A NEW CHARACTERIZATION OF GAPS BETWEEN TWO SUBSPACES

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ABSTRACT. In this note, a new characterization of gaps between two subspaces of a Hilbert space is established.

Let $\mathcal{M}$ and $\mathcal{N}$ be two closed linear subspaces of a Hilbert space $\mathcal{H}$. The gap between $\mathcal{M}$ and $\mathcal{N}$ (see [2]) is defined by

$$(1) \quad \text{gap}(\mathcal{M}, \mathcal{N}) = \max\{\delta(\mathcal{M}, \mathcal{N}), \delta(\mathcal{N}, \mathcal{M})\},$$

where

$$\delta(\mathcal{M}, \mathcal{N}) = \sup\{\text{dist}(x, \mathcal{N}) : x \in \mathcal{M}, \|x\| \leq 1\},$$

and the conorm between $\mathcal{M}$ and $\mathcal{N}$ is defined by

$$(2) \quad \gamma(\mathcal{M}, \mathcal{N}) = \inf\{\|x\| : x \in \mathcal{M}, \text{dist}(x, \mathcal{N}) \geq 1\}.$$

In this note, the specific representations of the gap and the conorm between two subspaces are established.

For two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$, denote $\mathcal{H}_1 = \mathcal{M} \cap \mathcal{N}$, $\mathcal{H}_2 = \mathcal{M} \cap \mathcal{N}^\perp$, $\mathcal{H}_3 = \mathcal{M}^\perp \cap \mathcal{N}$, $\mathcal{H}_4 = \mathcal{M}^\perp \cap \mathcal{N}^\perp$ and let $\mathcal{H}_5 = \mathcal{M} \cap (\mathcal{H} \ominus (\bigoplus_{i=1}^4 \mathcal{H}_i))$ and $\mathcal{H}_6 = (\mathcal{H} \ominus (\bigoplus_{i=1}^4 \mathcal{H}_i)) \ominus \mathcal{H}_5$. It is clear that $\mathcal{H}_i \perp \mathcal{H}_j$, $j \neq i$ and $1 \leq i, j \leq 6$. If $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are orthogonal projections onto $\mathcal{M}$ and $\mathcal{N}$, respectively, we have the following lemma which is useful later.

Lemma 1 (see [1]). *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. Then $P_{\mathcal{M}}\mathcal{H}_i \subseteq \mathcal{H}_i$ and $P_{\mathcal{N}}\mathcal{H}_i \subseteq \mathcal{H}_i$, $1 \leq i \leq 4$, and $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the following operator matrices:*

$$(3) \quad P_{\mathcal{M}} = \begin{pmatrix} I & & & & & \\ & I & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & I & \\ & & & & & 0 \end{pmatrix}, \quad P_{\mathcal{N}} = \begin{pmatrix} I & & & & & \\ & 0 & & & & \\ & & I & & & \\ & & & 0 & & \\ & & & & Q & Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}D \\ & & & & D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}} & D^*(I-Q)D \end{pmatrix},$$

with respect to the space decomposition $\mathcal{H} = \sum_{i=1}^6 \mathcal{H}_i$, respectively, where Q is a positive contraction on $\mathcal{H}_5$, 0 and 1 are not eigenvalues of Q , and D is a unitary operator from $\mathcal{H}_6$ onto $\mathcal{H}_5$.

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If $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices (3), for $x \in \mathcal{M}$, we have $x = x_1 + x_2 + x_5$ with $x_i \in \mathcal{H}_i$, $i = 1, 2, 5$, and $P_{\mathcal{N}}x = x_1 + Qx_5 + D^*Q^{\frac{1}{2}}(I - Q)^{\frac{1}{2}}x_5$. Observing that $\text{dist}(x, \mathcal{N})^2 = \|x\|^2 - \|P_{\mathcal{N}}x\|^2$, then

$$\begin{aligned} \text{dist}(x, \mathcal{N})^2 &= \|x_1\|^2 + \|x_2\|^2 + \|x_5\|^2 - \|x_1\|^2 \\ &\quad - \|Qx_5\|^2 - \|D^*Q^{\frac{1}{2}}(I - Q)^{\frac{1}{2}}x_5\|^2 \\ &= \|x_2\|^2 + \|(I - Q)^{\frac{1}{2}}x_5\|^2 \leq \|x_2\|^2 + \|x_5\|^2 \\ &\leq \|x\|^2, \end{aligned}$$

that is,

$$(4) \quad \text{dist}(x, \mathcal{N}) = (\|x_2\|^2 + \|(I - Q)^{\frac{1}{2}}x_5\|^2)^{\frac{1}{2}} \leq \|x\|.$$

