PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 8, Pages 2401–2408
S 0002-9939(05)07790-7
Article electronically published on March 4, 2005

ALMOST AUTOMORPHIC SOLUTIONS OF SEMILINEAR EVOLUTION EQUATIONS

JEROME A. GOLDSTEIN AND GASTON M. N'GUÉRÉKATA

(Communicated by Carmen C. Chicone)

ABSTRACT. We are concerned with the semilinear differential equation in a Banach space $\mathbb{X}$,

$$x'(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R},$$

where A generates an exponentially stable C_0-semigroup and $F(t, x) : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a function of the form $F(t, x) = P(t)Q(x)$. Under appropriate conditions on P and Q, and using the Schauder fixed point theorem, we prove the existence of an almost automorphic mild solution to the above equation.

1. INTRODUCTION

Consider in a Banach space $(\mathbb{X}, \|\cdot\|)$ the semilinear differential equation

$$x'(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where the linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ generates an exponentially stable C_0-semigroup $\mathcal{T} = (T(t))_{t\geq 0}$; that is, $\mathcal{T}$ satisfies the estimate

$$\|T(t)\| \leq Me^{-\epsilon t}, \tag{1.2}$$

for some constants $M > 0, \epsilon > 0$ and all $t \geq 0$. Let $F : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be jointly continuous. A *mild solution* to (1.1) is a function $x \in C(\mathbb{R}, \mathbb{X})$ satisfying the integral equation

$$x(t) = T(t-a)x(a) + \int_a^t T(t-s)F(s, x(s))ds \tag{1.3}$$

for every $a \in \mathbb{R}$ and every $t \geq a$.

A fundamental problem is the existence of almost automorphic mild solutions to (1.1). Recently, G. M. N'Guérékata [5] showed, using the Banach fixed point theorem, that if

i) F is Lipschitzian in $x \in \mathbb{X}$, uniformly in $t \in \mathbb{R}$, that is,

$$\|F(t, x) - F(t, y)\| \leq L\|x - y\| \tag{1.4}$$

for all $x, y \in \mathbb{X}$, and $t \geq 0$, and L is sufficiently small, namely $L < \dfrac{\epsilon}{M}$, where ϵ and M are as in (1.2), and

Received by the editors February 11, 2004 and, in revised form, April 12, 2004.

2000 *Mathematics Subject Classification.* Primary 34A05, 34K05, 47D60, 34G20.

ii) $F(t, x)$ is almost automorphic in $t \in \mathbb{R}$ for each $x \in \mathbb{X}$,

then problem (1.1) has a unique almost automorphic mild solution.

In this paper, we are going to prove the existence of almost automorphic mild solutions to (1.1), F being not necessarily Lipschitzian. But first, let us recall some definitions.

Definition 1.1. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be *almost automorphic* if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that

$$g(t) = \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

It is well known that the range $\mathcal{R}_f = \{f(t)|t \in \mathbb{R}\}$ of an almost automorphic function f is relatively compact in $\mathbb{X}$, thus bounded in norm (see [6], Theorem 2.13). The function g in the definition is also bounded and strongly measurable. Also, the set $AA(\mathbb{X})$ of all almost automorphic functions $f : \mathbb{R} \to \mathbb{X}$ equipped with the sup-norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|,$$

is a Banach space (see [6], page 20).

Also, given two Banach spaces $(\mathbb{X}_1, \|\cdot\|_1)$ and $(\mathbb{X}_2, \|\cdot\|_2)$, $B(\mathbb{X}_1, \mathbb{X}_2)$ will denote the Banach space of bounded linear operators $L : \mathbb{X}_1 \to \mathbb{X}_2$, $BC(\mathbb{R}, \mathbb{X}_1)$ is the Banach space of all continuous and bounded functions $f : \mathbb{R} \to \mathbb{X}_1$, and $BUC(\mathbb{R}, \mathbb{X}_1)$ is the Banach space of all bounded and uniformly continuous functions $f : \mathbb{R} \to \mathbb{X}_1$.

