A REMARK ON THE HOMOMORPHISM ON $C(X)$

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ABSTRACT. Let X be a real compact space. Without using the axiom of choice we present a simple and direct proof that a non-zero homomorphism on $C(X)$ is determined by a point.

As usual $C(X)$ stands for the algebra of real valued continuous functions on a topological space X under the pointwise operations, that is, for each $f, g, h \in C(X)$ and $\alpha \in \mathbb{R}$,

$$(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x).$$

A map $\varphi : C(X) \longrightarrow \mathbb{R}$ is called an (*algebra*) *homomorphism* if

$$\varphi(fg + \alpha h) = \varphi(f)\varphi(g) + \alpha\varphi(h)$$

for each $f, g, h \in C(X)$ and $\alpha \in \mathbb{R}$. In [4] it is shown that a map $\varphi : C(X) \longrightarrow \mathbb{R}$ is a homomorphism if and only if it is a *ring homomorphism*, that is,

$$\varphi(fg + h) = \varphi(f)\varphi(g) + \varphi(h)$$

for each $f, g, h \in C(X)$. For each $r \in \mathbb{R}$, $\mathbf{r} : X \longrightarrow \mathbb{R}$ is defined by $\mathbf{r}(x) = r$. The characteristic function of $A \subset X$ is denoted by χ_A .

A Hausdorff topological space X is called a *real compact space* if it is homeomorphic to a closed subspace of the product space $\prod_{i \in I} \mathbb{R}$ for some index set I . For each $j \in I$, $P_j : \prod_{i \in I} \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $P_j((x_i)) = x_j$ and $e_j = (x_i)$ with $x_j = 1$ and $x_i = 0$ if $i \neq j$.

The following theorem is well known.

Theorem 1. *Let X be a real compact space and let $\varphi : C(X) \longrightarrow \mathbb{R}$ be a nonzero map. Then φ is a homomorphism if and only if there exists a unique $c \in X$ such that $\varphi(f) = f(c)$ for each $f \in C(X)$.*

The known proofs of the above theorem depend on Stone-Cech compactification or (equivalently) on the maximal ideal theory (see [2], p. 241, or [3], p. 142). In [1] a simple and direct proof of the above theorem is given for $X = \mathbb{R}$ by only using properties of continuous functions. More precisely, Aron and Fricke proved that

$$\varphi(f) = f(\varphi(i)),$$

where $i : \mathbb{R} \longrightarrow \mathbb{R}$ is the identity map, that is, $i(x) = x$. By following the technique of [1] we can reprove the above theorem simply and directly without using any

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compactification or ideal theory. We believe that this proof may be of interest because of its simplicity and accessibility.

Proof of Theorem 1. First we prove the theorem for $X = \prod_{i \in I} \mathbb{R}$. Let

$$c = (c_i) = (\varphi(P_i)).$$

We show that for each $f \in C(X)$

$$\varphi(f) = f(c).$$

For each $f \in C(X)$ let

$$k_f : X \longrightarrow \mathbb{R} \quad \text{be defined by} \quad k_f = f - f(c)\mathbf{1}.$$

Then $\varphi(k_f) = \varphi(f) - f(c)$ and

$$\varphi(k_f) = 0 \iff \varphi(f) = f(c).$$

Hence it is enough to show that

$$f(c) = 0 \implies \varphi(f) = 0.$$

We claim that if $f|_U \equiv 0$ for some open set U which contains c , then $\varphi(f) = 0$. There exists a family $(U_i)_{i \in I}$ of open subsets of $\mathbb{R}$ such that

$$c \in \prod_i U_i = V \subset U \quad \text{and} \quad F = \{i \in I : U_i \neq \mathbb{R}\} \quad \text{is finite.}$$

Let

$$h = \sum_{i \in F} (P_i - \mathbf{c}_i)^2.$$

It is clear that $h(x) \neq 0$ whenever $x \notin V$. Define

$$g : X \longrightarrow \mathbb{R}, \quad g(x) = \frac{f(x)}{h(x)} \chi_{X \setminus V}.$$

Then $g \in C(X)$ and

$$f = gh.$$

This implies that

$$\varphi(f) = \varphi(g)\varphi(h) = \varphi(g)(\sum_{i \in F} (\varphi(P_i) - c_i)^2) = 0.$$

Now suppose that there exists $f \in C(X)$ with $\varphi(f) = 2$ and $f(c) = 0$. Let

$$g : \mathbb{R} \longrightarrow \mathbb{R} \quad \text{be defined by} \quad g(x) = -\chi_{(-\infty, -1]}(x) + x\chi_{(-1, 1]}(x) + \chi_{(1, \infty)}(x).$$

Then $g \in C(\mathbb{R})$. Let $U = f^{-1}(-1, 1)$. Then $c \in U$ and

$$f - g \circ f = 0 \quad \text{on} \quad U.$$

From the above observation $2 = \varphi(f) = \varphi(g \circ f)$. This implies that

$$1 = \varphi((\mathbf{2} - g \circ f)^{-1}(\mathbf{2} - g \circ f)) = \varphi((\mathbf{2} - g \circ f)^{-1})(\varphi(\mathbf{2}) - \varphi(g \circ f)) = 0,$$

which is a contradiction. Clearly this shows that $\varphi(f) = 0$ whenever $f(c) = 0$. Now suppose that X is homeomorphic to a closed subset $Y = \prod_{i \in I} \mathbb{R}$ for some set I . Let

$$\psi : C(Y) \longrightarrow C(X) \quad \text{be defined by} \quad \psi(f) = f|_X,$$

where $f|_X$ denotes the restriction of f to X . Then there exists $c \in Y$ such that

$$\varphi \circ \psi(f) = f(c) \quad \text{for each} \quad f \in C(Y).$$

Suppose that $c \notin X$. As Y is a completely regular Hausdorff space, there exists $f \in C(Y)$ such that

$$f(c) = 1 \quad \text{and} \quad f|_X \equiv 0.$$

This implies that

$$1 = f(c) = \varphi \circ \psi(f) = \varphi(f|_X) = 0,$$

which is a contradiction and completes the proof.

The proofs of the following corollaries immediately follow from the theorem.

Corollary 2. *Let $F(\mathbb{N})$ be the algebra of real valued functions on $\mathbb{N}$ with respect to the pointwise operations. For each non-zero homomorphism $\varphi : F(\mathbb{N}) \rightarrow \mathbb{R}$ there exists a unique $n \in \mathbb{N}$ such that $\varphi(f) = f(n)$ for each $f \in F(\mathbb{N})$.*

Corollary 3. *Let X be a real compact Hausdorff space and let Y be a completely regular Hausdorff space. Let $\varphi : C(X) \rightarrow C(Y)$ be a non-zero homomorphism. Then there exists a continuous function $\sigma : Y \rightarrow X$ such that*

$$\varphi(f) = f \circ \sigma.$$

The following corollary is known as the *Banach-Stone theorem*.

Corollary 4. *Let X be a real compact space and let Y be a completely regular Hausdorff space. Then $C(X)$ and $C(Y)$ are ring isomorphic if and only if X and Y are homeomorphic.*

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