Proposition 2. Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$, and let $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices (3). If $\mathcal{M} \cap \mathcal{N}^{\perp} \neq \{0\}$, then $\delta(\mathcal{M}, \mathcal{N}) = 1$. If $\mathcal{M} \cap \mathcal{N}^{\perp} = \{0\}$, then $\delta(\mathcal{M}, \mathcal{N}) = \|(I - Q)^{\frac{1}{2}}\|$.

Proof. By Lemma 1, $0 \leq Q \leq I$. Then $0 \leq I - Q \leq I$. So $I - Q$ is also a positive contraction.

If $\mathcal{M} \cap \mathcal{N}^{\perp} \neq \{0\}$, and for a vector $x \in \mathcal{M}$, x has the decomposition $x = x_1 + x_2 + x_5$ with $x_i \in \mathcal{H}_i$, $i = 1, 2, 5$, then

$$\begin{aligned} \delta(\mathcal{M}, \mathcal{N}) &= \sup\{\text{dist}(x, \mathcal{N}) : x \in \mathcal{M}, \|x\| \leq 1\} \\ &= \sup\{(\|x_2\|^2 + \|(I - Q)^{\frac{1}{2}}x_5\|^2)^{\frac{1}{2}} : x_i \in \mathcal{H}_i, i = 1, 2, 5, \\ &\quad \text{and } \sum_{i=1,2,5} \|x_i\|^2 \leq 1\} \\ &\leq 1. \end{aligned}$$

To prove the reverse inequality, choose $x_0 = x_2 \in \mathcal{H}_2$ with $\|x_2\| = 1$, $\text{dist}(x_0, \mathcal{N}) = \|x_0\| = 1$, i.e., $\delta(\mathcal{M}, \mathcal{N}) \geq 1$. Hence $\delta(\mathcal{M}, \mathcal{N}) = 1$.

If $\mathcal{M} \cap \mathcal{N}^{\perp} = \{0\}$, then $\mathcal{H}_2 = \{0\}$. In this case, for a vector $x \in \mathcal{M}$, x has the decomposition $x = x_1 + x_5$ with $x_1 \in \mathcal{H}_1$ and $x_5 \in \mathcal{H}_5$. We have

$$\begin{aligned} \delta(\mathcal{M}, \mathcal{N}) &= \sup\{\text{dist}(x, \mathcal{N}) : x \in \mathcal{M}, \|x\| \leq 1\} \\ &= \sup\{\|(I - Q)^{\frac{1}{2}}x_5\| : x_i \in \mathcal{H}_i, i = 1, 5, \text{ and } \sum_{i=1,5} \|x_i\|^2 \leq 1\} \\ &= \sup\{\|(I - Q)^{\frac{1}{2}}x_5\| : x_5 \in \mathcal{H}_5, \|x_5\| \leq 1\} \\ &= \|(I - Q)^{\frac{1}{2}}\|. \end{aligned}$$

□

Proposition 3. Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$, and let $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices (3). If $\mathcal{M}^{\perp} \cap \mathcal{N} \neq \{0\}$, then $\delta(\mathcal{N}, \mathcal{M}) = 1$. If $\mathcal{M}^{\perp} \cap \mathcal{N} = \{0\}$, then $\delta(\mathcal{N}, \mathcal{M}) = \|(I - Q)^{\frac{1}{2}}\|$.