2. Preliminaries

In this paper $(\mathbb{Y}, |\cdot|)$ will denote a Banach space algebraically contained in $\mathbb{X}$ such that the canonical injection $\mathbb{Y} \to \mathbb{X}$ is compact. An example of such a space $\mathbb{Y}$ is an abstract Sobolev space that we construct as follows:

Let A be as in (1.1), (1.2). By (1.2), $0 \in \rho(A)$, so that the fractional powers $(-A)^\alpha$, $0 < \alpha < 1$, are well defined. Also, since $0 \in \rho(A)$, the norm

$$|f| = \|(-A)^\alpha f\| \tag{2.1}$$

is equivalent to the graph norm

$$\|f\|_\alpha = \|(-A)^\alpha f\| + \|f\|.$$

Now we take $\mathbb{X} = L^p(\Omega)$, where $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain in $\mathbb{R}^n$. Let A be a linear uniformly elliptic operator (with suitable boundary conditions), of order $2m$. Then let $\mathbb{Y}$ be the domain of $(-A)^\alpha$ with norm (2.1); we have

$$W_0^{2m\alpha,p}(\Omega) \subset \mathbb{Y} \subset W^{2m\alpha,p}(\Omega)$$

and the norm $|\cdot|$ in $\mathbb{Y}$ is equivalent to the usual norm in $W^{2m\alpha,p}(\Omega)$. Also, the injection $\mathbb{Y} \to \mathbb{X}$ is compact in this case, by Sobolev embedding.

3. Main results

Now let $\mathbb{Y} = D((-A)^\alpha)$, the domain of $(-A)^\alpha$, with norm

$$|y| = \|(-A)^\alpha y\|, \quad y \in D((-A)^\alpha),$$

where $0 < \alpha < 1$ is fixed. We get

$$|T(t)y| = \|T(t)(-A)^\alpha y\| \le Me^{-\epsilon t}\|(-A)^\alpha y\| = Me^{-\epsilon t}|y| \tag{3.1}$$

for each $y \in \mathbb{Y}$ and every $t \ge 0$, by (1.2).

We also make the following assumptions:

$$F(t,x) = P(t)Q(x), \quad \text{for all} \quad t \in \mathbb{R}, x \in \mathbb{X}, \tag{3.2}$$

where $P(t) \in AA(\mathbb{Z})$ for each $t \in \mathbb{R}$ with $\mathbb{Z} = B(\mathbb{X}, \mathbb{Y})$; P is continuous from $\mathbb{R}$ to $AA(\mathbb{Z})$, and $Q : BC(\mathbb{R}, \mathbb{X}) \to BC(\mathbb{R}, \mathbb{X})$ is continuous and satisfies the estimate

$$\|Q\varphi\|_\infty \le \mathcal{M}(\|\varphi\|_\infty), \tag{3.3}$$

where $\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|$ and $\mathcal{M} \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies

$$\lim_{r\to\infty} \frac{\mathcal{M}(r)}{r} = 0. \tag{3.4}$$

Note that $\mathcal{M}$ can be unbounded but must grow slower than a linear function. Let

$$[P] := \sup_{t\in\mathbb{R}} \|P(t)\|_{\mathbb{Z}} < \infty. \tag{3.5}$$

Define $G : BC(\mathbb{R}, \mathbb{X}) \to BC(\mathbb{R}, \mathbb{Y})$ by

$$(G\varphi)(t) = \int_{-\infty}^{t} T(t-s)F(s, \varphi(s))ds. \tag{3.6}$$

For $\varphi \in BC(\mathbb{R}, \mathbb{X})$, this integral exists. Indeed, we have

$$\begin{aligned} |(G\varphi)(t)| &\le \int_{-\infty}^{t} |T(t-s)||P(t)Q(\varphi(s))|ds \\ &\le \int_{-\infty}^{t} Me^{-\epsilon(t-s)}[P]\mathcal{M}(\|\varphi\|_\infty)ds \end{aligned}$$

using (3.1), (3.3) and (3.5). Consequently

$$\begin{aligned} |G\varphi|_\infty &= \sup_{t\in\mathbb{R}} |(G\varphi)(t)| \\ &\le M\epsilon^{-1}[P]\mathcal{M}(\|\varphi\|_\infty). \end{aligned} \tag{3.7}$$

Continuity of G is straightforward by virtue of continuity of both P and Q. Thus we have

$$G(BC(\mathbb{R}, \mathbb{X})) \subset BC(\mathbb{R}, \mathbb{Y}).$$

Finally, for $0 < \delta \le 1$, let

$$BC^\delta(\mathbb{R}, \mathbb{Y}) \equiv \{f \in BC(\mathbb{R}, \mathbb{Y}) : |f|_{\delta,\mathbb{Y}} < \infty\},$$

where

$$|f|_{\delta,\mathbb{Y}} \equiv \sup_{t\in\mathbb{R}} |f(t)| + \delta \sup_{t,s\in\mathbb{R}, t\ne s} \frac{|f(t) - f(s)|}{|t-s|^\delta}.$$

With the norm $|\cdot|_{\delta,\mathbb{Y}}$, $BC^\delta(\mathbb{R}, \mathbb{Y})$ turns out to be a Banach space of all bounded Hölder continuous $\mathbb{Y}$-valued functions on $\mathbb{R}$ of Hölder exponent δ.

