EXTENSION OF SIMONS' INEQUALITY

KERSTI KIVISOO AND EVE OJA

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ABSTRACT. We prove the following extended version of Simons' inequality and present its applications. Let S be a set and T be a subset of S . Let C be a subset of a Hausdorff topological vector space which is invariant under infinite convex combinations. Let $f : C \times S \rightarrow \mathbb{R}$ be a bounded function such that the functions $f(\cdot, t) : C \rightarrow \mathbb{R}$ are convex for all $t \in T$ and $f(\lambda x, s) = \lambda f(x, s)$ whenever $\lambda > 0$, $x, \lambda x \in C$ and $s \in S$. Let (x_n) be a sequence in C . Assume that, for every $x \in C_1 = \{\sum_{n=1}^{\infty} \lambda_n x_n : \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1\}$, there exists $t \in T$ satisfying $f(x, t) = \sup_{s \in S} f(x, s)$. Then

$$\inf_{x \in C_1} \sup_{s \in S} f(x, s) \leq \sup_{t \in T} \limsup_n f(x_n, t).$$

If $-C_1 \subset C$, then the set C_1 in the above inequality can be replaced by $\text{conv}\{x_1, x_2, \dots\}$.

1. INTRODUCTION AND THE MAIN RESULT

The remarkable inequality of Simons [S1, Lemma 2] (see also e.g. [HHZ, p. 49]) has important applications to real analysis and the geometry of Banach spaces (see e.g. [AG], [FG], [FHHMPZ], [G1], [G2], [GZ], [HHZ], [O2], [O3], [S1], [S2]).

Theorem 1 (Simons' inequality). *Let S be a set and let T be a subset of S . Let (x_n) be a bounded sequence in $l_{\infty}(S)$, the Banach space of bounded real functions on S . Assume that, for every*

$$x \in C_1 = \left\{ \sum_{n=1}^{\infty} \lambda_n x_n : \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1 \right\},$$

there exists $t \in T$ satisfying

$$(1) \quad x(t) = \sup_{s \in S} x(s).$$

Then the following inequality holds:

$$(2) \quad \inf_{x \in C_0} \sup_{s \in S} x(s) \leq \sup_{t \in T} \limsup_n x_n(t),$$

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where C_0 denotes the convex hull of the sequence (x_n) , that is,

$$C_0 = \text{conv}\{x_1, x_2, \dots\} = \left\{ \sum_{n=1}^m \lambda_n x_n : m \in \mathbb{N}, \lambda_n \geq 0, \sum_{n=1}^m \lambda_n = 1 \right\}.$$

Recently, Deville and Finet [DF] extended Theorem 1 by taking instead of $l_\infty(S)$ a topological vector space and by introducing a function $f : C_1 \times S \rightarrow \mathbb{R}$ in conditions (1) and (2).

Theorem 2 (see [DF]). *Let S be a set and let C be a subset of a Hausdorff topological vector space which is invariant under infinite convex combinations. Let $f : C \times S \rightarrow \mathbb{R}$ be a bounded function such that the functions $f(\cdot, s) : C \rightarrow \mathbb{R}$, $s \in S$, are convex and there exists a neighbourhood $\mathcal{U}$ of 0 such that*

$$|f(x, s) - f(y, s)| \leq p_{\mathcal{U}}(x - y), \quad x, y \in C, s \in S,$$

where $p_{\mathcal{U}}$ is the Minkowski functional of $\mathcal{U}$. Assume that, for every $x \in C$, there exists $t \in S$ satisfying

$$f(x, t) = \sup_{s \in S} f(x, s).$$

Then, for every sequence (x_n) in C , the following inequality holds:

$$\inf_{x \in C} \sup_{s \in S} f(x, s) \leq \sup_{s \in S} \limsup_n f(x_n, s).$$

In the present article, inspired by Theorem 2, we shall prove the following extension of Theorem 1, where the continuity assumption on $f(\cdot, s)$, $s \in S$, in Theorem 2 is replaced by a purely algebraic one. Moreover, like in Theorem 1, the subset T is not necessarily equal to S (cf. Theorem 2), a fact that has been crucial in applications of Simons' inequality (see e.g. [AG], [FG], [G1], [G2], [GZ], [O3], [S1]).