Proof. If $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices as the formula (3), since $\mathcal{M} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_5$, $\mathcal{N} = \mathcal{H}_1 \oplus \mathcal{H}_3 \oplus R(P_{\mathcal{N}}|_{\mathcal{H}_5 \oplus \mathcal{H}_6})$ and

$$\begin{aligned} &\begin{pmatrix} Q \\ D^*Q^{\frac{1}{2}}(I - Q)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} Q & Q^{\frac{1}{2}}(I - Q)^{\frac{1}{2}}D \end{pmatrix} \\ &= \begin{pmatrix} Q^2 & Q^{\frac{3}{2}}(I - Q)^{\frac{1}{2}}D \\ D^*Q^{\frac{3}{2}}(I - Q)^{\frac{1}{2}} & D^*Q(I - Q)D \end{pmatrix} \\ &= \begin{pmatrix} Q & Q^{\frac{1}{2}}(I - Q)^{\frac{1}{2}}D \\ D^*Q^{\frac{1}{2}}(I - Q)^{\frac{1}{2}} & D^*(I - Q)D \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & D^*QD \end{pmatrix}, \end{aligned}$$

we obtain

$$\begin{aligned} \overline{R\left(\begin{pmatrix} Q \\ D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}} \end{pmatrix}\right)} &= R\left(\begin{pmatrix} Q & Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}D \\ D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}} & D^*(I-Q)D \end{pmatrix}\right) \\ &= R(P_{\mathcal{N}}|_{\mathcal{H}_5 \oplus \mathcal{H}_6}) \end{aligned}$$

and by Lemma 1 that Q is injective and positive, where $R(T)$ denotes the range of an operator T and $\overline{K}$ the closure of a set K . Denoting

$$\mathcal{N}^0 = \mathcal{H}_1 \oplus \mathcal{H}_3 \oplus R\left(\begin{pmatrix} Q \\ D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}} \end{pmatrix}\right),$$

it is obvious that $\overline{\mathcal{N}^0} = \mathcal{N}$, so $\delta(\mathcal{N}, \mathcal{M}) = \delta(\mathcal{N}^0, \mathcal{M})$. For a vector $y \in \mathcal{N}^0$, there exist vectors $y_1 \in \mathcal{H}_1$, $y_3 \in \mathcal{H}_3$ and $y_5 \in \mathcal{H}_5$ such that $y = y_1 + y_3 + Qy_5 + D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5$ and $P_{\mathcal{M}}y = y_1 + Qy_5$. Observing that $\|y\|^2 = \|y_1\|^2 + \|y_3\|^2 + \|Qy_5\|^2 + \|D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5\|^2 = \|y_1\|^2 + \|y_3\|^2 + \|Q^{\frac{1}{2}}y_5\|^2$ and $\|P_{\mathcal{M}}y\|^2 = \|y_1\|^2 + \|Qy_5\|^2$, we have

$$\begin{aligned} \text{dist}(y, \mathcal{M})^2 &= \|y\|^2 - \|P_{\mathcal{M}}y\|^2 \\ &= \|y_3\|^2 + \|D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5\|^2 \\ &= \|y_3\|^2 + \|Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \delta(\mathcal{N}, \mathcal{M}) &= \delta(\mathcal{N}^0, \mathcal{M}) \\ &= \sup\{\text{dist}(y, \mathcal{M}) : y \in \mathcal{N}^0, \|y\| \leq 1\} \\ &= \sup\{(\|y_3\|^2 + \|Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5\|^2)^{\frac{1}{2}} : \\ &\quad \|y_1\|^2 + \|y_3\|^2 + \|Q^{\frac{1}{2}}y_5\|^2 \leq 1, y_i \in \mathcal{H}_i, i = 1, 3, 5\} \\ &\leq 1. \end{aligned}$$

If $\mathcal{M}^\perp \cap \mathcal{N} \neq \{0\}$, then $\mathcal{H}_3 \neq \{0\}$ and $\mathcal{H}_3 \subset \mathcal{N}^0$; hence $\delta(\mathcal{N}^0, \mathcal{M}) \geq \sup\{\|y_3\| : y_3 \in \mathcal{H}_3, \|y_3\| \leq 1\} = 1$. Therefore, $\delta(\mathcal{N}, \mathcal{M}) = 1$.

If $\mathcal{M}^\perp \cap \mathcal{N} = \{0\}$, $\mathcal{H}_3 = \{0\}$ and $y_3 = 0$, then

$$\begin{aligned} \delta(\mathcal{N}, \mathcal{M}) &= \delta(\mathcal{N}^0, \mathcal{M}) \\ &= \sup\{\|Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5\| : \|y_1\|^2 + \|Q^{\frac{1}{2}}y_5\|^2 \leq 1, y_1 \in \mathcal{H}_1, y_5 \in \mathcal{H}_5\} \\ &= \sup\{\|(I-Q)^{\frac{1}{2}}Q^{\frac{1}{2}}y_5\| : y_5 \in \mathcal{H}_5, \|Q^{\frac{1}{2}}y_5\| \leq 1\} \\ &= \|(I-Q)^{\frac{1}{2}}\|. \end{aligned}$$

