A CHARACTERIZATION OF THE PROJECTIVE LINE

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ABSTRACT. Let X be a set (with at least three different points) and let G be a group of bijections of X . If the action of G on X satisfies three natural conditions, then X admits a canonical structure of a projective line over a commutative field, such that G is the group of all projective transformations of X .

1. INTRODUCTION

Given a commutative field K , let us recall that the projective line $\mathbb{P}_1$ over K is defined to be the set of all 1-dimensional subspaces of some 2-dimensional vector space V over K . A projective transformation is a bijection of $\mathbb{P}_1$ induced by some linear automorphism of V .

The purpose of this note is to prove the following group-theoretic characterization of the projective line.

Theorem 1.1. *Let $G \times X \rightarrow X$ be an action of a group G on a set X (with at least three different points) satisfying the following conditions:*

- A₁ Given three different points $x_1, x_2, x_3 \in X$ and another three different points $y_1, y_2, y_3 \in X$, there exists a unique transformation $g \in G$ such that $g(x_i) = y_i$ for any $i = 1, 2, 3$.*
- A₂ Given three different points $p, x, y \in X$, there exists a unique transformation $g \in G$ such that $g(x) = y$ and p is the unique fixed point of g .*
- B Let $g \in G$. If there exist two different points $x, y \in X$ such that $g(x) = y$ and $g(y) = x$, then g is an involution: $g^2 = \text{Id}$.*

Then X admits a natural structure of a projective line over a commutative field K , such that G is the group of all projective transformations of X .

Axiom A₁ implies that any transformation $g \in G$ with three fixed points is the identity map. We may think of axiom A₂ as a degenerate case of A₁ imagining p as a double-counted point.

As a preliminary step in the proof of Theorem 1.1, we give a similar characterization of the affine line.

In the case of dimension $n > 1$, there exist group-theoretic definitions of the affine and projective geometries in terms of the group of dilatations (see [2] and [3]).

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2. CHARACTERIZATION OF THE AFFINE LINE

As is well known, there exist two elementary definitions for the affine space: the synthetic definition and the algebraic one. For dimension 1, only the algebraic definition makes sense. Let us recall it.

Let K be a commutative field. An *affine space* is a set $\mathbb{A}$ endowed with an action $\mu: V \times \mathbb{A} \rightarrow \mathbb{A}$ of a K -vector space V on $\mathbb{A}$. It is assumed that the action satisfies the following property: Given $p, q \in \mathbb{A}$, there exists a unique vector $v \in V$ such that $q = p + v$ (we denote $p + v = \mu(v, p)$). The *dimension* of $\mathbb{A}$ is, by definition, the dimension of the vector space V .

A bijection $\varphi: \mathbb{A} \rightarrow \mathbb{A}$ is said to be an *affine transformation* if there exists a (necessarily unique) K -linear automorphism $\phi: V \rightarrow V$ such that

$$\varphi(p + v) = \varphi(p) + \phi(v)$$

for any $p \in \mathbb{A}$ and $v \in V$.

Of course, an *affine line* $\mathbb{A}$ is an affine space of dimension 1. Once we fix a point p_0 (the *origin*) in the affine line $\mathbb{A}$ and a vector $v \neq 0$ in V , we may write any point $p \in \mathbb{A}$ in the form $p = p_0 + xv$ for some scalar $x \in K$; the scalar x is said to be the affine coordinate of p . In terms of affine coordinates, any affine transformation $\varphi: \mathbb{A} \rightarrow \mathbb{A}$ may be written in the form

$$\varphi(x) = ax + b \quad \text{for some } a, b \in K.$$

Theorem 2.1. *Let $G \times X \rightarrow X$ be an action of a group G on a set X satisfying the following conditions:*

- C_1 *Given two different points $x_1, x_2 \in X$ and another two different points $y_1, y_2 \in X$, there exists a unique transformation $g \in G$ such that $g(x_i) = y_i$ for any $i = 1, 2$.*
- C_2 *Given two different points $x, y \in X$, there exists a unique transformation $g \in G$, without fixed points, such that $g(x) = y$.*
- D *For each point $x \in X$, the isotropy subgroup H_x is commutative.*

Then X admits a natural structure of an affine line over a commutative field K such that G is the group of all affine transformations of X .

