WEAK L^1 NORMS OF RANDOM SUMS

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ABSTRACT. Let $\{g_j\}$ denote a sequence of measurable functions on $\mathbf{R}^n$, and let $\|\cdot\|_{WL^1}$ denote the weak L^1 norm. It is shown that

$$\left\| \mathbb{E} \left(\sum_{j=1}^N \epsilon_j g_j \right) \right\|_{WL^1} \lesssim \sum_{j=1}^N \|g_j\|_{WL^1},$$

where $\{\epsilon_j\}$ is a sequence of independent random variables taking on values $+1$ and -1 with equal probability. Moreover, it is shown that

$$\left\| \mathbb{E} \left(\sum_{j=1}^N \epsilon_j g_j \right) \right\|_{WL^1} \lesssim \mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{WL^1} \right).$$

The paper concludes by providing an example indicating that, if $\|g_1\|_{WL^1} = \cdots = \|g_N\|_{WL^1} = 1$, then the estimate

$$\mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{WL^1} \right) \lesssim N \log N$$

is the best possible.

1. INTRODUCTION

The weak L^1 norm of a measurable function f supported on $\mathbf{R}^n$ is defined by

$$\|f\|_{WL^1} = \sup_{\alpha > 0} \alpha |\{x \in \mathbf{R}^n : |f(x)| > \alpha\}|.$$

This norm is of frequent occurrence in modern-day mathematics, finding use in probability theory, harmonic analysis, and ergodic theory. In spite of its many applications, however, the weak L^1 norm has one decidedly inconvenient drawback: it is not a “norm” in the strictest sense of the word as it does not satisfy the triangle inequality. For example, the functions $\frac{1}{x}\chi_{(0,1)}(x)$ and $\frac{1}{1-x}\chi_{(0,1)}(x)$ both have weak L^1 norms of 1, but their sum has a weak L^1 norm of 4.

The best-known result relating the weak L^1 norm of a sum of functions to their respective individual weak L^1 norms is the following due to E. M. Stein and N. J. Weiss [1].

Theorem 1. *Let $\{g_j\}$ denote a sequence of measurable functions on $\mathbf{R}^n$ such that $\|g_j\|_{WL^1} = 1$ for each j . Let $\{c_j\}$ be a sequence of positive numbers with $\sum_j c_j = 1$,*

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let $s > 0$, and set $K = \sum_j c_j \log \left(\frac{1}{c_j} \right)$. Then $\left| \left\{ x \in \mathbf{R}^n : \sum_j c_j g_j(x) > s \right\} \right| < \frac{2(K+2)}{s}$.

The bound of $\left\| \sum c_j g_j \right\|_{WL^1}$ in this theorem is essentially optimal: the log term may not be removed. For example, letting $c_1 = \cdots = c_N = \frac{1}{N}$ and $g_j(x) = \frac{1}{2} \frac{1}{|x - \frac{j}{N}|}$, one may readily check that $\|g_j\|_{WL^1} = 1$ for each j and that

$$\left\| \sum_{j=1}^N c_j g_j(x) \right\|_{WL^1} \sim \sum_{j=1}^N c_j \log \left(\frac{1}{c_j} \right) \sim \log N.$$

Given the prevalence of randomization techniques in modern harmonic analysis, it is natural to conduct the admittedly Baconian experiment of determining the expected value of $\left\| \sum_{j=1}^N \epsilon_j c_j g_j(x) \right\|_{WL^1}$, where the c_j and $g_j(x)$ are as above and $\{\epsilon_j\}$ is a sequence of independent random variables taking on the values of $+1$ and -1 with equal probability. It turns out that a large majority of the 2^N associated weak L^1 norms are close to one, exhibiting significant improvement over the log N bound above because of cancellation between the $\epsilon_j c_j g_j$'s. In fact, one can calculate that $\mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j c_j g_j(x) \right\|_{WL^1} \right) \sim 1$. Although not difficult, the techniques involved in this calculation are not immediately transparent and are accordingly given below.