Here the last equation holds since $\overline{R(Q^{\frac{1}{2}})} = \mathcal{H}_5$. $\square$

Proposition 4. Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$, and let $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices (3). If $\mathcal{M} \cap \mathcal{N}^\perp \neq \{0\}$, then $\gamma(\mathcal{M}, \mathcal{N}) = 1$. If $\mathcal{M} \cap \mathcal{N}^\perp = \{0\}$, then $\gamma(\mathcal{M}, \mathcal{N}) = \|(I-Q)^{\frac{1}{2}}\|^{-1}$.

Proof. Let $x \in \mathcal{M}$. Then there exist vectors $x_1 \in \mathcal{H}_1$, $x_2 \in \mathcal{H}_2$ and $x_5 \in \mathcal{H}_5$ such that $x = x_1 + x_2 + x_5$, $\|x\|^2 = \sum_{i=1,2,5} \|x_i\|^2$ and

$$\begin{aligned} \text{dist}(x, \mathcal{N})^2 &= \|x\|^2 - \|P_{\mathcal{N}}x\|^2 \\ &= \|x_2\|^2 + \|x_5\|^2 - \|Qx_5\|^2 - \|D^*Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}x_5\|^2 \\ &= \|x_2\|^2 + \|(I-Q)^{\frac{1}{2}}x_5\|^2. \end{aligned}$$

Hence,

$$\begin{aligned}\gamma(\mathcal{M}, \mathcal{N}) &= \inf\{\|x\|: x \in \mathcal{M}, \text{dist}(x, \mathcal{N}) \geq 1\} \\ &= \inf\{\|x\|: x = \sum_{i=1,2,5} x_i, x_i \in \mathcal{H}_i, i = 1, 2, 5, \\ &\quad \|x_2\|^2 + \|(I - Q)^{\frac{1}{2}}x_5\|^2 \geq 1\}.\end{aligned}$$

The operator Q being a positive contraction, $\text{dist}(x, \mathcal{N}) \geq 1$ implies that $1 \leq \|x_2\|^2 + \|(I - Q)^{\frac{1}{2}}x_5\|^2 \leq \|x_2\|^2 + \|x_5\|^2 \leq \|x\|^2$. Hence $\gamma(\mathcal{M}, \mathcal{N}) \geq 1$.

If $\mathcal{M} \cap \mathcal{N}^\perp \neq \{0\}$, then it follows upon choosing $x_0 = x_2 \in \mathcal{H}_2$ with $\|x_2\| = 1$ that $\text{dist}(x_0, \mathcal{N}) = 1$, i.e., $\gamma(\mathcal{M}, \mathcal{N}) = 1$.

If, instead, $\mathcal{M} \cap \mathcal{N}^\perp = \{0\}$, then

$$\begin{aligned}\gamma(\mathcal{M}, \mathcal{N}) &= \inf\{\|x_1 + x_5\|: x_1 \in \mathcal{H}_1, x_5 \in \mathcal{H}_5, \|(I - Q)^{\frac{1}{2}}x_5\| \geq 1\} \\ &= \inf\{\|x_5\|: x_5 \in \mathcal{H}_5, \|(I - Q)^{\frac{1}{2}}x_5\| \geq 1\}.\end{aligned}$$