Proposition 3.1. *The function G defined above maps bounded sets of $BC(\mathbb{R},\mathbb{X})$ into bounded sets of $BC^{\delta}(\mathbb{R},\mathbb{Y})$ for any $\delta > 0$ satisfying $\delta < \alpha$, where $0 < \alpha < 1$ is the exponent defining $\mathbb{Y} = D(-A)^{-\alpha}$.*

Proof. The proof is basically a modification of the above remarks. Let $0 < \beta < \alpha$. Then

$$\begin{aligned} |(G\varphi)(t)| &= |\int_{-\infty}^{t} T(t-s)(-A)^{\beta}(-A)^{-\beta}F(s,\varphi(s))ds| \\ &\le \int_{-\infty}^{t} |T(t-s)(-A)^{\beta}||(-A)^{-\beta}P(s)||Q(\varphi(s)|ds. \end{aligned} \tag{3.8}$$

Now, by semigroup theory (see for instance [4]), there exists a constant M_1 such that

$$\|T(r)(-A)^{\beta}\| \le \frac{M_1 e^{-\epsilon r}}{r^{\beta}}$$

for all $r > 0$. Thus we obtain, as previously,

$$|T(r)(-A)^{\beta}| \le M_1 e^{-\epsilon r} r^{-\beta}, \quad r > 0. \tag{3.9}$$

Next, we observe that the function $s \mapsto (-A)^{-\beta}P(s)$ is a uniformly bounded function $\mathbb{R} \to B(\mathbb{X}, D((-A)^{\alpha-\beta})$. Indeed, it is the composition of $P(\cdot) : \mathbb{R} \to B(\mathbb{X}, D((-A)^{\alpha}))$, which is bounded by $[P]$, with $(-A)^{-\beta}$, an isometry from $D((-A)^{\alpha})$ onto $D((-A)^{\alpha-\beta})$. Thus

$$\sup_{t\in\mathbb{R}} \|P(t)\|_{B(\mathbb{X},D((-A)^{\alpha-\beta}))} \le [P].$$

Now combining the estimates in (3.8) and (3.9), we deduce

$$|(G\varphi)(t)| \le \int_{-\infty}^{t} M_1 e^{-\epsilon(t-s)}(t-s)^{-\beta}[P]\mathcal{M}(\|\varphi\|_{\infty})ds.$$

Letting $r = t - s$ in the integral gives

$$|(G\varphi(t)| \le \int_{0}^{\infty} M_1 e^{-r} r^{-\beta}[P]\mathcal{M}(\|\varphi\|)dr;$$

that is,

$$|(G\varphi)(t)| \le C_1(\beta)\mathcal{M}(\|\varphi\|_{\infty}), \tag{3.10}$$

where $C_1(\beta)$ depends on β, M_1, ϵ and $[P]$. Next, for $t_2 > t_1$, we have

$$\begin{aligned} &|(G\varphi)(t_2) - (G\varphi)(t_1)| \\ &\quad \le |(\int_{-\infty}^{t_2} - \int_{-\infty}^{t_1})T(t_2-s)(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))ds| \\ &\qquad + |\int_{-\infty}^{t_1}(T(t_2-s)-T(t_1-s))(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))ds| \\ &\quad \le \int_{t_1}^{t_2} |T(t_2-s)(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\qquad + \int_{-\infty}^{t_1} |(T(t_2-t_1)-I)T(t_1-s)(-A)^{\beta}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\quad = J_1 + J_2. \end{aligned}$$

By the same argument leading to (3.10) we get

$$\begin{aligned} J_1 &\leq \int_0^{t_2-t_1} M_1 e^{-\epsilon r} r^{-\beta}[P]\mathcal{M}(\|\varphi\|_\infty)dr \\ &\leq C_2(\beta)\mathcal{M}(\|\varphi\|_\infty)(t_2-t_1)^{1-\beta}. \end{aligned}$$

