ON MORDELL-TORNHEIM ZETA VALUES

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ABSTRACT. We prove that the Mordell-Tornheim zeta value of depth r can be expressed as a rational linear combination of products of the Mordell-Tornheim zeta values of lower depth than r when r and its weight are of different parity.

1. INTRODUCTION

For complex numbers $s_1, s_2, \dots, s_r, s$, Matsumoto defined the Mordell-Tornheim r -fold zeta function $\zeta_{MT,r}(s_1, s_2, \dots, s_r; s)$ by

$$(1.1) \quad \sum_{m_1, m_2, \dots, m_r=1}^{\infty} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s},$$

where the sum is over r -tuples of positive integers (see [4, 5]). He showed that this function can be continued meromorphically to the whole r -dimensional complex space. The origin of this function goes back to Tornheim and Mordell. Tornheim investigated the properties of $\zeta_{MT,2}(k_1, k_2; k)$ for positive integers k_1, k_2, k and discovered some relations (see [8]). Later Mordell independently considered $\zeta_{MT,2}(k, k; k)$ for any even positive integer k and also studied the values

$$\sum_{m_1, m_2, \dots, m_r=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_r (m_1 + \cdots + m_r + a)}$$

with $a > r$ (see [6], see also [7]). By using this result, Hoffman gave some evaluation formulas for $\zeta_{MT,r}(1, \dots, 1; k)$ for any positive integer k (see [1]).

In the present paper, we aim to consider $\zeta_{MT,r}(k_1, \dots, k_r; k)$ for positive integers $k_1, \dots, k_r, k$ with $k \geq 2$. We call it the Mordell-Tornheim zeta value of depth r and of weight $\sum_{j=1}^r k_j + k$ as well as the multiple zeta values defined by

$$(1.2) \quad \sum_{1 \leq m_1 < m_2 < \cdots < m_r} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}}$$

for positive integers $k_1, k_2, \dots, k_r$ with $k_r \geq 2$ (see, for example, [1]).

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The main aim of this paper is to prove

Theorem 1.1. *The Mordell-Tornheim zeta value of depth r with $r \geq 2$ and of weight w can be expressed as a rational linear combination of products of the Mordell-Tornheim zeta values of lower depth than r , when its depth r and its weight w are of different parity.*

Note that this theorem is an analogue of the result on multiple zeta values, which was proved by Zagier (see [3]), and has recently been proved in a different method by the author (see [9]).

The case $r = 2$ of Theorem 1.1 was proved in [8] and the explicit formulas were given in [2]. The case $r = 3$ was proved in [10].

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2. PRELIMINARIES

We use the same notation as in [9]. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, and $\mathbb{R}$ the field of real numbers. Throughout this paper we fix $\delta \in \mathbb{R}$ with $\delta > 0$. For $u \in \mathbb{R}$ with $1 \leq u \leq 1 + \delta$, we define

$$(2.1) \quad \phi(s; u) = \sum_{m=1}^{\infty} \frac{(-u)^{-m}}{m^s} \quad (s \in \mathbb{R}).$$

If $u > 1$, then $\phi(s; u)$ is convergent for any $s \in \mathbb{R}$. In the case when $u = 1$, let $\phi(s) = \phi(s; 1) = (2^{1-s} - 1)\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. Corresponding to $\phi(s; u)$, we define a set of numbers $\{\varepsilon_m(u)\}$ by

$$(2.2) \quad F(x; u) = \frac{(1+u)e^x}{e^x + u} = \sum_{m=0}^{\infty} \varepsilon_m(u) \frac{x^m}{m!} \quad (|x| < \pi).$$

Note that

$$(2.3) \quad \varepsilon_{2j}(1) = 0 \quad (j \in \mathbb{N}).$$

It follows from (2.2) that if $u \in [1, 1 + \delta]$, then

$$(2.4) \quad \liminf_{m \rightarrow \infty} \left(\frac{|\varepsilon_m(u)|}{m!} \right)^{-1/m} \geq \pi \quad \text{and} \quad \frac{|\varepsilon_n(u)|}{n!} \leq \frac{M}{\gamma^n} \quad (n \in \mathbb{N}_0)$$

for any γ with $0 < \gamma < \pi$, where M is the constant independent of n and u .