The proof of this theorem is a modification of the introduction of coordinates in the affine space used by Artin [1].

Under the hypotheses of the above theorem, let us introduce some terminology. A transformation $\tau \in G$ is said to be a translation if it has no fixed points or it is the identity. Given points $x, y \in X$, axiom C_2 states that there exists a unique translation $\tau \in G$ such that $\tau(x) = y$.

A transformation $\sigma \in G$ with a fixed point p_0 is said to be a homothety with centre p_0 . Axiom D states that the subgroup of all homotheties with centre p_0 is commutative.

Lemma 2.2. *The set T of all translations is a normal subgroup of G .*

Proof. T is a subgroup: Since the identity is a translation and the inverse of a translation is clearly a translation, we only need to show that the composition of two translations $\tau_1\tau_2$ is a translation, i.e., $\tau_1\tau_2$ has no fixed points or it is the identity. If x is a fixed point of $\tau_1\tau_2$, then $\tau_2(x) = \tau_1^{-1}(x)$. By the uniqueness condition of C_2 , we obtain that $\tau_2 = \tau_1^{-1}$; hence $\tau_1\tau_2 = Id$.

It is immediate to check that T is normal in G . □

Lemma 2.3. *Let τ_1, τ_2 be translations different from Id . Given a point $p_0 \in X$, there exists a unique homothety σ with centre p_0 such that $\sigma\tau_1\sigma^{-1} = \tau_2$.*

Proof. Let $p_1 = \tau_1(p_0)$ and $p_2 = \tau_2(p_0)$. By C_1 , there exists a unique homothety σ with centre p_0 such that $\sigma(p_1) = p_2$. Both translations $\sigma\tau_1\sigma^{-1}$ and τ_2 map p_0 to p_2 ; hence they coincide: $\sigma\tau_1\sigma^{-1} = \tau_2$. Conversely, if σ is a homothety with centre p_0 such that $\sigma\tau_1\sigma^{-1} = \tau_2$, then $(\sigma\tau_1\sigma^{-1})(p_0) = \tau_2(p_0)$; hence $\sigma(p_1) = p_2$. $\square$

Proposition 2.4. *The group T of all translations is commutative.*

Proof. Let τ_1, τ_2 be translations different from Id . By Lemma 2.3, there exists a homothety σ such that

$$\sigma(\tau_2\tau_1\tau_2^{-1})\sigma^{-1} = \tau_1.$$

If $\sigma\tau_2$ has some fixed point (i.e., it is a homothety), then the uniqueness part of Lemma 2.3 used in the equality

$$(\sigma\tau_2)\tau_1(\tau_2^{-1}\sigma^{-1}) = \tau_1$$

implies that $\sigma\tau_2 = Id$; hence $\sigma = \tau_2^{-1}$ has no fixed point, contradicting that σ is a homothety. Therefore, $\sigma\tau_2$ has no fixed point (i.e., it is a translation); hence σ is a translation. Since σ is a homothety, we conclude that $\sigma = Id$ and then

$$\tau_2\tau_1\tau_2^{-1} = \tau_1$$

or, equivalently, $\tau_2\tau_1 = \tau_1\tau_2$. $\square$

Since T is a normal subgroup of G , we may consider the action of G on T by conjugation. Let us consider the following ring with unity:

$$K := \text{End}_G(T).$$

Each element $\lambda \in K$ is an endomorphism of T which commutes with the action of G . The product operation in K is the composition of endomorphisms. Let us show that K is a field.

