ON SUCCESSIVE COEFFICIENTS OF ODD UNIVALENT FUNCTIONS

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ABSTRACT. The relative growth of successive coefficients of odd univalent functions is investigated. We prove that a conjecture of Hayman is true.

1. INTRODUCTION

Let S be the class of functions $f(z) = z + a_2z^2 + \dots$ regular and univalent in $|z| < 1$ and let S_2 be the subclass of odd functions $f(z) = \sum_{k=0}^{\infty} b_k z^{2k+1}$ in S . In the investigation of the relative growth of successive coefficients of functions in S_2 , Goluzin [1] has established the inequality

$$(1) \quad ||b_n| - |b_{n-1}|| \leq An^{-1/4} \ln n \quad (n = 2, 3, \dots),$$

where A is an absolute constant. In 1963, W. Hayman [2] formulated the conjecture that

$$(2) \quad ||b_n| - |b_{n-1}|| \leq A(\varepsilon)n^{-1/2+\varepsilon}$$

for every $\varepsilon > 0$. Lucas [3] and Huke [4], coming close to this conjecture, proved respectively the estimates

$$||b_n| - |b_{n-1}|| = O(n^{-b}),$$

where $b = \sqrt{2} - 1$ and $b = 0.42667$. In this paper we shall prove Hayman's conjecture. If $f_2 \in S_2$, $f_2(z) = \sqrt{f(z^2)} = \sum_{k=0}^{\infty} b_k z^{2k+1}$, where f is in S . Thus we only need to study the successive coefficients of the function $(f(z)/z)^{1/2} = \sum_{k=0}^{\infty} b_k z^k$, where f is in S . Our main result is

Theorem 1.1. *Let $f \in S$ and let the coefficients b_k ($k = 0, 1, 2, \dots$) be defined by the expansion*

$$(3) \quad (f(z)/z)^{1/2} = \sum_{k=0}^{\infty} b_k z^k.$$

Then for $n = 2, 3, \dots$,

$$(4) \quad ||b_n| - |b_{n-1}|| \leq An^{-1/2} \log n,$$

where A is an absolute constant.

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The exponent $-\frac{1}{2}$ is sharp in Theorem 1.1, since

$$f(z) = z(1 - z^4)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} b_{2k+1} z^{2k+1}$$

is univalent in $|z| < 1$ and

$$b_{4k+1} \sim k^{-\frac{1}{2}}/\sqrt{\pi}, k \rightarrow \infty,$$

while

$$b_{4k-1} = 0.$$

2. AN AUXILIARY RESULT

Assume, as usual, that the logarithmic coefficients $2\gamma_k$ ($k = 1, 2, \dots$) of the function $f(z) \in S$ are defined by the expansion

$$(5) \quad \log(f(z)/z) = 2 \sum_{k=1}^{\infty} \gamma_k z^k.$$

Theorem 2.1. *Let $f \in S$. Let $r_1 = (1 - 1/n)^{\frac{1}{2}}, r_2 = (1 - 1/2n)^{\frac{1}{2}}$ ($n = 2, 3, \dots$). If t is chosen such that $|f(t)| = \max\{|f(z)| : |z| = r_2\}$ and if $z = r_1 e^{i\theta}$, then*

$$(6) \quad \int_0^{2\pi} \left| \frac{f(z)}{z} (1 - t^{-1}z) \right| d\theta \leq A \log n,$$

where A is an absolute constant.

Proof. We consider the function $g(z) = 1/f(z^{-1})$ ($|z| > 1$) in Σ which satisfies

$$(7) \quad 1/g(z) = f(z^{-1}) = z^{-1} + \sum_{n=2}^{\infty} a_n z^{-n}.$$

Let b_{kl} ($k, l = 1, 2, \dots$) be the Grunsky coefficients of $g(z)$. We define

$$(8) \quad \alpha_k = \alpha_k(s) = \sum_{l=1}^{\infty} b_{kl} s^{-l} \quad (|s| > 1, k = 1, 2, \dots),$$

$$(9) \quad h = h(z, s) = \frac{z - s}{g(z) - g(s)} = \sum_{\nu=0}^{\infty} \beta_{\nu}(s) z^{-\nu} \quad (|z|, |s| > 1).$$