$$(2.5) \quad \phi(-k; u) = -\frac{1}{1+u} \varepsilon_k(u)$$

for $k \in \mathbb{N}_0$ and $u \in (1, 1 + \delta]$ (see [9], Lemma 1). For simplicity we let

$$(2.6) \quad \mathcal{R}_n(\theta; k_1, \dots, k_n; u) = \sum_{m_1, \dots, m_n=1}^{\infty} \frac{(-u)^{-\sum_{\nu=1}^n m_{\nu}} e^{i(\sum_{\nu=1}^n m_{\nu})\theta}}{m_1^{k_1} \cdots m_n^{k_n}}$$

for $k_1, \dots, k_n \in \mathbb{N}$, $u \in [1, 1 + \delta]$ and $\theta \in \mathbb{R}$, where $i = \sqrt{-1}$. We define $\lambda_j = (1 + (-1)^j)/2$ for $j \in \mathbb{Z}$ and

$$(2.7) \quad H(\theta; k; u) = \mathcal{R}_1(\theta; k; u) - \sum_{\nu=0}^k \phi(k - \nu; u) \lambda_{k-\nu} \frac{(i\theta)^{\nu}}{\nu!}$$

for $k \in \mathbb{N}$, $\theta \in \mathbb{R}$ and $u \in [1, 1 + \delta]$. It follows from (2.5) and (2.6) that

$$(2.8) \quad H(\theta; k; u) = \sum_{\nu=0}^k \phi(k - \nu; u) \lambda_{k-\nu+1} \frac{(i\theta)^\nu}{\nu!} - \frac{1}{1+u} \sum_{N=1}^{\infty} \varepsilon_N(u) \frac{(i\theta)^{N+k}}{(N+k)!},$$

when $u \in (1, 1 + \delta]$. From (2.4) we can see that each side of (2.8) is uniformly convergent with respect to $u \in (1, 1 + \delta]$ if $\theta \in (-\pi, \pi)$ and $k \in \mathbb{N}$. So (2.8) holds for $u = 1$. Hence it follows from (2.3) that if k is even (resp. odd), then $H(\theta; k; 1)$ is odd (resp. even) function. Namely

$$(2.9) \quad H(-\theta; k; 1) = (-1)^{k+1} H(\theta; k; 1).$$

For $k_1, \dots, k_r \in \mathbb{N}$, $\theta \in \mathbb{R}$ and $u \in [1, 1 + \delta]$, we define

$$(2.10) \quad G(\theta; k_1, \dots, k_r; u) = \prod_{j=1}^r H(\theta; k_j; u).$$

By (2.9), we have

$$(2.11) \quad G(-\theta; k_1, \dots, k_r; 1) = (-1)^{\sum_{j=1}^r k_j + r} G(\theta; k_1, \dots, k_r; 1).$$

Now we prepare some notation. Let

$$\Delta_s = \{J = \{j_1, \dots, j_s\} \mid J \subset \{1, 2, \dots, r\}\} \text{ and } \Delta = \bigcup_{s=1}^{r-1} \Delta_s.$$

For $J = \{j_1, \dots, j_s\} \in \Delta$, let $\overline{J} = \{1, \dots, r\} \setminus J$. Let $\Omega(u)$ be the $\mathbb{Q}$ -algebra generated by $\{\phi(2j; u) \mid j \in \mathbb{N}_0\}$. Note that $\Omega(1) = \mathbb{Q}[\pi^2]$ because $\phi(s; 1) = (2^{1-s} - 1)\zeta(s)$.

By (2.7) and (2.10), we have the expansion

$$(2.12) \quad \begin{aligned} G(\theta; k_1, \dots, k_r; u) &= \mathcal{R}_r(\theta; k_1, \dots, k_r; u) \\ &+ \sum_{\substack{J=\{j_1, \dots, j_s\} \\ \in \Delta}} \sum_{\mu}^* B(J; \mu; u) \frac{(i\theta)^\mu}{\mu!} \mathcal{R}_s(\theta; k_{j_1}, \dots, k_{j_s}; u) \\ &+ \sum_{\nu}^* C(\nu; u) \frac{(i\theta)^\nu}{\nu!}, \end{aligned}$$

where the sums $\sum_{\mu}^*$ and $\sum_{\nu}^*$ are taken over all $\mu, \nu \in \mathbb{N}_0$ with

$$0 \leq \mu \leq \sum_{j \in \overline{J}} k_j, \quad 0 \leq \nu \leq \sum_{l=1}^r k_l,$$

respectively, and $B(J; \mu; u), C(\nu; u) \in \Omega(u)$. Note that

$$(2.13) \quad B(J; \mu; 1), C(\nu; 1) \in \Omega(1) = \mathbb{Q}[\pi^2].$$

3. SOME LEMMAS

We define the multiple series

$$(3.1) \quad \rho_n(k_1, \dots, k_n; s; u) = \sum_{m_1, m_2, \dots, m_n=1}^{\infty} \frac{(-u)^{-\sum_{\nu=1}^n m_\nu}}{m_1^{k_1} \cdots m_n^{k_n} (m_1 + \cdots + m_n)^s}$$

