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ALMOST-DISJOINT CODING AND STRONGLY SATURATED IDEALS

PAUL B. LARSON

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Abstract. We show that Martin's Axiom plus $\mathfrak{c} = \aleph_2$ implies that there is no $(\aleph_2, \aleph_2, \aleph_0)$-saturated σ-ideal on ω_1.

Given cardinals λ, κ and γ, a σ-ideal I on a set X is said to be $(\lambda, \kappa, \gamma)$-*saturated* if for every set $\{A_\alpha : \alpha < \lambda\} \subset \mathcal{P}(X) \setminus I$ there exists a set $Y \in [\lambda]^\kappa$ such that for all $Z \in [Y]^\gamma$, $\bigcap\{A_\alpha : \alpha \in Z\} \notin I$. Laver [6] was the first to show the consistency of an $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal on ω_1, using a huge cardinal. Shelah [10] later showed that the nonstationary ideal on ω_1 restricted to a given stationary set can be $(\aleph_2, \aleph_2, \aleph_0)$-saturated, using a supercompact cardinal.

The cardinal characteristic $\mathfrak{ap}$ is defined to be the least κ such that there exist an *almost disjoint family* $\{e_\alpha : \alpha < \kappa\}$ (i.e., each e_α is an infinite subset of ω, and for each distinct pair $\alpha, \beta < \kappa$, $e_\alpha \cap e_\beta$ is finite) and a set $A \subset \kappa$ such that for no $x \subset \omega$ does it hold for all $\alpha < \kappa$ that $\alpha \in A$ if and only if $e_\alpha \cap x$ is infinite (in [5] we called this $\mathfrak{q}$, but [1] shows that we should not have, as consistently every set of reals of cardinality $\mathfrak{ap}$ is a Q-set). We let $\mathfrak{c}$ denote the cardinality of the continuum. It follows easily that $2^\gamma = \mathfrak{c}$ for every infinite $\gamma < \mathfrak{ap}$.

Given a cardinal γ, MA_γ is the variant of Martin's Axiom that says that if P is a c.c.c. partial order and D_α ($\alpha < \gamma$) are dense subsets of P, then there is a filter $G \subset P$ such that $G \cap D_\alpha$ is nonempty for each $\alpha < \gamma$. It is a standard fact that MA_γ implies that $\mathfrak{ap} > \gamma$ [4].

In this note, we show that the statement $\mathfrak{ap} = \mathfrak{c} = \aleph_2$ implies that there is no countably complete $(\aleph_2, \aleph_2, \aleph_0)$-saturated σ-ideal on ω_1. This contradicts statements in [7, 8] to the effect that the axiom PFA (see [10]) had been shown to be consistent with the existence of a stationary subset of ω_1 such that the nonstationary ideal restricted to this set is $(\aleph_2, \aleph_2, \aleph_0)$-saturated. This situation is addressed by Nyikos in [9] and in another corrigendum to appear.

For a fixed cardinal κ, an ideal on a set X is κ-*dense* if there is a subset $\mathcal{A}$ of $\mathcal{P}(X) \setminus I$ of cardinality κ such that every I-positive subset of X contains a member of $\mathcal{A}$ modulo I. It follows easily that every $\aleph_1$-dense σ-ideal is $(\aleph_2, \aleph_2, \aleph_0)$-saturated.

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It is a classical fact due to Ulam that there is no $\aleph_0$-dense σ-ideal on ω_1 (see, for instance, Lemma 10.13 of [3]). Taylor [11] showed that under $\mathrm{MA}_{\aleph_1}$ there is no $\aleph_1$-dense σ-ideal on ω_1. The proof of Theorem 18 in [2] shows that $\mathfrak{ap} > \aleph_1$ suffices, i.e., that the following holds (a version of the argument appears also in [5]).

Fact 0.1. If $\mathfrak{ap} > \aleph_1$, then there is no $\aleph_1$-dense σ-ideal on ω_1.

For the rest of this note, we fix an almost disjoint family $\{e_\alpha : \alpha < \omega_1\}$. For each $n \in \omega$ and each (possibly finite) $\sigma \subset \omega$, we let $E^n_\sigma = \{\alpha < \omega_1 \mid |e_\alpha \cap \sigma| \geq n\}$, and we let $F^n_\sigma = \omega_1 \setminus E^n_\sigma$.