Proposition 2.5. *Let $\lambda: T \rightarrow T$ be an endomorphism which commutes with the action of G . Then λ is the zero map or it is a bijection.*

As a consequence, it follows that K is a field.

Proof. By Lemma 2.3, G acts transitively on $T - \{0\}$ (additive notation: 0 represents the identity translation). Since λ commutes with the action of G , we conclude that $\ker \lambda = 0$ or $\ker \lambda = T$, and $\text{Im } \lambda = T$ or $\text{Im } \lambda = 0$. $\square$

Now we shall show that the product in K is commutative.

Lemma 2.6. *Fix a point $p_0 \in X$. Given a transformation $g \in G$, there exists a unique translation τ and a unique homothety σ with centre p_0 such that $g = \sigma\tau$.*

Proof. Let $p_1 \in X$ such that $g(p_1) = p_0$ and let τ be the unique translation such that $\tau(p_1) = p_0$. Then $\sigma := g\tau^{-1}$ is the desired homothety with centre p_0 . $\square$

Let H be the subgroup of G of all homotheties with centre p_0 . H is a commutative subgroup by axiom D. The above lemma implies that the natural morphism

$$H \longrightarrow G/T$$

is an isomorphism. Since the action by conjugation $G \rightarrow \text{End}(T)$ factors through the quotient $G/T = H$ and H is commutative, we conclude that any element $g \in G$ defines, acting by conjugation on T , a scalar $\lambda_g \in K = \text{End}_G(T)$.

Proposition 2.7. *The field $K = \text{End}_G(T)$ is commutative, and T is a K -vector space of dimension 1.*

Proof. Given two non-zero translations $\tau_1, \tau_2 \in T$, there exists a homothety $\sigma \in H$ such that $\sigma\tau_1\sigma^{-1} = \tau_2$ (see Lemma 2.3); hence $\lambda_\sigma \cdot \tau_1 = \tau_2$. This fact implies that T has dimension 1 and that the morphism $H \rightarrow K^*$, $\sigma \mapsto \lambda_\sigma$, is surjective. Since H is commutative (axiom D), we conclude that K is a commutative field. $\square$

Proof of Theorem 2.1. Since T is a K -vector space of dimension 1, X inherits the structure of an affine line via the action of T on X .

Given $\sigma \in G$, for any translation $\tau \in T$ we have $\sigma\tau\sigma^{-1} = \lambda_\sigma \cdot \tau$; hence $\sigma\tau = (\lambda_\sigma \cdot \tau)\sigma$. Rewriting this last equality using the additive notation $\tau(p) = p + v$, we obtain that

$$\sigma(p + v) = \sigma(p) + \lambda_\sigma v;$$

hence σ is an affine transformation of X . Conversely, using axiom C₁, it is immediate that any affine transformation of X coincides with some element of G . $\square$

3. PROOF OF THE MAIN RESULT

With the hypotheses of Theorem 1.1, let us fix a point $p_\infty \in X$. Let

$$X' = X - \{p_\infty\}, \quad G' = \{g \in G : g(p_\infty) = p_\infty\}.$$

Lemma 3.1. *The action of G' on X' satisfies all the conditions of Theorem 2.1.*

Proof. It is clear that axioms A₁ and A₂ imply C₁ and C₂.

To prove condition D, given any point $p_0 \in X'$, let us consider a transformation $g \in G$ such that $g(p_0) = p_\infty$ and $g(p_\infty) = p_0$. By axiom B, we have that $g^2 = Id$. Let

$$H_{p_0} = \{\sigma \in G' : \sigma(p_0) = p_0\} = \{\sigma \in G : \sigma(p_0) = p_0 \text{ and } \sigma(p_\infty) = p_\infty\}$$

be the isotropy subgroup (in G') of p_0 . By conjugation with g , we obtain an automorphism of groups

$$\varphi: H_{p_0} \longrightarrow H_{p_0}, \quad \sigma \mapsto g\sigma g.$$

For any $\sigma \in H_{p_0}$, we have that $g(\sigma(p_0)) = p_\infty$ and $g(\sigma(p_\infty)) = p_0$; hence $g\sigma g\sigma = Id$ by axiom B. This implies that $\varphi(\sigma) = \sigma^{-1}$ and, since φ is an automorphism of groups, we conclude that the isotropy group H_{p_0} is commutative. $\square$