Observing $1 \leq \|(I - Q)^{\frac{1}{2}}x_5\| \leq \|(I - Q)^{\frac{1}{2}}\| \|x_5\|$, it follows that $\|x_5\| \geq \|(I - Q)^{\frac{1}{2}}\|^{-1}$, and hence that $\|(I - Q)^{\frac{1}{2}}\|^{-1} \leq \gamma(\mathcal{M}, \mathcal{N})$. To prove the reverse inequality, we let Q have the spectral representation $Q = \int_{\mu_0}^{\nu_0} \lambda dE_\lambda$, where $\mu_0 = \min\{\lambda : \lambda \in \sigma(Q)\} \geq 0$, $\nu_0 = \max\{\lambda : \lambda \in \sigma(Q)\} \leq 1$, and $\sigma(Q)$ denotes the spectrum of Q . For $\epsilon > 0$ small enough, choose a vector $x_\epsilon \in E([\mu_0, \mu_0 + \epsilon])\mathcal{H}_5$ with $1 + \epsilon \geq \|(I - Q)^{\frac{1}{2}}x_\epsilon\| \geq 1$. Then $1 + \epsilon \geq \|(I - Q)^{\frac{1}{2}}x_\epsilon\| = \|\int_{\mu_0}^{\nu_0} (1 - \lambda)^{\frac{1}{2}} dE_\lambda x_\epsilon\| = \|\int_{\mu_0}^{\mu_0 + \epsilon} (1 - \lambda)^{\frac{1}{2}} dE_\lambda x_\epsilon\| \geq (1 - \mu_0 - \epsilon)^{\frac{1}{2}} \|x_\epsilon\|$, i.e., $\|x_\epsilon\| \leq (1 - \mu_0 - \epsilon)^{-\frac{1}{2}}(1 + \epsilon)$. Since ϵ is arbitrary and $\|(I - Q)^{\frac{1}{2}}\|^{-1} = (1 - \mu_0)^{-\frac{1}{2}}$, $\gamma(\mathcal{M}, \mathcal{N}) \leq \|(I - Q)^{\frac{1}{2}}\|^{-1}$. Hence $\gamma(\mathcal{M}, \mathcal{N}) = \|(I - Q)^{\frac{1}{2}}\|^{-1}$. $\square$

Proposition 5. Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$, and let $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices (3). If $\mathcal{M}^\perp \cap \mathcal{N} \neq \{0\}$, then $\gamma(\mathcal{N}, \mathcal{M}) = 1$. If $\mathcal{M}^\perp \cap \mathcal{N} = \{0\}$, then $\gamma(\mathcal{N}, \mathcal{M}) = \|(I - Q)^{\frac{1}{2}}\|^{-1}$.

Proof. Similar to the proof of Proposition 3, we can show that

$$\gamma(\mathcal{N}, \mathcal{M}) = \gamma(\mathcal{N}^0, \mathcal{M}).$$

To complete the proof, it is enough to consider the quantum $\gamma(\mathcal{N}^0, \mathcal{M})$. Since

$$\begin{aligned}\gamma(\mathcal{N}^0, \mathcal{M}) &= \inf\{\|y\|: y = y_1 + y_3 + Qy_5 + D^*Q^{\frac{1}{2}}(I - Q)^{\frac{1}{2}}y_5, \\ &\quad y_i \in \mathcal{H}_i, i = 1, 3, 5, \|y_3\|^2 + \|D^*Q^{\frac{1}{2}}(I - Q)^{\frac{1}{2}}y_5\|^2 \geq 1\},\end{aligned}$$

it is clear that $\gamma(\mathcal{N}^0, \mathcal{M}) \geq 1$ in general.

Let $\mathcal{M}^\perp \cap \mathcal{N} \neq \{0\}$, choosing $y_0 = y_3 \in \mathcal{H}_3$ with $\|y_3\| = 1$, so that $y_0 \in \mathcal{N}^0$, $\|y_0\| = 1$ and $\text{dist}(y_0, \mathcal{M}) = 1$, i.e., $\gamma(\mathcal{N}^0, \mathcal{M}) \leq 1$. Hence $\gamma(\mathcal{N}^0, \mathcal{M}) = 1$.

If $\mathcal{M}^\perp \cap \mathcal{N} = \{0\}$, then $\mathcal{H}_3 = \{0\}$, so

$$\begin{aligned}\gamma(\mathcal{N}^0, \mathcal{M}) &= \inf\{\|y\|: y = y_1 + Qy_5 + D^*Q^{\frac{1}{2}}(I - Q)^{\frac{1}{2}}y_5, \\ &\quad y_i \in \mathcal{H}_i, i = 1, 5, \|D^*Q^{\frac{1}{2}}(I - Q)^{\frac{1}{2}}y_5\|^2 \geq 1\}.\end{aligned}$$