for $k_1, \dots, k_n \in \mathbb{N}$, $s \in \mathbb{R}$ and $u \in [1, 1 + \delta]$. In particular when $n = 1$, we have $\rho_1(k_1; s; u) = \phi(k_1 + s; u)$. If $u \in (1, 1 + \delta]$, then

$$(3.2) \quad \mathcal{R}_n(\theta; k_1, \dots, k_n; u) = \sum_{N=0}^{\infty} \rho_n(k_1, \dots, k_n; -N; u) \frac{(i\theta)^N}{N!}.$$

For $N \in \mathbb{Z}$ and $u \in (1, 1 + \delta]$ we define

$$(3.3) \quad \begin{aligned} \mathcal{A}_N(k_1, \dots, k_r; u) &= \rho_r(k_1, \dots, k_r; -N; u) \\ &+ \sum_{J=\{j_1, \dots, j_s\} \atop \in \Delta} \sum_{\mu}^* B(J; \mu; u) \binom{N}{\mu} \rho_s(k_{j_1}, \dots, k_{j_s}; \mu - N; u). \end{aligned}$$

Furthermore we define

$$(3.4) \quad \begin{aligned} \tilde{\mathcal{A}}_N(k_1, \dots, k_r; u) &= \begin{cases} \mathcal{A}_N(k_1, \dots, k_r; u) + C(N; u) & (\text{if } 0 \leq N \leq \sum_{j=1}^r k_j), \\ \mathcal{A}_N(k_1, \dots, k_r; u) & (\text{otherwise}). \end{cases} \end{aligned}$$

Note that if $N \leq -1$, then we can define

$$\tilde{\mathcal{A}}_N(k_1, \dots, k_r; 1) = \lim_{u \rightarrow 1} \tilde{\mathcal{A}}_N(k_1, \dots, k_r; u).$$

By combining (2.12), (3.2) and (3.4), we have

Lemma 3.1. *With the above notation,*

$$(3.5) \quad G(\theta; k_1, \dots, k_r; u) = \sum_{N=0}^{\infty} \tilde{\mathcal{A}}_N(k_1, \dots, k_r; u) \frac{(i\theta)^N}{N!}.$$

By (2.2), (2.8), (2.10) and (3.5), we have

Lemma 3.2. *With the above notation and for $u \in (1, 1 + \delta]$,*

$$(3.6) \quad \liminf_{N \rightarrow \infty} \left(\frac{|\tilde{\mathcal{A}}_N(k_1, \dots, k_r; u)|}{N!} \right)^{-1/N} \geq \pi \quad \text{and} \quad \frac{|\tilde{\mathcal{A}}_n(k_1, \dots, k_r; u)|}{n!} \leq \frac{\tilde{M}}{\gamma^n}$$

for any $\gamma \in (0, \pi)$ and $n \in \mathbb{N}_0$, where $\tilde{M}(> 0)$ is independent of n and u . Further

$$(3.7) \quad \lim_{u \rightarrow 1} \tilde{\mathcal{A}}_N(k_1, \dots, k_r; u) \lambda_{N+1+r+\sum_{j=1}^r k_j} = 0$$

for $N \in \mathbb{N}_0$.

For simplicity we let

$$(3.8) \quad \mathcal{S}_n^p(\theta; k_1, \dots, k_n; s; u) = \sum_{m_1, \dots, m_n=1}^{\infty} \frac{(-u)^{-\sum_{\nu=1}^n m_\nu} \sin^{(p)}((\sum_{\nu=1}^n m_\nu) \theta)}{m_1^{k_1} \cdots m_n^{k_n} (m_1 + \cdots + m_n)^s}$$

for $p \in \mathbb{N}_0$, $s \in \mathbb{R}$, $k_1, \dots, k_n \in \mathbb{N}$ and $u \in [1, 1 + \delta]$, where we denote the l th derivative of a function $f(\theta)$ by $f^{(l)}(\theta)$. Then we have the following lemma.