Theorem 3. *Let S be a set and let T be a subset of S . Let C be a subset of a Hausdorff topological vector space which is invariant under infinite convex combinations. Let $f : C \times S \rightarrow \mathbb{R}$ be a bounded function such that*

$$(3) \quad \text{the functions } f(\cdot, t) : C \rightarrow \mathbb{R}, t \in T, \text{ are convex}$$

and

$$(4) \quad f(\lambda x, s) = \lambda f(x, s) \text{ whenever } \lambda > 0, x, \lambda x \in C \text{ and } s \in S.$$

Let (x_n) be a sequence in C . Assume that, for every

$$x \in C_1 = \left\{ \sum_{n=1}^{\infty} \lambda_n x_n : \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1 \right\},$$

there exists $t \in T$ satisfying

$$f(x, t) = \sup_{s \in S} f(x, s).$$

Then the following inequality holds:

$$(5) \quad \inf_{x \in C_1} \sup_{s \in S} f(x, s) \leq \sup_{t \in T} \limsup_n f(x_n, t).$$

If, moreover, $-C_1 \subset C$, then the set C_1 in (5) can be replaced by $\text{conv}\{x_1, x_2, \dots\}$.

Remark 1. Let (x_n) be a bounded sequence in a Banach space and let C be the closed absolutely convex hull of (x_n) . It is straightforward to verify that C is invariant under infinite convex combinations. Therefore Theorem 1 immediately follows from Theorem 3 if one sets

$$f(x, s) = x(s), \quad x \in C, s \in S.$$

The beautiful technical proof of Theorem 2 due to Deville and Finet [DF] uses ideas inherent in the original proof of Theorem 1 (see [S1] or [HHZ, pp. 49–51] or [G2]). In [O1] (see [FHHMPZ, pp. 80–81]), the second-named author found a simple direct proof of Theorem 1 that, unfortunately, does not work for Theorem 2. In contrast, Theorem 3 can and will be proved (in Section 2) relying on ideas of the proof of Theorem 1 in [O1].

The main application on a pointwise convergent sequence of vector-valued functions deduced from Theorem 2 in [DF] (see Corollary 2 in Section 3 below) is also immediate from our Theorem 3, thus providing an easier proof for this result.

2. PROOF OF THEOREM 3

The proof of Theorem 3 will be given as a consequence of some elementary lemmas. Let us recall that S is a set, T is a subset of S , and C is a subset of a Hausdorff topological vector space. The set C is assumed to be invariant under infinite convex combinations, meaning that

$$\sum_{n=1}^{\infty} \lambda_n x_n \in C$$

for every sequence (x_n) in C and for every sequence of non-negative numbers (λ_n) such that $\sum_{n=1}^{\infty} \lambda_n = 1$. In particular, C is a convex set. We also have a bounded function $f : C \times S \rightarrow \mathbb{R}$ satisfying conditions (3) and (4).

We now fix some more notation. Let

$$\mathcal{C} = \bigcup_{\alpha \geq 1} \alpha C.$$

We extend f from $C \times S$ to $\mathcal{C} \times S$ defining

$$f(\alpha x, s) = \alpha f(x, s), \quad \alpha \geq 1, x \in C, s \in S.$$

Clearly, the function

$$f : \mathcal{C} \times S \rightarrow \mathbb{R}$$

is well-defined (by (4)), bounded on every subset $\alpha C \times S$, where $\alpha \geq 1$, and

$$(6) \quad f(\lambda z, s) = \lambda f(z, s), \quad \lambda \geq 1, z \in \mathcal{C}, s \in S.$$

Lemma 1. *If $z_1, z_2 \in \mathcal{C}$, then $z_1 + z_2 \in \mathcal{C}$ and*

$$f(z_1 + z_2, t) \leq f(z_1, t) + f(z_2, t), \quad t \in T,$$

meaning that the functions $f(\cdot, t) : \mathcal{C} \rightarrow \mathbb{R}$, $t \in T$, are subadditive.

Proof. If $z_1, z_2 \in \mathcal{C}$, then $z_1 = \alpha x$ and $z_2 = \beta y$ with $\alpha, \beta \geq 1$ and $x, y \in C$. Hence

$$\gamma := \alpha + \beta \geq 1, \quad \lambda := \frac{\alpha}{\gamma} \in (0, 1), \quad \text{and } \beta = \gamma(1 - \lambda).$$