Noting that

$$\begin{aligned}\|y\|^2 &= \|y_1\|^2 + \|Qy_5\|^2 + \|D^*Q^{\frac{1}{2}}(I - Q)^{\frac{1}{2}}y_5\|^2 \\ &= \|y_1\|^2 + \|Q^{\frac{1}{2}}y_5\|^2 \geq \|Q^{\frac{1}{2}}y_5\|^2\end{aligned}$$

and $\overline{R(Q^{\frac{1}{2}})} = \mathcal{H}_5$, then

$$\begin{aligned}\gamma(\mathcal{N}^0, \mathcal{M}) &= \inf\{\|Q^{\frac{1}{2}}y_5\|: y_5 \in \mathcal{H}_5, \|Q^{\frac{1}{2}}(I-Q)^{\frac{1}{2}}y_5\| \geq 1\} \\ &= \inf\{\|z\|: z \in \mathcal{H}_5, \|(I-Q)^{\frac{1}{2}}z\| \geq 1\} \\ &= \|(I-Q)^{\frac{1}{2}}\|^{-1}.\end{aligned}$$

So $\gamma(\mathcal{N}^0, \mathcal{M}) = \|(I-Q)^{\frac{1}{2}}\|^{-1}$. $\square$

By Propositions 2–5, the following consequences are clear.

Corollary 6. *If $\mathcal{M}$ and $\mathcal{N}$ are two closed subspaces of $\mathcal{H}$, then $\delta(\mathcal{M}, \mathcal{N})\gamma(\mathcal{M}, \mathcal{N}) = 1$.*

Combining Proposition 2 with Proposition 3, we get

Corollary 7. *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. If $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ as projections onto $\mathcal{M}$ and $\mathcal{N}$, respectively, have the formulas (3) of operator matrices, then*

$$\text{gap}(\mathcal{M}, \mathcal{N}) = \begin{cases} 1, & \mathcal{M} \cap \mathcal{N}^{\perp} \neq \{0\} \text{ or } \mathcal{M}^{\perp} \cap \mathcal{N} \neq \{0\}; \\ \|(I-Q)^{\frac{1}{2}}\|, & \mathcal{M} \cap \mathcal{N}^{\perp} = \{0\} \text{ and } \mathcal{M}^{\perp} \cap \mathcal{N} = \{0\}. \end{cases}$$

Remarks. (1) $\delta(\mathcal{M}, \mathcal{N}) = \|P_{\mathcal{M}}(I - P_{\mathcal{N}})\|$.

Proof. Observing that $\|P_{\mathcal{M}}(I - P_{\mathcal{N}})\|^2 = \|P_{\mathcal{M}}(I - P_{\mathcal{N}})P_{\mathcal{M}}\|$, from the formula (2), we obtain

$$P_{\mathcal{M}}(I - P_{\mathcal{N}})P_{\mathcal{M}} = \begin{pmatrix} 0 & & & & \\ & I & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & I - Q & \\ & & & & & 0 \end{pmatrix}.$$

If $\mathcal{H}_2 \neq \{0\}$, then $\|P_{\mathcal{M}}(I - P_{\mathcal{N}})\| = \max\{1, \|(I-Q)^{\frac{1}{2}}\|\} = 1$ since Q is a positive contraction. If $\mathcal{H}_2 = \{0\}$, then $\|P_{\mathcal{M}}(I - P_{\mathcal{N}})\| = \|(I-Q)^{\frac{1}{2}}\|$. That is, $\delta(\mathcal{M}, \mathcal{N}) = \|P_{\mathcal{M}}(I - P_{\mathcal{N}})\|$. $\square$

(2) Proposition 3 can be seen as a corollary of Proposition 2. In fact, from Remark (1) and Proposition 2, we have $\delta(\mathcal{N}, \mathcal{M}) = \|P_{\mathcal{N}}(I - P_{\mathcal{M}})\|$. Then

$$\delta(\mathcal{N}, \mathcal{M}) = \|P_{\mathcal{N}}(I - P_{\mathcal{M}})\| = \begin{cases} 1, & \text{if } \mathcal{M}^{\perp} \cap \mathcal{N} \neq \{0\}, \\ \|(I-Q)^{\frac{1}{2}}\|, & \text{if } \mathcal{M}^{\perp} \cap \mathcal{N} = \{0\}. \end{cases}$$

(3) Similarly, we have $\gamma(\mathcal{M}, \mathcal{N}) = \|P_{\mathcal{M}}(I - P_{\mathcal{N}})\|^{-1}$. Proposition 5 can also be derived as a corollary of Proposition 4.

But here we still give the direct proofs of Propositions 3 and 5 because the direct proofs give us more information of the geometrical structures between two subspaces.

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