It follows that for a fixed s (see [7], p. 82),

$$(10) \quad \frac{d}{dz} \log h = \frac{1}{z - s} - \frac{g'(z)}{g(z) - g(s)} = - \sum_{k=0}^{\infty} k \alpha_k(s) z^{-k-1}$$

and

$$(11) \quad \frac{d}{dz} h = \frac{1}{g(z) - g(s)} - \frac{(z - s)g'(z)}{(g(z) - g(s))^2} = - \sum_{\nu} \nu \beta_{\nu}(s) z^{-\nu-1}.$$

The Grunsky inequality shows that (see [7], p. 83)

$$(12) \quad \sum_{k=1}^{\infty} k |\alpha_k(r)|^2 \leq \sum_{k=1}^{\infty} \frac{r^{-2k}}{k} = \log \frac{1}{1 - r^{-2}}.$$

Write $\beta_\nu = \beta_\nu(s)$. Let $|s| = \rho = r_2^{-1}$. Applying Milin's theorem (see [7], p. 80), we obtain from (12) that for $n = 1, 2, \dots$,

$$(13) \quad |\beta_n(r)|^2 \leq \exp\left\{\sum_{k=1}^n k|\alpha_k|^2\right\} \leq n^{-1} \exp\left\{\sum_{k=1}^{\infty} k|\alpha_k|^2\right\} \leq n^{-1} \frac{1}{1-\rho^{-2}} \leq A,$$

where A is an absolute constant. We obtain from (9), (10) and (11) that (see [7], p. 82)

$$(14) \quad \frac{d}{dz} \frac{z-s}{g(z)} = \left(1 - \frac{g(s)}{g(z)}\right) h \frac{d}{dz} \log h - g(s) h \frac{d}{dz} \frac{1}{g(z)}.$$

We choose a fixed s such that

$$(15) \quad |g(s)| = \min\{|g(z)| : |z| = \rho\} = \frac{1}{M(\frac{1}{\rho})},$$

where

$$(16) \quad M\left(\frac{1}{\rho}\right) = \max\{|f(z)| : |z| = \frac{1}{\rho}\}.$$

Write $z = re^{i\theta}$, $r \in [\rho, 2]$, $\theta \in [0, 2\pi]$ and $\lambda = r_1^{-1}$ ($n = 1, 2, \dots$). We obtain from (14) that

$$(17) \quad \int_0^{2\pi} \int_{\lambda}^2 \left| \frac{d}{dz} \frac{z-s}{g(z)} \right| dr d\theta \\ \leq \int_0^{2\pi} \int_{\lambda}^2 \left| 1 - \frac{g(s)}{g(z)} \right| \left| h \frac{d}{dz} \log h \right| dr d\theta + \int_0^{2\pi} \int_{\lambda}^2 |g(s)| \left| h \frac{d}{dz} \frac{1}{g(z)} \right| dr d\theta.$$

Now we estimate the two terms I_1 and I_2 on the right-hand side of (17).

(a) Since $|g(s)| \leq |g(z)|$, by (15) and (16), we obtain from Schwarz's inequality and (9) and (10) that

$$(18) \quad |I_1| \leq 2 \left[\int_0^{2\pi} \int_{\lambda}^2 |h|^2 d\theta dr \right]^{\frac{1}{2}} \left[\int_0^{2\pi} \int_{\lambda}^2 \left| \frac{d}{dz} \log h \right|^2 d\theta dr \right]^{\frac{1}{2}} \\ \leq 4\pi \left[\int_{\lambda}^2 \sum_{\nu=0}^{\infty} |\beta_\nu|^2 r^{-2\nu} dr \right]^{\frac{1}{2}} \left[\int_{\lambda}^2 \sum_{\nu=1}^{\infty} \nu^2 |\alpha_\nu|^2 r^{-2\nu-2} \right]^{\frac{1}{2}} \\ \leq 4\pi \left[2 + \sum_{\nu=1}^{\infty} \frac{1}{2\nu-1} |\beta_\nu|^2 \lambda^{-2\nu+1} \right]^{\frac{1}{2}} \left[\sum_{\nu=1}^{\infty} \frac{\nu}{2\nu+1} \nu |\alpha_\nu|^2 \lambda^{-2\nu-1} \right]^{\frac{1}{2}}.$$

It follows from (12), (13) and (18) that

$$(19) \quad |I_1| \leq A \log \frac{1}{1-\lambda^{-2}} \leq A \log n,$$

where A is an absolute constant.