Also, we have

$$\begin{aligned} J_2 &\leq \int_{-\infty}^{t_1} |(T(t_2-t_1)-I)(-A)^{-\gamma}(T(t_1-s)(-A)^{(\beta-\gamma)}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\leq \int_{-\infty}^{t_1} |(T(t_2-t_1)-I)(-A)^{-\gamma}| \\ &\qquad \cdot|(T(t_1-s)(-A)^{(\beta-\gamma)}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\leq |(T(t_2-t_1)-I)(-A)^{-\gamma}| \\ &\qquad \cdot\int_{-\infty}^{t_1} |T(t_1-s)(-A)^{(\beta-\gamma)}(-A)^{-\beta}P(s)Q(\varphi(s))|ds \\ &\leq |(T(t_2-t_1)-I)(-A)^{-\gamma}|C_3(\beta,\gamma)\mathcal{M}(\|\varphi\|_\infty) \end{aligned}$$

provided $0<\gamma<\beta$. Next recall that $(T(r)-I)g=\int_0^r T(s)Agds$ for $g\in D(A)$, by the fundamental theorem of calculus. Thus, for $f\in\mathbb{Y}$,

$$\begin{aligned} |(T(r)-I)(-A)^{-\gamma}f| &= \|\int_0^r T(s)(-A)^{1-\gamma-\alpha}(-A)^{\alpha}fds\| \\ &\leq \|(-A)^{\alpha}f\|\int_0^r M_1e^{-\epsilon s}s^{1-\gamma-\alpha}ds \\ &= C_4(\gamma,\epsilon,M_1)r^{2-\gamma-\alpha}|f|, \end{aligned}$$

since $1-\gamma-\alpha>-1$, because $0<\gamma<\beta<\alpha<1$.

In other words, $|(T(r)-I)(-A)^{-\gamma}|\leq C_4 r^{2-\gamma-\alpha}$; consequently,

$$J_2\leq C_4(t_2-t_1)^{2-\gamma-\alpha}C_3\mathcal{M}(\|\varphi\|_\infty).$$

For $\delta=\min(2-\gamma-\alpha,1-\beta)>0$, it follows that

$$|(G\varphi)(t_2)-(G\varphi(t_1)|\leq C_5|t_2-t_1|^{\delta}\mathcal{M}(\|\varphi\|_\infty), \tag{3.11}$$

where C_5 depends on $\epsilon, M_1, [P], \alpha, \beta, \gamma$ and $\mathbb{Y}$, that is, on parameters of the problem.

It follows that, for $\varphi\in BC(\mathbb{R},\mathbb{X})$ with $\|\varphi(t)\|\leq R$ for all $t\in\mathbb{R}$, then $G\varphi\in BC^{\delta}(\mathbb{R},\mathbb{Y})$ with $\|G\varphi(t)\|\leq R_1$ for all $t\in\mathbb{R}$ and some R_1 that depends on R. This completes the proof. $\square$

Proposition 3.2. *The function G maps bounded sets of $AA(\mathbb{X})$ into bounded sets of $BC^{\delta}(\mathbb{R},\mathbb{Y})\cap AA(\mathbb{X})$ for $0<\delta<\alpha$.*

Proof. We just need to check that

$$G(AA(\mathbb{X}))\subset AA(\mathbb{X}).$$

To this end, let $\varphi\in AA(\mathbb{X})$. Then given a sequence $(s'_n)\subset\mathbb{R}$, there exists a subsequence $(s_n)\subset(s'_n)$ such that

$$\psi(t)=\lim_{n\to\infty}\varphi(t+s_n)$$

is well defined for each $t\in\mathbb{R}$ and

$$\lim_{n\to\infty}\psi(t-s_n)=\varphi(t)$$

for each $t \in \mathbb{R}$. Since $\psi \in BC(\mathbb{R}, \mathbb{X})$, then

$$(G\varphi)(t+s_n) = \int_{-\infty}^{t+s_n} T(t+s_n-s)P(s)Q(\varphi(s))ds.$$

Let $\sigma = s - s_n$. Then

$$\begin{aligned}(G\varphi)(t+s_n) &= \int_{-\infty}^{t} T(t-\sigma)P(\sigma+s_n)Q(\varphi(\sigma+s_n))d\sigma \\ &= \int_{-\infty}^{t} T(t-\sigma)P_n(\sigma)Q_n(\sigma)d\sigma,\end{aligned}$$

where $P_n(\sigma) = P(\sigma+s_n), Q_n(\sigma) = Q(\varphi(\sigma+s_n))$, $n = 1, 2, \cdots$, $\sigma \in \mathbb{R}$.