Example 1. Let $c_1 = \cdots = c_N = \frac{1}{N}$ and $g_j(x) = \frac{1}{2} \frac{1}{|x - \frac{j}{N}|}$. Then

$$\mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j c_j g_j(x) \right\|_{WL^1} \right) \sim \sum_{j=1}^N \frac{1}{N} = 1.$$

We first observe that

$$2^{2^N} \left| \left\{ x : \sum_{j=1}^N |\epsilon_j c_j g_j(x)| > 2^{2^N} \right\} \right| \sim 2^{2^N} \cdot N \cdot N^{-1} \cdot 2^{-2^N} = 1$$

for any sequence $\{\epsilon_j\}$ consisting of -1 's and $+1$'s, and hence the result that

$$\mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j c_j g_j(x) \right\|_{WL^1} \right) \gtrsim \sum_{j=1}^N \frac{1}{N} = 1$$

easily holds. We accordingly turn our attention to the opposite inequality.

Let $\tilde{g}_j(x) = g_j(x) \chi_{\{|x - \frac{j}{N}| > 2N\}}(x)$. Let $h_j(x) = g_j(x) - \tilde{g}_j(x)$. Now,

$$\left\| \sum_{j=1}^N \epsilon_j c_j g_j \right\|_{WL^1} \leq 2 \left[\left\| \sum_{j=1}^N \epsilon_j c_j \tilde{g}_j \right\|_{WL^1} + \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{WL^1} \right].$$

As the $\tilde{g}_j$ have disjoint supports,

$$\left\| \sum_{j=1}^N \epsilon_j c_j \tilde{g}_j \right\|_{WL^1} \leq \sum_{j=1}^N \|c_j \tilde{g}_j\|_{WL^1} \leq 1.$$

So it suffices to show that $\mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{WL^1} \right) \lesssim 1$.

Note that, letting $E = [-1, 2]$ and $\tilde{E} = (-\infty, -1) \cup (2, \infty)$, we have that

$$\begin{aligned} \mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{WL^1} \right) \\ \leq \mathbb{E} \left(\left\| \left(\sum_{j=1}^N \epsilon_j c_j h_j \right) \chi_E \right\|_{WL^1} \right) \\ + \mathbb{E} \left(\left\| \left(\sum_{j=1}^N \epsilon_j c_j h_j \right) \chi_{\tilde{E}} \right\|_{WL^1} \right). \end{aligned}$$

Now, for $x \in \tilde{E}$, $\left| \sum_{j=1}^N \epsilon_j c_j h_j(x) \right| \sim \left| \left(\sum_{j=1}^N \epsilon_j \right) \cdot \frac{1}{|Nx|} \right| \leq \frac{1}{|x|}$, and so

$$\mathbb{E} \left(\left\| \left(\sum_{j=1}^N \epsilon_j c_j h_j \right) \chi_{\tilde{E}} \right\|_{WL^1} \right) \lesssim 1.$$

Hence it suffices to show that

$$\mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j c_j h_j \chi_{[-1,2]} \right\|_{WL^1} \right) \lesssim 1.$$

To see this, note that

$$\begin{aligned} \mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j c_j h_j \chi_{[-1,2]} \right\|_{WL^1} \right) &\leq \mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j c_j h_j \chi_{[-1,2]} \right\|_{L^1} \right) \\ &= \left\| \mathbb{E} \left[\sum_{j=1}^N \epsilon_j c_j h_j \chi_{[-1,2]} \right] \right\|_{L^1} \\ &\lesssim \left(\sum_{j=1}^N |c_j h_j(1)|^2 \right)^{1/2} \\ &\lesssim \left(\sum_{j=1}^N \left(\frac{1}{j} \right)^2 \right)^{1/2} \sim 1, \end{aligned}$$

as desired.