(b) We apply Schwarz's inequality again to estimate I_2 . It follows from (7) and (13) that

$$\begin{aligned}
 |I_2| &\leq |g(s)| \left[\int_{\lambda}^2 \int_0^{2\pi} |h|^2 d\theta dr \right]^{\frac{1}{2}} \left[\int_{\lambda}^2 \int_0^{2\pi} \left| \frac{d}{dz} \frac{1}{g(z)} \right|^2 d\theta dr \right]^{\frac{1}{2}} \\
 (20) \quad &\leq 2\pi |g(s)| \left[2 + 2 + \sum_{\nu=1}^{\infty} \frac{1}{2\nu-1} |\beta_{\nu}|^2 \lambda^{-2\nu+1} \right]^{\frac{1}{2}} \left[\int_{\lambda}^2 \sum_{\nu=1}^{\infty} \nu^2 |\alpha_{\nu}|^2 r^{-2\nu} dr \right]^{\frac{1}{2}} \\
 &\leq A |g(s)| \left[\log \frac{1}{1-\rho^{-2}} \right]^{\frac{1}{2}} \left[\sum_{\nu=1}^{\infty} \frac{\nu}{2\nu-1} \nu |\alpha_{\nu}|^2 \lambda^{-2\nu+1} \right]^{\frac{1}{2}},
 \end{aligned}$$

where A is an absolute constant. Because $f(z)$ is univalent in $|z| < 1$, we obtain from (15) and (16) that

$$(21) \quad \sum_{\nu=1}^{\infty} \frac{\nu}{2\nu-1} \nu |\alpha_{\nu}|^2 \lambda^{-2\nu} \leq \frac{1}{\pi} \int \int_{|z| < 1/\rho} |f'(z)|^2 d\sigma \leq (M(\frac{1}{\rho}))^2 = |g(s)|^2.$$

We obtain from (20) and (21) that

$$(22) \quad |I_2| \leq A \log \frac{1}{1-\rho^{-2}} \leq A \log n,$$

where A is an absolute constant. The estimate obtained in (a) and (b) for the terms on the right-hand side of (17) shows that there exists an absolute constant A such that

$$(23) \quad \int_0^{2\pi} \int_{\lambda}^2 \left| \frac{d}{dz} \frac{z-s}{g(z)} \right| dr d\theta \leq A \log n.$$

Let $z_1 = 2e^{i\theta}$, $z_2 = \lambda e^{i\theta}$ and $z = re^{i\theta}$, $r \in [\rho, 2]$. We obtain from (23) that

$$\begin{aligned}
 &\int_0^{2\pi} \left| \frac{z_2-s}{g(z_2)} \right| d\theta - \int_0^{2\pi} \left| \frac{z_1-s}{g(z_1)} \right| d\theta \leq \int_0^{2\pi} \int_{\lambda}^2 \left| \frac{\partial}{\partial r} \left| \frac{z-s}{g(z)} \right| \right| dr d\theta \\
 (24) \quad &\leq \int_0^{2\pi} \int_{\lambda}^2 \left| \frac{d}{dz} \frac{z-s}{g(z)} \right| dr d\theta \leq A \log n.
 \end{aligned}$$

It is clear that

$$(25) \quad \int_0^{2\pi} \left| \frac{z_1-s}{g(z_1)} \right| d\theta \leq 2\pi M\left(\frac{1}{2}\right)(|z_1| + |s|) \leq A,$$

where A is an absolute constant. We obtain from (24) and (25) that

$$(26) \quad \int_0^{2\pi} |f(z_2^{-1})| |z_2^{-1} - s^{-1}| d\theta = \frac{1}{|z_2 s|} \int_0^{2\pi} \left| \frac{z_2-s}{g(z_2)} \right| d\theta \leq A \log n,$$

where A is an absolute constant. The inequality (26) implies for $z = r_1 e^{i\theta}$, $|t| = r_2$ and $|f(t)| = \max\{|f(z)| : |z| = r_2\}$ that

$$(27) \quad \int_0^{2\pi} |f(z)| |z-t| d\theta \leq A \log n.$$

□

3. PROOF OF THEOREM 1.1

Proof. We define two functions and give their power series expansions (we choose single-valued branch with value 1 at $z = 0$) when $|z| < r_2$:

$$(28) \quad u(z) = (f(z)/z)^{1/2}(1 - t^{-1}z)^{1/2} = \sum_{k=0}^{\infty} d_k z^k,$$

$$(29) \quad v(z) = u(z)(1 - t^{-1}z)^{1/2} = (f(z)/z)^{1/2}(1 - t^{-1}z) = \sum_{k=0}^{\infty} c_k z^k,$$

where $|t| = r_2 = [1 - 1/(2n)]^{\frac{1}{2}}$ and $|f(t)| = \max\{|f(z)| : |z| = r_2\}$. Theorem 2.1 shows that

$$(30) \quad \sum_{k=0}^{\infty} |d_k|^2 r_1^{2k} = \frac{1}{2\pi} \int_0^{2\pi} |u(r_1 e^{i\theta})|^2 d\theta \leq A \log n.$$

Hence, we obtain from (30) that

$$(31) \quad \sum_{k=0}^{\infty} |c_k|^2 r_1^{2k} = \frac{1}{2\pi} \int_0^{2\pi} |v(r_1 e^{i\theta})|^2 d\theta \leq \frac{2}{2\pi} \int_0^{2\pi} |u(r_1 e^{i\theta})|^2 d\theta \leq A \log n,$$

where A is an absolute constant. Let

$$(32) \quad \log v(z) = \sum_{k=1}^{\infty} A_k z^k$$

for $|z| < r_2$. Then it follows from (29) that

$$(33) \quad A_k = \gamma_k - t^{-k}/k \quad (k = 1, 2, \dots).$$

It is clear that $v'(z) = v(z)[\log v(z)]'$. Comparing coefficients, we obtain the recursion formula

$$(34) \quad nc_n = \sum_{k=1}^n k A_k c_{n-k}.$$

Applying Schwarz's inequality to the recursion formula (34), we obtain that

$$(35) \quad \begin{aligned} n^2 |c_n|^2 r_1^{2n} &\leq \sum_{k=1}^n k^2 |A_k|^2 r_1^{2k} \sum_{k=0}^n |c_k|^2 r_1^{2k} \leq n \sum_{k=1}^n k |A_k|^2 r_1^{2k} \sum_{k=0}^n |c_k|^2 r_1^{2k} \\ &\leq n \sum_{k=1}^{\infty} k |A_k|^2 r_1^{2k} \sum_{k=0}^{\infty} |c_k|^2 r_1^{2k}. \end{aligned}$$

It follows by (33) that

$$(36) \quad \sum_{k=1}^{\infty} k |A_k|^2 r_1^{2k} \leq 2 \left(\sum_{k=1}^{\infty} k |\gamma_k|^2 r_1^{2k} + \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{r_1}{r_2} \right)^{2k} \right).$$

Since $r_1/r_2 < r_2$, it follows that

$$(37) \quad \sum_{k=1}^{\infty} k^{-1} \left(\frac{r_1}{r_2} \right)^{2k} < \sum_{k=1}^{\infty} k^{-1} r_2^{2k} = \log \frac{1}{1 - r_2^2} < \log 2n.$$

On the other hand, we have (see [6], pp. 283-290)

$$(38) \quad \sum_{k=1}^{\infty} k |\gamma_k|^2 r_1^{2k} \leq \max\{\log |f(z)/z| : |z| = r_1\} \leq \log \frac{1}{(1-r_1)^2} < 2 \log n.$$

Combining (37) and (38), we obtain from (36) that

$$(39) \quad \sum_{k=1}^{\infty} k |A_k|^2 r_1^{2k} \leq 6 \log 2n.$$

Thus, it follows from (31), (39) and (35) that

$$(40) \quad |c_n| \leq A n^{-1/2} \log n,$$

where A is an absolute constant. It follows from (29) that $c_n = b_n - t^{-1}b_{n-1}$. Hence, we have

$$(41) \quad ||b_n| - |b_{n-1}|| \leq |c_n| + \left(1 - \frac{1}{2n}\right)^{-\frac{1}{2}} - 1 |b_{n-1}|.$$

It is well known that $|b_n| < 1.17$ ($n = 1, 2, \dots$) (see [5]). We obtain from (40) and (41) that

$$(42) \quad ||b_n| - |b_{n-1}|| \leq A n^{-\frac{1}{2}} \log n + O\left(\frac{1}{n}\right).$$

Finally, Theorem 1 follows from (42). $\square$

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