Since $P \in AA(\mathbb{Z})$, there exists a subsequence of (s_n), which we still denote by (s_n), such that

$$\hat{P}(\sigma) = \lim_{n\to\infty} P_n(\sigma)$$

exists for each $\sigma \in \mathbb{R}$ and

$$\lim_{n\to\infty} \hat{P}(\sigma - s_n) = P(\sigma)$$

for each $\sigma \in \mathbb{R}$. Clearly we also have, by passing to a subsequence if necessary,

$$\lim_{n\to\infty} \varphi(t+s_n) = \psi(t)$$

and

$$\lim_{n\to\infty} \psi(t-s_n) = \varphi(t),$$

for each $t \in \mathbb{R}$. By the Bochner integral version of Lebesgue's dominated convergence theorem, we get

$$(G\varphi)(t+s_n) = \int_{-\infty}^{t} T(t-\sigma)P_n(\sigma)Q_n(\sigma)d\sigma$$

$$\longrightarrow \int_{-\infty}^{t} T(t-\sigma)\hat{P}(\sigma)Q(\varphi(\sigma))d\sigma = \chi(t)$$

for each $t \in \mathbb{R}$, and

$$\begin{aligned}\chi(t-s_n) &= \int_{-\infty}^{t-s_n} T(t-s_n-\sigma)\hat{P}(\sigma)Q(\psi(\sigma))d\sigma \\ &= \int_{-\infty}^{t} T(t-r)\hat{P}(r-s_n)Q(\psi(r-s_n))dr\end{aligned}$$

by letting $r = \sigma + s_n$. Thus we obtain

$$\chi(t-s_n) \longrightarrow \int_{-\infty}^{t} T(t-r)P(r)Q(\varphi(r))dr = (G\varphi)(t),$$

again by Lebesgue's dominated convergence theorem. This shows that $G(AA(\mathbb{X})) \subset AA(\mathbb{X})$, and the proof is now complete. □

Proposition 3.3. *$BC^{\delta}(\mathbb{R}, \mathbb{Y})$ is compactly contained in $BC(\mathbb{R}, \mathbb{X})$; in other words, the canonical injection $id : BC^{\delta}(\mathbb{R}, \mathbb{Y}) \to BC(\mathbb{R}, \mathbb{X})$ is compact, which implies that*

$$id : BC^{\delta}(\mathbb{R}, \mathbb{Y}) \cap AA(\mathbb{X}) \to AA(\mathbb{X})$$

is compact too.

Proof. We show that id maps bounded sets of $BC^\delta(\mathbb{R},\mathbb{Y})$ into relatively compact sets of $BC(\mathbb{R},\mathbb{X})$. To this end, let (φ_ν) be a bounded sequence in $BC^\delta(\mathbb{R},\mathbb{Y})$. Let $\mathbb{Q}=\{r_n\}$ be the set of all rational numbers. Then $(\varphi_\nu(r_n))$ is a bounded sequence in $\mathbb{Y}$, for each n. By the well-known Cantor diagonalization process, there exists a subsequence (φ_{ν_k}) such that

$$\varphi_{\nu_k}(r_n)\to\varphi(r_n),$$

as $k\to\infty$ in $\mathbb{X}$, for each n, and some $\varphi:\mathbb{Q}\to\mathbb{X}$. But the sequence (φ_n) is an equicontinuous family in $BUC(\mathbb{R},\mathbb{Y})\subset BUC(\mathbb{R},\mathbb{X})$, because of the uniform Hölder condition. Thus, as in the proof of the Arzela-Ascoli theorem, there is a further subsequence (which we still denote by (φ_{ν_k})) satisfying

$$\varphi_{\nu_k}(t)\to\varphi(t),\ \text{as } k\to\infty \tag{3.12}$$

in $\mathbb{X}$, for all $t\in\mathbb{R}$. In addition the convergence is uniform in $t\in\mathbb{R}$. Note that $BUC(\mathbb{R},\mathbb{X})$ can be identified with $C(K,\mathbb{X})$ for a suitable Hausdorff compactification K of $\mathbb{R}$ (see for instance [3]). Thus the convergence $\varphi_{\nu_k}\to\varphi$ holds in $BUC(\mathbb{R},\mathbb{X})\subset BC(\mathbb{R},\mathbb{X})$. This completes the proof. $\square$