Thus $\lambda x + (1 - \lambda)y \in C$, and so

$$z_1 + z_2 = \gamma(\lambda x + (1 - \lambda)y) \in \mathcal{C}.$$

Since the functions $f(\cdot, t)$, $t \in T$, are convex on C ,

$$\begin{aligned} f(z_1 + z_2, t) &= f(\gamma(\lambda x + (1 - \lambda)y), t) \\ &= \gamma f(\lambda x + (1 - \lambda)y, t) \\ &\leq \gamma \lambda f(x, t) + \gamma(1 - \lambda)f(y, t) \\ &= \alpha f(x, t) + \beta f(y, t) \\ &= f(z_1, t) + f(z_2, t). \end{aligned}$$

□

Let $(x_n) = (x_n)_{n=1}^\infty$ be a sequence in C . Below, we use the notation

$$C_k = \left\{ \sum_{n=k}^\infty \lambda_n x_n : \lambda_n \geq 0, \sum_{n=k}^\infty \lambda_n = 1 \right\}, \quad k \in \mathbb{N}.$$

Clearly $C \supset C_1 \supset C_2 \supset \dots$

Lemma 2. *If $y \in C_k$, $k \in \mathbb{N}$, and $t \in T$, then*

$$f(y, t) \leq \sup_{n \geq k} f(x_n, t).$$

Proof. Let

$$y = \sum_{n=k}^\infty \lambda_n x_n, \quad \lambda_n \geq 0, \quad \sum_{n=k}^\infty \lambda_n = 1.$$

If there exists an $l \in \mathbb{N}$ such that $\sum_{n=k}^l \lambda_n = 1$, then the claim is immediate from the convexity of $f(\cdot, t) : C \rightarrow \mathbb{R}$ (assumption (3)).

Let us assume that $\sum_{n=k}^l \lambda_n < 1$ for all $l \in \mathbb{N}$. For $\epsilon \in (0, 1/2)$, choose $l \in \mathbb{N}$ so that

$$\lambda := \sum_{n=k}^l \lambda_n > 1 - \epsilon.$$

Since $y \in C \subset \mathcal{C}$ and $1/(1 - \lambda) > 1$, we have, by (6),

$$\frac{1}{1 - \lambda} f(y, t) = f\left(\frac{y}{1 - \lambda}, t\right) = f(z_1 + z_2, t),$$

where

$$z_1 = \frac{\lambda}{1 - \lambda} \sum_{n=k}^l \frac{\lambda_n}{\lambda} x_n \in \frac{\lambda}{1 - \lambda} C \subset \mathcal{C}$$

(notice that $\lambda/(1 - \lambda) > (1 - \epsilon)/\epsilon > 1$) and

$$z_2 = \sum_{n=l+1}^\infty \frac{\lambda_n}{1 - \lambda} x_n \in C \subset \mathcal{C}.$$

Using Lemma 1, (6), and (3), and denoting

$$M = \sup_{x \in C} |f(x, t)|,$$

we get that

$$f(z_1 + z_2, t) \leq \frac{\lambda}{1 - \lambda} \sup_{n \geq k} f(x_n, t) + M.$$

Hence

$$\begin{aligned} f(y, t) &\leq \lambda \sup_{n \geq k} f(x_n, t) + (1 - \lambda) M \\ &\leq \max \left\{ \sup_{n \geq k} f(x_n, t); (1 - \epsilon) \sup_{n \geq k} f(x_n, t) \right\} + \epsilon M, \end{aligned}$$

and the claim follows since $\epsilon \in (0, 1/2)$ is arbitrary. $\square$

Lemma 3. Let $k \in \mathbb{N}$. If $\mu_m \geq 0$, $\sum_{m=k}^{\infty} \mu_m = 1$, and $z_m \in C_m$, then

$$y := \sum_{m=k}^{\infty} \mu_m z_m \in C_k.$$

Proof. Notice that $y \in C$ since $z_m \in C_m \subset C$. Let

$$z_m = \sum_{n=m}^{\infty} \lambda_{nm} x_n, \quad \lambda_{nm} \geq 0, \quad \sum_{n=m}^{\infty} \lambda_{nm} = 1.$$