It is natural at this point to consider whether or not this example is indicative of more general inequalities involving the expectation operator and the weak L^1 norm. The purpose of this paper is to show that the techniques involved in the above example may indeed be generalized to prove that, for $\{g_j\}$ and $\{c_j\}$ satisfying the hypotheses of Theorem 1, $\|\mathbb{E}(|\sum c_j g_j|)\|_{WL^1} \lesssim \sum c_j$ and $\|\mathbb{E}(|\sum \epsilon_j c_j g_j|)\|_{WL^1} \lesssim \mathbb{E}(\|\sum \epsilon_j c_j g_j\|_{WL^1})$. It will also be shown, however, that the *a priori* Stein-Weiss estimate $\mathbb{E} \|\sum \epsilon_j c_j g_j\|_{WL^1} \lesssim \sum c_j \log \left(\frac{1}{c_j} \right)$ is the best possible.

2. WEAK L^1 NORMS OF EXPECTATIONS OF RANDOM SUMS

This section is devoted to the proof of the following.

Theorem 2. *Let $\{g_j\}$ denote a sequence of measurable functions on $\mathbf{R}^n$, and let $\{\epsilon_j\}$ be a sequence of independent random variables taking on values $+1$ and -1 with equal probability. Then*

$$\left\| \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j(x) \right| \right) \right\|_{WL^1} \lesssim \sum_{j=1}^N \|g_j\|_{WL^1}.$$

Proof. We first note that

$$\mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j(x) \right| \right) \sim \left(\sum_{j=1}^N |g_j(x)|^2 \right)^{1/2},$$

as follows from elementary probability theory. So it suffices to show that

$$\left\| \left(\sum_{j=1}^N |g_j(x)|^2 \right)^{1/2} \right\|_{WL^1} \lesssim \sum_{j=1}^N \|g_j\|_{WL^1}.$$

Let $\alpha > 0$. Let $g_{j,\alpha}(x) = g_j(x) \chi_{\{|g_j(x)| \leq \alpha\}}(x)$. Note that

$$(1) \quad \left\{ x : \left(\sum_{j=1}^N |g_j(x)|^2 \right)^{1/2} > \alpha \right\} \subset \left(\bigcup_j \{x : |g_j(x)| > \alpha\} \right) \cup \left\{ x : \sum_{j=1}^N |g_{j,\alpha}(x)|^2 > \alpha^2 \right\}.$$

By the definition of the weak L^1 norm, we have

$$(2) \quad |\{x : |g_j(x)| > \alpha\}| \leq \frac{1}{\alpha} \|g_j\|_{WL^1}.$$

Also,

$$\left| \left\{ x : \sum_{j=1}^N |g_{j,\alpha}(x)|^2 > \alpha^2 \right\} \right| \leq \frac{1}{\alpha^2} \sum_{j=1}^N \int_{\mathbf{R}^n} |g_{j,\alpha}|^2 dx$$

and

$$\begin{aligned} \int_{\mathbf{R}^n} |g_{j,\alpha}(x)|^2 dx &\leq \int_{y=\frac{1}{\alpha}\|g_j\|_{WL^1}}^{\infty} \left(\frac{\|g_j\|_{WL^1}}{y} \right)^2 dy \\ &\lesssim \|g_j\|_{WL^1}^2 \cdot \left[-y^{-1} \right]_{\frac{1}{\alpha}\|g_j\|_{WL^1}}^{\infty} \\ &\sim \alpha \|g_j\|_{WL^1}. \end{aligned}$$

Hence $\left| \left\{ x : \sum_{j=1}^N |g_{j,\alpha}(x)|^2 > \alpha^2 \right\} \right| \lesssim \frac{1}{\alpha^2} \cdot \alpha \sum_{j=1}^N \|g_j\|_{WL^1} = \frac{1}{\alpha} \sum_{j=1}^N \|g_j\|_{WL^1}$.