Lemma 0.2. *Assume that $\mathfrak{ap} > \aleph_1$. Let I be a σ-ideal on ω_1 and let β be a cardinal such that $\mathcal{P}(\omega_1)/I$ is not β-dense. Let $\{A_\alpha : \alpha < \beta\}$ be a subset of $\mathcal{P}(\omega_1) \setminus I$. Then there exist an $x \subset \omega$ and an $n \in \omega$ such that*

- $F^n_x \notin I$,
- *for each $\alpha < \beta$ there exists an $m \in \omega$ such that $E^n_{x\cap m} \cap A_\alpha \notin I$.*

Proof. Since $\mathcal{P}(\omega_1)/I$ is not β-dense, we may fix $\{B_\alpha : \alpha < \beta\} \subset \mathcal{P}(\omega_1) \setminus I$ and $D \in \mathcal{P}(\omega_1) \setminus I$ such that each $B_\alpha \subset A_\alpha$ and each $B_\alpha \cap D = \emptyset$. Since $\mathfrak{ap} > \aleph_1$, there exists an $x \subset \omega$ such that $e_\gamma \cap x$ is infinite for each $\gamma \in \bigcup\{B_\alpha : \alpha < \beta\}$ and $e_\gamma \cap x$ is finite for each $\gamma \in D$. Since $D \subset \bigcup\{F^n_x : n < \omega\}$, we may fix an $n \in \omega$ such that $F^n_x \notin I$. Similarly, for each $\alpha < \beta$, since $B_\alpha \subset \bigcup\{E^n_{x\cap m} : m < \omega\}$, there is an $m \in \omega$ such that $E^n_{x\cap m} \cap A_\alpha \notin I$. □

The following theorem shows that $\mathfrak{ap} > \aleph_1$ implies that there is no σ-ideal I on ω_1 which is $(\gamma, \gamma, \aleph_0)$-saturated, where γ is the least cardinality of a dense subset of $\mathcal{P}(\omega_1)/I$. In particular, if $\mathfrak{ap} = \mathfrak{c} = \aleph_2$, then there is no $(\aleph_2, \aleph_2, \aleph_0)$-saturated σ-ideal on ω_1.

Theorem 0.3. *Assume that $\mathfrak{ap} > \aleph_1$, and let I be a σ-ideal on ω_1. Let γ be the least cardinal such that there exists a dense (modulo I) subset of $\mathcal{P}(\omega_1) \setminus I$ of cardinality γ. Then there is a sequence $\langle D_\alpha : \alpha < \gamma\rangle$ of members of $\mathcal{P}(\omega_1) \setminus I$ such that for every cofinal $X \subset \gamma$ there exists a countable $y \subset X$ such that $\bigcap\{D_\alpha : \alpha \in y\} \in I$.*

Proof. Let $\{A_\alpha : \alpha < \gamma\}$ enumerate a dense subset of $\mathcal{P}(\omega_1) \setminus I$ modulo I. For each $\beta < \gamma$, apply Lemma 0.2 to $\{A_\alpha : \alpha < \beta\}$, obtaining x_β, n_β and $D_\beta = F^{n_\beta}_{x_\beta}$ such that $D_\beta \notin I$ and such that for each $\alpha < \beta$ there exists an $m \in \omega$ such that $A_\alpha \cap E^{n_\beta}_{x_\beta \cap m} \notin I$.

Now let $X \subset \gamma$ be cofinal. Let Z be the set of pairs (n, σ) ($n \in \omega$, $\sigma \subset \omega$ finite) such that there exists a $\beta \in X$ with $E^n_\sigma \cap D_\beta \in I$. We claim that $\{E^n_\sigma : (n, \sigma) \in Z\}$ is predense in $\mathcal{P}(\omega_1) \setminus I$, i.e., that for every $\alpha < \gamma$ there exist a $\beta \in X$, an $n \in \omega$ and a finite $\sigma \subset \omega$ such that $E^n_\sigma \cap D_\beta \in I$ and $E^n_\sigma \cap A_\alpha \notin I$. To verify this, fix $\alpha < \gamma$ and let β be any member of X greater than α. Then $D_\beta = F^{n_\beta}_{x_\beta}$ and there exists an m such that $E^{n_\beta}_{x_\beta \cap m} \cap A_\alpha \notin I$, so β, n_β and $x_\beta \cap m$ suffice for α.

Now, for each $(n, \sigma) \in Z$, choose $\beta_{(n,\sigma)} \in X$ such that $E^n_\sigma \cap D_{\beta_{(n,\sigma)}} \in I$. Then since $\{E^n_\sigma : (n, \sigma) \in Z\}$ is predense in $\mathcal{P}(\omega_1) \setminus I$, $\bigcap\{D_{\beta_{(n,\sigma)}} : (n, \sigma) \in Z\} \in I$. □

We do not know whether some forcing axiom implies that the nonstationary ideal on ω_1 is not $(\aleph_2, \aleph_1, \aleph_0)$-saturated. On the other hand, for all we know, some forcing axiom implies that the nonstationary ideal on ω_1 *is* $(\aleph_2, \aleph_1, \aleph_0)$-saturated.

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Department of Mathematics, Miami University, Oxford, Ohio 45056
E-mail address: larsonpb@muohio.edu