Denote

$$a_{nm} = \mu_m \lambda_{nm}, \quad m \geq k, n \geq m;$$

then

$$y = \sum_{m=k}^{\infty} \sum_{n=m}^{\infty} a_{nm} x_n.$$

Let $(b_i)_{i=1}^{\infty}$ be any rearrangement of the elements of the matrix (a_{nm}) as a sequence. Since

$$\sum_{m=k}^{\infty} \sum_{n=m}^{\infty} a_{nm} = 1,$$

we also have

$$\sum_{i=1}^{\infty} b_i = 1.$$

In particular,

$$(7) \quad \sum_{n=k}^{\infty} \sum_{m=k}^n a_{nm} = 1.$$

Let us now denote

$$u_{nm} = a_{nm} x_n, \quad m \geq k, n \geq m,$$

and let $(v_i)_{i=1}^{\infty}$ be any rearrangement of the elements of the matrix (u_{nm}) as a sequence. Since $v_i = b_i x_{n(i)}$ with $\sum_{i=1}^{\infty} b_i = 1$, $b_i \geq 0$, and $x_{n(i)} \in C$, the series $\sum_{i=1}^{\infty} v_i$ converges in C . This implies that the series $\sum_{i=1}^{\infty} v_{\pi(i)}$ converges for every permutation π of the natural numbers. It is well known and can be easily verified that then the sum of $\sum_{i=1}^{\infty} v_{\pi(i)}$ does not depend on the permutation π . (In fact, if two rearrangements of the series gave different sums x and y , then one could combine these two rearrangements to obtain a rearrangement of the series such that a subsequence of its partial sums would converge to x and another to y .)

Denote this uniquely determined sum by x . In particular, by (7), we have that

$$x = \sum_{n=k}^{\infty} \sum_{m=k}^n a_{nm} x_n \in C_k.$$

Therefore it suffices to prove that $y = x$.

Recall that

$$y = \sum_{m=k}^{\infty} \sum_{n=m}^{\infty} u_{nm}.$$

Let $\mathcal{U}$ be any closed neighbourhood of 0 and let $\mathcal{V}$ be a neighbourhood of 0 satisfying $\mathcal{V} - \mathcal{V} \subset \mathcal{U}$. Choose $M_1 \geq k$ so that

$$\sum_{m=k}^M \sum_{n=m}^{\infty} u_{nm} - y \in \mathcal{V}$$

whenever $M \geq M_1$. Next choose $M_2 > M_1$ so that

$$\sum_{m=k}^{M_1} \sum_{n=M_2+1}^{\infty} u_{nm} \in \mathcal{V}.$$

We continue in an obvious manner to obtain a sequence of indices $k \leq M_1 < M_2 < M_3 < \dots$ so that

$$\sum_{m=k}^{M_j} \sum_{n=M_{j+1}+1}^{\infty} u_{nm} \in \mathcal{V}, \quad j \in \mathbb{N},$$

and therefore

$$\begin{aligned} \sum_{m=k}^{M_j} \sum_{n=M_{j+1}}^{M_{j+1}} u_{nm} - y &= \sum_{m=k}^{M_j} \sum_{n=m}^{\infty} u_{nm} - y - \sum_{m=k}^{M_j} \sum_{n=M_{j+1}+1}^{\infty} u_{nm} \\ (8) \quad &\in \mathcal{V} - \mathcal{V} \subset \mathcal{U}, \quad j \in \mathbb{N}. \end{aligned}$$

We can choose inductively a rearrangement (v_i) of the matrix (u_{nm}) so that

$$(9) \quad \sum_{m=k}^{M_j} \sum_{n=m}^{M_{j+1}} u_{nm} = \sum_{i=1}^{N_j} v_i, \quad j \in \mathbb{N},$$

for some sequence $N_1 < N_2 < \dots$ of integers. Indeed, we start by picking the elements u_{nm} with $m \in \{k, \dots, M_1\}$ and $n \in \{m, \dots, M_2\}$. Next we add the elements u_{nm} with $m \in \{M_1 + 1, \dots, M_2\}$ and $n \in \{m, \dots, M_3\}$, and also those with $(m, n) \in \{k, \dots, M_1\} \times \{M_2 + 1, \dots, M_3\}$. We continue in an obvious way.

Since

$$\lim_{j \rightarrow \infty} \sum_{i=1}^{N_j} v_i = x,$$

we get from (8) and (9) that $x - y \in \mathcal{U}$. This means that $x - y$ is contained in all neighbourhoods of 0. Hence $x = y$ as desired. $\square$

Remark 2. Lemma 3 holds in the stronger form with $z_m \in C_k$ (instead of $z_m \in C_m$). (In fact, an obvious modification of the above proof (look at $n \geq k$ instead of $n \geq m$) shows that $y = x$, where x is the uniquely determined sum of the rearrangements (as in the above proof). Let $z = \sum_{n=k}^{\infty} \sum_{m=k}^{\infty} a_{nm} x_n$. Then $z = x$, by the same proof. Hence $y = z$. Since clearly $z \in C_k$, we have $y \in C_k$ as desired.) This form of Lemma 3 extends a result by Galán and Simons [GS, Lemma 2] from real sequentially complete Hausdorff locally convex spaces to general Hausdorff topological vector spaces.