This inequality together with (1), (2) implies that $\left| \left\{ x : \left(\sum_{j=1}^N |g_j(x)|^2 \right)^{1/2} > \alpha \right\} \right| \lesssim \frac{1}{\alpha} \sum_{j=1}^N \|g_j\|_{WL^1}$. So $\left\| \left(\sum_{j=1}^N |g_j|^2 \right)^{1/2} \right\|_{WL^1} \lesssim \sum_{j=1}^N \|g_j\|_{WL^1}$, as desired. $\square$

3. EXPECTATIONS OF WEAK L^1 NORMS OF RANDOM SUMS

The theory involving the relationship between the expectation operator and the weak L^1 norm would be very nice indeed if the comparability $\left\| \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j \right| \right) \right\|_{WL^1} \sim \mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{WL^1} \right)$ held. In this section we shall see that the inequality $\left\| \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j \right| \right) \right\|_{WL^1} \lesssim \mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{WL^1} \right)$ does hold. However, afterwards we shall provide an example indicating that the opposite inequality is false.

Theorem 3. *Let $\{g_j\}$ denote a sequence of measurable functions on $\mathbf{R}^n$, and let $\{\epsilon_j\}$ be a sequence of independent random variables taking on values $+1$ and -1 with equal probability. Then*

$$\left\| \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j \right| \right) \right\|_{WL^1} \lesssim \mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{WL^1} \right).$$

Proof. It suffices to show that there exist universal constants $0 < c, C < \infty$ such that, for any $\alpha > 0$,

$$(3) \quad \left| \left\{ x : \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j(x) \right| \right) > \alpha \right\} \right| \leq C \mathbb{E} \left(\left| \left\{ x : \left| \sum_{j=1}^N \epsilon_j g_j(x) \right| > c\alpha \right\} \right| \right).$$

To see this, suppose (3) held. Let β be such that $\beta \cdot \left| \left\{ x : \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j(x) \right| \right) > \beta \right\} \right| \sim \left\| \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j \right| \right) \right\|_{WL^1}$. (We may assume that $\left\| \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j \right| \right) \right\|_{WL^1} < \infty$ without loss of generality.) Then

$$\begin{aligned} \left\| \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j \right| \right) \right\|_{WL^1} &\sim \beta \cdot \left| \left\{ x : \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j(x) \right| \right) > \beta \right\} \right| \\ &\leq C\beta \mathbb{E} \left(\left| \left\{ x : \left| \sum_{j=1}^N \epsilon_j g_j(x) \right| > c\beta \right\} \right| \right) \\ &= C \mathbb{E} \left(\beta \left| \left\{ x : \left| \sum_{j=1}^N \epsilon_j g_j(x) \right| > c\beta \right\} \right| \right) \\ &\leq C \mathbb{E} \left(\frac{1}{c} \left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{WL^1} \right) \\ &\leq \frac{C}{c} \mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{WL^1} \right), \end{aligned}$$

the desired result.

We accordingly turn our attention to (3). To prove (3), it suffices to show that there exist universal constants $0 < c, C < \infty$ such that

$$(4) \quad \left| \left\{ x : \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j g_j(x) \right| \right) > 1 \right\} \right| \leq C \mathbb{E} \left(\left| \left\{ x : \left| \sum_{j=1}^N \epsilon_j g_j(x) \right| > c \right\} \right| \right).$$

The proof of (4) will be facilitated by the use of Rademacher functions.

Definition 1. Let $r_n(t)$ denote the Rademacher functions on $\mathbf{R}$, defined by

$$r_n(t) = r_0(2^n t),$$

where $r_0(t) = 1$ if $0 \leq t \leq \frac{1}{2}$, -1 if $\frac{1}{2} < t < 1$, and $r_0(t+1) = r_0(t)$.