Proposition 3.4. *The function G has a fixed point in $AA(\mathbb{X})$.*

Proof. Let us recall that the estimates (3.10)-(3.11), $|G\varphi|_\infty\le C_1(\beta)\mathcal{M}(\|\varphi\|_\infty)$ and $|(G\varphi)(t_2)-(G\varphi(t_1)|\le C_5|t_2-t_1|\delta\mathcal{M}(\|\varphi\|_\infty)$, hold for all $\varphi\in BC(\mathbb{R},\mathbb{Y})$ and all $t_1,t_2\in\mathbb{R}$ with t_2 not equal to t_1. It follows that there exists a constant $C_6=C_6(\epsilon,M,M_1,\alpha,\beta,\gamma)$ such that

$$\begin{aligned}\varphi\in BC(\mathbb{R},\mathbb{X})\quad&\text{and}\quad\|\varphi\|_\infty<R\text{ imply}\\ G\varphi\in BC^\delta(\mathbb{R},\mathbb{Y})\quad&\text{and}\quad|G\varphi|<R_1,\end{aligned}$$

where $R_1=C_6\mathcal{M}(R)$.

Since $\mathcal{M}(R)/R\to0$ as $R\to\infty$, and since $\|y\|\le C_7|y|$ holds for some constant C_7 and all $y\in\mathbb{Y}$, it follows that there exists $\rho>0$ such that for all $R\ge\rho$, we have

$$G(B_{AA(\mathbb{X})}(0,R))\subset B_{BC^\delta(\mathbb{R},\mathbb{Y})}(0,R)\cap B_{AA(\mathbb{X})}(0,R). \tag{3.13}$$

Since G leaves $AA(\mathbb{X})\subset BC(\mathbb{R},\mathbb{X})$ invariant, the estimate (3.13) along with the continuity properties of G imply that G is a continuous, compact mapping $S\to S$, where S is the ball of radius R in $AA(\mathbb{X})$ and $R\ge\rho$. By the Schauder fixed point theorem, G has a fixed point in S, φ_0. Obviously, φ_0 is a mild solution of (1.1). $\square$

Finally, the above results can be summarized as follows.

Theorem 3.5. *Let A generate an exponentially stable C_0-semigroup $\mathcal{T}$ in $\mathcal{B}(\mathbb{X})$. Assume assumptions* (1.1) *and* (3.2)-(3.5). *Then* (1.1) *has a mild solution in $AA(\mathbb{X})$.*

Now we end this paper with the following

Example of nonuniqueness. Let $\mathbb{X}=\mathbb{R}$, $A=-1$ and

$$u(t)=\begin{cases}t^{3/2}e^{1-t}, & \text{for } t\in[0,\frac32],\\ 0, & \text{for } t\in[-\frac32,0].\end{cases}$$

Then for $t\in[0,\frac32]$ we have

$$u'(t)=-u(t)+\frac32t^{1/2}e^{(1-t)}=-u(t)+\frac32u(t)^{1/3}e^{\frac23(1-t)}=-u(t)+f(t,u(t))$$

where

$$f(t,\varphi)=\begin{cases}\frac{3}{2}\varphi^{1/3}e^{\frac{2}{3}(1-t)}, & \text{for } t\in[0,\frac{3}{2}]\times\mathbb{R},\\ \frac{3}{2}\varphi^{1/3}e^{2/3}, & \text{for } t\in[-\frac{3}{2},0]\times\mathbb{R}.\end{cases}$$

Note that $u'(\frac{3}{2})=0$ and $u(\frac{3}{2})=(\frac{3}{2})^{\frac{3}{2}}e^{-\frac{3}{2}}$.

Now let $f(t,\varphi)=f(\frac{3}{2},\varphi)$ on $[\frac{3}{2},3]\times\mathbb{R}$ and $f(t,\varphi)=f(\frac{9}{2}-t,\varphi)$ on $[3,\frac{9}{2}]\times\mathbb{R}$; let $u(t)=u(\frac{3}{2})$ on $[\frac{3}{2},3]$, and $u(t)=u(\frac{9}{2}-t)$ on $[3,\frac{9}{2}]$. Then $u'=-u+f(t,u)$ on $[-\frac{3}{2},\frac{9}{2}]$, together with $u(0)=0$.

Extend u to be a periodic function of period 6 (hence an almost automorphic function). Then u and $v\equiv 0$ both satisfy

$$\frac{dx}{dt}=-x+f(t,x),\quad x(0)=0.$$

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Department of Mathematical Sciences, University of Memphis, Memphis, Tennessee 38152-3240

E-mail address: jgoldste@memphis.edu

Department of Mathematics, Morgan State University, Baltimore, Maryland 21251

E-mail address: gnguerek@jewel.morgan.edu