\frac{1}{q}} \right\} \\ &= q \mu_{1,q}(A_1, g)^{\frac{1}{q}} \\ &= q \lambda_{1,q}(M, g)^{\frac{1}{q}}. \end{aligned}$$

□

Finally, Proposition 2.6 implies that for any $p > 2$,

$$\lambda_{1,p}(M, g) > \left(\frac{2}{p} \right)^p (\lambda_{1,2}(M, g))^{\frac{p}{2}}.$$

Since the functional $g \rightarrow \lambda_{1,2}(M, g)$ is not bounded on $\mathcal{M}(M)$ it follows that for any $p > 2$ the functional $g \rightarrow \lambda_{1,p}(M, g)$ is also not bounded on $\mathcal{M}(M)$. This ends the proof of Theorem 2.2. □

Remark 2.7. Proposition 2.6 provides a simple method for obtaining lower bounds for $\lambda_{1,p}(M, g)$, $p > 2$, in terms of other geometric invariants. For example, it is known that $\lambda_{1,2}(M, g) > \frac{h_{(M,g)}^2}{4}$ where $h_{(M,g)}$ is the Cheeger isoperimetric constant of (M, g) . We then have

$$\lambda_{1,p}(M, g) > \left(\frac{2}{p}\right)^p (\lambda_{1,2}(M, g))^{\frac{p}{2}} > \left(\frac{h_{(M,g)}}{p}\right)^p \quad \text{for any } p > 2.$$

Note that in [8] we proved that $\lambda_{1,p}(M, g) > \left(\frac{h_{(M,g)}}{p}\right)^p$ for any $p > 1$.

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