We will also need the following lemma:

Lemma 1 ([2]). *Let $\sum_{j=0}^{\infty} |a_j|^2 < \infty$, and let $F(t) = \sum_{j=0}^N a_j r_j(t)$ be a Rademacher series. Let $0 < p < \infty$. Then there exist finite, positive constants $A(p)$, $B(p)$ such that*

$$A(p) \|F\|_{L^p([0,1])} \leq \left(\sum_{j=0}^N |a_j|^2 \right)^{1/2} \leq B(p) \|F\|_{L^p([0,1])}.$$

We now fix x and let $a_j = g_j(x)$. To prove (4), it suffices to show that

$$(5) \quad \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j a_j \right| \right) = 1 \text{ implies that } \left| \left\{ t \in [0, 1] : \left| \sum_{j=1}^N a_j r_j(t) \right| > \frac{1}{2} \right\} \right| \geq \tilde{c}$$

for some universal constant $0 < \tilde{c} < \infty$. Well, suppose $\mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j a_j \right| \right) = 1$. Then

$$(6) \quad \left\| \sum_{j=1}^N a_j r_j(t) \right\|_{L^1([0,1])} = 1.$$

By the above lemma we then realize that for some universal constant $0 < C_2 < \infty$ we have

$$(7) \quad \left\| \sum_{j=1}^N a_j r_j(t) \right\|_{L^2([0,1])} \leq C_2.$$

Equations (6) and (7) imply that $\left| \sum_{j=1}^N a_j r_j(t) \right| > \frac{1}{2}$ on a set of measure at least $\tilde{c} > 0$, where $\tilde{c}$ is a universal constant, and so (5) holds, as desired. $\square$

We now provide an example indicating that the inequality opposite to that of Theorem 3 is false. Let $\phi_1, \phi_2, \dots, \phi_{2^N}$ denote the 2^N functions from $\{1, \dots, N\}$ to $\{-1, +1\}$. For $j = 1, \dots, N$, let

$$h_j(x) = \sum_{k=1}^{2^N} \phi_k(j) 2^k \chi_{[2^{-(k+1)}, 2^{-k})}(x).$$

Note that

$$\mathbb{E} \left(\left| \sum_{k=1}^N \epsilon_k h_k(x) \right| \right) \sim N^{1/2} \sum_{k=1}^{2^N} 2^k \chi_{[2^{-(k+1)}, 2^{-k})}(x),$$

and so $\left\| \sum_{k=1}^N \epsilon_k h_k \right\|_{W^{L^1}} \sim N^{1/2}$. However, if $\{\epsilon_j\}_{j=1}^N$ is a sequence of -1 's and $+1$'s, $\phi_\ell(j) = \epsilon_j$ for some $\ell \in \{1, \dots, 2^N\}$. Hence $\sum_{j=1}^N \epsilon_j h_j(x) = N \cdot 2^\ell$ on $[2^{-(\ell+1)}, 2^{-\ell})$, and so $\left\| \sum_{k=1}^N \epsilon_k h_k \right\|_{W^{L^1}} \gtrsim N$. As $\{\epsilon_k\}_{k=1}^N$ is an arbitrary sequence

of -1 's and $+1$'s, we see that $\mathbb{E} \left(\left\| \sum_{k=1}^N \epsilon_k h_k \right\|_{WL^1} \right) \gtrsim N$. As N may be arbitrarily large, we see that the inequality opposite to that of Theorem 3 cannot hold.

4. A COUNTEREXAMPLE REGARDING EXPECTATIONS OF WEAK L^1 NORMS OF RANDOM SUMS

Perhaps the most interesting inequality we could wish for regarding expectations of weak L^1 norms of random sums would be that $\mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{WL^1} \right) \lesssim \sum_{j=1}^N \|g_j\|_{WL^1}$, providing via Theorem 3 an improvement to Theorem 2. Note that the functions h_j constructed at the end of the previous section, although not satisfying $\mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j h_j \right\|_{WL^1} \right) \lesssim \left\| \mathbb{E} \left(\left| \sum_{j=1}^N \epsilon_j h_j \right| \right) \right\|_{WL^1}$, do satisfy such an estimate. However, we shall now construct a counterexample indicating that the inequality $\mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{WL^1} \right) \lesssim \sum_{j=1}^N \|g_j\|_{WL^1}$ in general does not hold, and that, if $\{c_j\}$ is a sequence of positive numbers such that $\sum c_j = 1$ and $\{g_j\}$ is a sequence of functions each having a weak L^1 norm of 1, then the estimate $\mathbb{E} \left(\left\| \sum \epsilon_j c_j g_j \right\|_{WL^1} \right) \lesssim \sum c_j \log \left(\frac{1}{c_j} \right)$ following from Theorem 1 is the best possible.