Proof of Theorem 3. Denote

$$\sigma(z) = \sup_{s \in S} f(z, s), \quad z \in \mathcal{C},$$

and notice that $\sigma(z) \in \mathbb{R}$ because if $z = \alpha x$ with $\alpha \geq 1$ and $x \in C$, then $\sigma(z) = \alpha \sigma(x)$ and the function f is bounded on $C \times S$. We also notice that, by (6),

$$\sigma(\lambda z) = \lambda \sigma(z), \quad \lambda \geq 1, z \in \mathcal{C}.$$

To establish inequality (5), that is,

$$\inf_{x \in C_1} \sigma(x) \leq \sup_{t \in T} \limsup_n f(x_n, t) =: \sigma_T,$$

it clearly suffices to prove that, for any $\epsilon > 0$, there exist $v \in C_1$, $y_m \in C_{m+1}$ with $m \in \mathbb{N}$, and $t \in T$ satisfying

$$(10) \quad \sigma(v) - \epsilon \leq f(y_m, t), \quad m \in \mathbb{N}.$$

Indeed, by Lemma 2, we would then have

$$\begin{aligned} \inf_{x \in C_1} \sigma(x) - \epsilon &\leq \sigma(v) - \epsilon \\ &= \lim_m \sup_{n \geq m+1} f(x_n, t) \\ &= \limsup_m f(x_m, t) \leq \sigma_T, \end{aligned}$$

yielding the desired inequality because $\epsilon > 0$ is arbitrary.

Let $\epsilon > 0$. Developing the idea of the proof of Simons' inequality in [O1], we shall define $v \in C_1$ and $y_m \in C_{m+1}$, $m \in \mathbb{N}$, by using some inductively constructed sequences $(v_k)_{k=0}^\infty$ and $(z_k)_{k=1}^\infty$. Their construction will be based on the simple observation that

$$\inf_{z \in 2^k C} \sigma(z) \in \mathbb{R}, \quad k = 0, 1, \dots$$

(since f is bounded on the sets $2^k C \times S$).

We start by taking $v_0 = 0$. Using the fact that

$$\inf_{z \in C_1} \sigma(z) \in \mathbb{R}$$

(because $C_1 \subset C$), we choose $z_1 \in C_1$ so that

$$\sigma(z_1) \leq \inf_{z \in C_1} \sigma(z) + \frac{\epsilon}{2}.$$

Once $z_k \in C_k$, $k \in \mathbb{N}$, has been chosen, we put

$$v_k = \sum_{n=1}^k \frac{z_n}{2^n}.$$

Then, for all $z \in C$, in particular, for all $z \in C_{k+1}$, we have

$$2^k v_k + z = 2^k \left(v_k + \frac{z_k}{2^k} \right) \in 2^k \left(\sum_{n=1}^k \left(\frac{1}{2^n} C \right) + \frac{1}{2^k} C \right) \subset 2^k C$$

(because C is convex) and therefore

$$\inf_{z \in C_{k+1}} \sigma(2^k v_k + z) \in \mathbb{R}.$$

This enables us to choose $z_{k+1} \in C_{k+1}$ so that

$$(11) \quad \sigma(2^k v_k + z_{k+1}) \leq \inf_{z \in C_{k+1}} \sigma(2^k v_k + z) + \frac{\epsilon}{2^{k+1}}.$$

Write

$$y_k = \sum_{n=k+1}^{\infty} 2^k \frac{z_n}{2^n}, \quad k = 0, 1, \dots,$$

and

$$v = y_0 = \sum_{n=1}^{\infty} \frac{z_n}{2^n}.$$

Since $z_n \in C_n$, $n \in \mathbb{N}$, we have, by Lemma 3, that $v \in C_1$ and $y_k \in C_{k+1}$, $k \in \mathbb{N}$, as needed, and by assumption, we have a $t \in T$ satisfying $f(v, t) = \sigma(v)$.