Lemma 3.3. *With the above notation and for $l \in \mathbb{N}$ with $l \geq 2$,*

$$\begin{aligned}
 (3.9) \quad & \mathcal{S}_r^p(\theta; k_1, \dots, k_r; l+p; u) \\
 & + \sum_{\substack{J=\{j_1, \dots, j_s\} \\ \in \Delta}} \sum_{\mu}^* B(J; \mu; u) (-1)^{\mu} \sum_{\sigma=0}^{\mu} \binom{l+p+\mu-1-\sigma}{\mu-\sigma} \\
 & \quad \times \mathcal{S}_s^{(\sigma+p)}(\theta; k_{j_1}, \dots, k_{j_s}; l+p+\mu-\sigma; u) \frac{(-\theta)^{\sigma}}{\sigma!} \\
 & + i^{p-1} \sum_{\nu}^* C(\nu; u) \lambda_{\nu+l+1} \frac{(i\theta)^{\nu+l+p}}{(\nu+l+p)!} \\
 & = i^{p-1} \sum_{n=-l-p}^{\infty} \tilde{\mathcal{A}}_n(k_1, \dots, k_r; u) \lambda_{n+l+1} \frac{(i\theta)^{n+l+p}}{(n+l+p)!}.
 \end{aligned}$$

Proof. It is known that

$$\begin{aligned}
 (3.10) \quad & \sum_{\nu=0}^b \binom{a-1+b-\nu}{b-\nu} \frac{(-\theta)^{\nu}}{\nu!} \frac{\sin^{(\nu+p)}(\theta x)}{x^{a+b-\nu}} \\
 & = i^{p-1} \sum_{N \geq 0} \binom{a-1+b-N}{b} \frac{(i\theta)^N}{N!} \lambda_{p+1+N} x^{-a-b+N}
 \end{aligned}$$

for $a, b \in \mathbb{N}_0$ (see [9], (2.16)). We assume that $u \in (1, 1+\delta]$ and apply (3.10) with $a = l+p$ and $b = 0$. Then we obtain

$$\begin{aligned}
 (3.11) \quad & \mathcal{S}_r^p(\theta; k_1, \dots, k_r; l+p; u) \\
 & = i^{p-1} \sum_{N=0}^{\infty} \rho_r(k_1, \dots, k_r; l+p-N; u) \lambda_{p+1+N} \frac{(i\theta)^N}{N!}.
 \end{aligned}$$

Applying (3.10) with $a = l+p$ and $b = \mu$, we have

$$\begin{aligned}
 (3.12) \quad & \sum_{\substack{J=\{j_1, \dots, j_s\} \\ \in \Delta}} \sum_{\mu}^* B(J; \mu; u) \sum_{\sigma=0}^{\mu} \binom{l+p+\mu-1-\sigma}{\mu-\sigma} \\
 & \quad \times \mathcal{S}_s^{\sigma+p}(\theta; k_{j_1}, \dots, k_{j_s}; l+p+\mu-\sigma; u) \frac{(-\theta)^{\sigma}}{\sigma!} \\
 & = i^{p-1} \sum_{N=0}^{\infty} \sum_{\substack{J=\{j_1, \dots, j_s\} \\ \in \Delta}} \sum_{\mu}^* B(J; \mu; u) \binom{l+p+\mu-1-N}{\mu} \\
 & \quad \times \rho_s(k_{j_1}, \dots, k_{j_s}; l+p+\mu-N; u) \lambda_{p+1+N} \frac{(i\theta)^N}{N!}.
 \end{aligned}$$

Let $n = N - l - p$. By (3.3), (3.4), (3.11), (3.12) and using the well-known relation

$$\binom{-X}{j} = (-1)^j \binom{X+j-1}{j},$$

we obtain (3.9). □

4. PROOF OF THEOREM 1.1

In this section, we give the proof of Theorem 1.1. Let $\mathcal{MT}_n$ be the $\mathbb{Q}$ -algebra generated by the Mordell-Tornheim zeta values of depth lower than or equal to n . Assume $r \geq 2$. Then we aim to prove that the Mordell-Tornheim zeta values of depth r and of weight w belongs to $\mathcal{MT}_{r-1}$ if r and w are of different parity.

By (3.6), we can see that (3.9) is uniformly convergent with respect to $u \in (1, 1 + \delta]$ if $l + p \geq 2$ and $\theta \in [-\pi, \pi]$. So we let $\theta = \pi$ and $u \rightarrow 1$ in both sides of (3.9). For simplicity we let

$$(4.1) \quad \mathcal{F}^p(k_1, \dots, k_r; l) = \sum_{J=\{j_1, \dots, j_s\} \in \Delta} \sum_{\mu}^* (-1)^\mu B(J; \mu; 1) \sum_{\sigma=0}^{\mu} \binom{l+p+\mu-1-\sigma}{\mu-\sigma} \\ \times \mathcal{S}_s^{\sigma+p}(\pi; k_{j_1}, \dots, k_{j_s}; l+p+\mu-\sigma; 1) \frac{(-\pi)^\sigma}{\sigma!} \\ + i^{p-1} \sum_{\nu}^* C(\nu; 1) \lambda_{\nu+l+1} \frac{(i\pi)^{\nu+l+p}}{(\nu+l+p)!}.$$