Theorem 4. *Let $N \geq 2$ be a positive integer. Then there exist functions $g_1, g_2, \dots, g_N$ supported on $[0, 1]$ such that $\|g_j\|_{WL^1} \sim 1$ for each j and such that*

$$(8) \quad \mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{WL^1} \right) \sim N \log N.$$

Proof. Define the function $h_{1,1}(x)$ on $[0, 1]$ by

$$h_{1,1}(x) = \begin{cases} \left(\frac{1}{N} + \left(\frac{1-\frac{1}{N}}{N} \right) \cdot j \right)^{-1} & \text{if } \frac{1}{N} + \frac{1-\frac{1}{N}}{N} \cdot (j-1) < x \leq \frac{1}{N} + \frac{(1-\frac{1}{N})}{N} j \\ & \text{for some } j = 1, \dots, N, \\ 0 & \text{otherwise.} \end{cases}$$

For $m = 2, \dots, N$ we define the functions $h_{m,1}(x)$ by

$$h_{m,1}(x) = \begin{cases} \left(\frac{1}{N} + \left(\frac{1-\frac{1}{N}}{N} \right) \cdot (N-m+j+1) \right)^{-1} & \text{if } \frac{1}{N} + \frac{1-\frac{1}{N}}{N} \cdot (j-1) < x \\ & \leq \frac{1}{N} + \frac{(1-\frac{1}{N})}{N} j \\ & \text{for some } j = 1, \dots, m-1, \\ \left(\frac{1}{N} + \left(\frac{1-\frac{1}{N}}{N} \right) \cdot (j-m+1) \right)^{-1} & \text{if } \frac{1}{N} + \frac{1-\frac{1}{N}}{N} \cdot (j-1) < x \\ & \leq \frac{1}{N} + \frac{(1-\frac{1}{N})}{N} j \\ & \text{for some } j = m, \dots, N, \\ 0 & \text{otherwise.} \end{cases}$$

For $k \geq 2$ and $m = 1, \dots, N$, let $h_{m,k}(x) = N^{k-1} h_{m,1}(N^{k-1}x)$. Note that $h_{m,k}$ is supported on $\left[\left(\frac{1}{N} \right)^k, \left(\frac{1}{N} \right)^{k-1} \right]$.

Now, we again let $\phi_1, \phi_2, \dots, \phi_{2^N}$ denote the 2^N functions from $\{1, \dots, N\}$ to $\{-1, 1\}$. For $k = 1, \dots, N$ we define the functions $g_k(x)$ by

$$g_k(x) = \sum_{j=1}^{2^N} \phi_j(k) h_{k,j}(x).$$

Note that $\|g_j\|_{WL^1} \sim \left\| \frac{1}{x} \right\|_{WL^1} \sim 1$ for $j = 1, \dots, N$. Now, if $\{\epsilon_j\}_{j=1}^N$ is a sequence of -1 's and $+1$'s, $\{\epsilon_j\}_{j=1}^N = \{\phi_\ell(j)\}_{j=1}^N$ for some $1 \leq \ell \leq 2^N$. So for $x \in (N^{-\ell}, N^{-\ell+1})$ we have

$$\sum_{j=1}^N \epsilon_j g_j(x) \sim N^\ell \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N} \right) \sim N^\ell \log N.$$

So

$$\left\| \left(\sum_{j=1}^N \epsilon_j g_j \right) \chi_{[N^{-\ell}, N^{-\ell+1}]} \right\|_{WL^1} \sim N^{-\ell+1} (N^\ell \log N) = N \log N.$$

Since $\{\epsilon_j\}_{j=1}^N$ is an arbitrary sequence of -1 's and $+1$'s, we may conclude that $\mathbb{E} \left(\left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{WL^1} \right) \sim N \log N$, as desired.

□

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