Proof of Theorem 1.1. Applying Theorem 2.1 to the action $G' \times X' \rightarrow X'$, we obtain that X' is an affine line over a commutative field K , and G' is the group of all affine transformations of X' . Taking an affine coordinate on X' , we may write $X = K \cup \{\infty\}$, i.e., X is a projective line over the field K .

Now we have to prove that G is the group of all projective transformations of X .

Let $g \in G$ be the unique transformation such that $g(0) = \infty$, $g(1) = 1$ and $g(\infty) = 0$ (note that g is an involution by axiom B). Now we shall prove that $g(x) = x^{-1}$ for any $x \in X$. Let $\sigma_\lambda \in G'$ be the homothety with centre 0 and ratio λ , i.e., $\sigma_\lambda(x) = \lambda x$. The same argument used in the proof of Lemma 3.1 gives that $g\sigma_\lambda g\sigma_\lambda = Id$; hence $g\sigma_\lambda g = \sigma_{\lambda^{-1}}$. On the other hand, $g\sigma_\lambda g = \sigma_{g(\lambda)}$ because $g\sigma_\lambda g$ is a homothety with centre 0 and $(g\sigma_\lambda g)(1) = g(\lambda)$. In conclusion, it follows that $g(\lambda) = \lambda^{-1}$.

Since G contains the transformation $x \mapsto ax + b$ (because G contains the group G' of the affine transformations) and it also contains the transformation $g : x \mapsto x^{-1}$, we conclude that G contains all the projective transformations $x \mapsto \frac{ax+b}{cx+d}$. Conversely, using the uniqueness part of axiom A_1 , it is immediate that any element of G is a projective transformation. $\square$

FURTHER COMMENTS

The projective line $\mathbb{P}_1$ over a commutative field K is usually defined as the set of all 1-dimensional subspaces of some 2-dimensional vector space V over K . This definition is not satisfactory since it presents the projective line as a set without any structure. This makes obscure the notion of an automorphism of $\mathbb{P}_1$. An automorphism of $\mathbb{P}_1$ should be a bijection preserving the “structure”, but which structure? If $\mathbb{P}_1$ were a mere set, then its automorphisms should be bijections of this set; then we ought to define the notion of projective transformation as an arbitrary bijection!

Of course, the question about the structure of $\mathbb{P}_1$ is illuminated by modern Algebraic Geometry: $\mathbb{P}_1$ *has the structure of a scheme*. Scheme automorphisms of $\mathbb{P}_1$ are just projective transformations in the sense of Staudt (= bijections of $\mathbb{P}_1$ induced by semilinear automorphisms of V), while ordinary projective transformations are automorphisms of $\mathbb{P}_1$ as a K -scheme, i.e., scheme automorphisms of $\mathbb{P}_1$ which define the identity on $H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}) = K$. A conceptually rigorous foundation of Projective Geometry may be considered as a first great result of Grothendieck’s theory of schemes.

Our characterization allows us to give an alternative “structural” definition of the projective line, in the spirit of Klein’s Erlangen Program: A projective line is a pair (X, G) where X is a set and G is a subgroup of $\text{Bij}(X)$ satisfying the conditions of Theorem 1.1. The automorphisms of this structure are the bijections $\varphi : X \rightarrow X$ preserving the subgroup G , i.e., $\varphi G \varphi^{-1} = G$. These automorphisms correspond to Staudt transformations in the conventional definition of $\mathbb{P}_1$, while the elements of G correspond to ordinary projective transformations.

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