By the facts $\mathcal{S}_s^{2m}(\pi; k_{j_1}, \dots, k_{j_s}; k; 1) = 0$, $\mathcal{S}_s^{2m+1}(\pi; k_{j_1}, \dots, k_{j_s}; k; 1) \in \mathcal{MT}_s$ and (2.13), we have

$$(4.2) \quad \pi^{p-1} \mathcal{F}^p(k_1, \dots, k_r; l) \in \mathcal{MT}_{r-1} \quad (p = 0, 1),$$

when $l + p \geq 2$, because $\pi^2 \subset \mathcal{MT}_1 \subset \mathcal{MT}_{r-1}$ for $r \geq 2$. Suppose $l \equiv \sum_{\eta=1}^r k_\eta + r \pmod{2}$ and $l + p \geq 2$. It follows from (3.7) that

$$(4.3) \quad \mathcal{S}_r^p(\pi; k_1, \dots, k_r; l+p; 1) + \mathcal{F}^p(k_1, \dots, k_r; l) \\ = i^{p-1} \sum_{\mu=0}^{l+p-1} \tilde{\mathcal{A}}_{\mu-l-p}(k_1, \dots, k_r; 1) \lambda_{\mu+p+1} \frac{(i\pi)^\mu}{\mu!}.$$

Let $p = 0$ and $l \geq 2$. Since $\mathcal{S}_r^0(\pi; k_1, \dots, k_r; l; 1) = 0$, we have

$$(4.4) \quad \mathcal{F}^0(k_1, \dots, k_r; l) = i^{-1} \sum_{\nu=0}^{[(l-2)/2]} \tilde{\mathcal{A}}_{2\nu+1-l}(k_1, \dots, k_r; 1) \frac{(i\pi)^{2\nu+1}}{(2\nu+1)!}.$$

Now we recall the following lemma.

Lemma 4.1 ([9], Lemma 7). *Suppose $\{P_m\}$ and $\{Q_m\}$ are sequences which satisfy the relation*

$$\sum_{j=0}^{[m/2]} P_{m-2j} \frac{(i\pi)^{2j}}{(2j+1)!} = Q_m,$$

for any $m \in \mathbb{N}_0$. Then the relation

$$P_m = -2 \sum_{\nu=0}^m \phi(m-\nu) \lambda_{m-\nu} Q_\nu$$

holds for any $m \in \mathbb{N}_0$.

Now we give the proof of Theorem 1.1. Applying Lemma 4.1 with

$$P_m = \tilde{\mathcal{A}}_{-m-1}(k_1, \dots, k_r; 1) \lambda_{\sum_{\eta=1}^r k_\eta + r + m}, \\ Q_m = \frac{1}{\pi} \mathcal{F}^0(k_1, \dots, k_r; m+2) \lambda_{\sum_{\eta=1}^r k_\eta + r + m},$$

it follows from (4.2) and (4.4) that

$$(4.5) \quad \begin{aligned} & \tilde{\mathcal{A}}_{-m-1}(k_1, \dots, k_r; 1) \\ &= -2 \sum_{\nu=0}^m \phi(m-\nu) \lambda_{m+\nu} \frac{1}{\pi} \mathcal{F}^0(k_1, \dots, k_r; m-\nu+2) \in \mathcal{MT}_{r-1} \end{aligned}$$

for any $m \in \mathbb{N}_0$ with $m \equiv \sum_{\eta=1}^r k_\eta + r \pmod{2}$. Furthermore let $k \in \mathbb{N}$ with $k \geq 2$ and $k \not\equiv \sum_{\eta=1}^r k_\eta + r \pmod{2}$. Namely $\sum_{\eta=1}^r k_\eta + k$ and r are of different parity. Applying (4.3) with $l = k - 1$ and $p = 1$, we have

$$(4.6) \quad \begin{aligned} & \zeta_{MT,r}(k_1, \dots, k_r; k) + \mathcal{F}^1(k_1, \dots, k_r; k-1) \\ &= \sum_{\nu=0}^{[(k-2)/2]} \tilde{\mathcal{A}}_{2\nu-k}(k_1, \dots, k_r; 1) \frac{(i\pi)^{2\nu}}{(2\nu)!}, \end{aligned}$$

because $\mathcal{S}_r^1(\pi; k_1, \dots, k_r; k; 1) = \zeta_{MT,r}(k_1, \dots, k_r; k)$. By combining (4.2), (4.5) and (4.6), we obtain the proof of Theorem 1.1.

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