To verify condition (10), we first observe that

$$2^k v_k + z_{k+1} = 2^k v_k + 2^{k+1} (v_{k+1} - v_k) = 2^{k+1} v_{k+1} - 2^k v_k, \quad k = 0, 1, \dots,$$

and

$$2^k v_k + y_k = 2^k v, \quad k = 0, 1, \dots.$$

Therefore, by (11),

$$\sigma(2^{k+1} v_{k+1} - 2^k v_k) \leq \sigma(2^k v) + \frac{\epsilon}{2^{k+1}} = 2^k \sigma(v) + \frac{\epsilon}{2^{k+1}}, \quad k = 0, 1, \dots.$$

Now, using the subadditivity of $f(\cdot, t) : \mathcal{C} \rightarrow \mathbb{R}$ (see Lemma 1), we can infer that

$$\begin{aligned} f(2^m v_m, t) &= f\left(\sum_{k=0}^{m-1} (2^{k+1} v_{k+1} - 2^k v_k), t\right) \\ &\leq \sum_{k=0}^{m-1} \sigma(2^{k+1} v_{k+1} - 2^k v_k) < \sum_{k=0}^{m-1} 2^k \sigma(v) + \epsilon \\ &= (2^m - 1) \sigma(v) + \epsilon = f(2^m v, t) - \sigma(v) + \epsilon \\ &= f(2^m v_m + y_m, t) - \sigma(v) + \epsilon \\ &\leq f(2^m v_m, t) + f(y_m, t) - \sigma(v) + \epsilon, \quad m \in \mathbb{N}. \end{aligned}$$

This means that (10) holds and thus completes the proof of the main assertion of Theorem 3.

For the *moreover part*, let us assume that $-C_1 \subset C$ and show that

$$\sigma_0 := \inf_{x \in C_0} \sigma(x) = \inf_{x \in C_1} \sigma(x) =: \sigma_1,$$

where $C_0 = \text{conv}\{x_1, x_2, \dots\}$, or equivalently, since $C_0 \subset C_1$, that

$$\sigma_0 \leq \sigma_1.$$

We shall use the observation that

$$\sigma(x) = \sup_{t \in T} f(x, t)$$

for every $x \in C_1$, which is clear from the assumption.

Let

$$y = \sum_{n=1}^{\infty} \lambda_n x_n, \quad \lambda_n \geq 0, \quad \sum_{n=1}^{\infty} \lambda_n = 1,$$

be any element of $C_1 \setminus C_0$ and let $\epsilon \in (0, 1/2)$. Fix $l \in \mathbb{N}$ so that

$$\lambda := \sum_{n=1}^l \lambda_n > 1 - \epsilon.$$

As in the proof of Lemma 2, we can write

$$\frac{y}{1-\lambda} = \frac{\lambda}{1-\lambda} x_0 + z$$

for some $x_0 \in C_0$ and $z \in C_1$. Since

$$\frac{\lambda}{1-\lambda} x_0 = \frac{y}{1-\lambda} + (-z), \quad x_0, y, (-z) \in C \subset \mathcal{C}, \quad \frac{\lambda}{1-\lambda}, \frac{1}{1-\lambda} > 1,$$

by Lemma 1,

$$\frac{\lambda}{1-\lambda} f(x_0, t) \leq \frac{1}{1-\lambda} f(y, t) + M$$

for every $t \in T$, where $M = \sup\{|f(x, t)| : x \in C, t \in T\}$. Hence,

$$\lambda f(x_0, t) \leq f(y, t) + \epsilon M \leq \sigma(y) + \epsilon M$$

for every $t \in T$, implying

$$\lambda \sigma_0 \leq \lambda \sigma(x_0) \leq \sigma(y) + \epsilon M.$$

Therefore

$$\min\{(1-\epsilon)\sigma_0, \sigma_0\} \leq \sigma(y) + \epsilon M.$$

This implies, since $\epsilon \in (0, 1/2)$ is arbitrary, that $\sigma_0 \leq \sigma(y)$ whenever $y \in C_1 \setminus C_0$. Since $\sigma_0 \leq \sigma(y)$ also for all $y \in C_0$, we conclude that $\sigma_0 \leq \sigma_1$ as needed. $\square$

3. APPLICATIONS

It is straightforward to verify that if C is a bounded closed convex subset of a Banach space, then C is invariant under infinite convex combinations and Theorem 3 applies.

We shall first apply Theorem 3 to study pointwise convergence of bounded sequences in Banach spaces.

Corollary 1. *Let (x_n) be a bounded sequence in a Banach space and let C denote its closed absolutely convex hull. Suppose that S, T , and f satisfy the conditions of Theorem 3. If $\sup_{s \in S} f(x, s) \geq 0$ for all $x \in C$ and $\lim_n f(x_n, t) = 0$ for all $t \in T$, then there exists a sequence (y_n) of convex combinations of (x_n) such that*

$$\limsup_n \sup_{s \in S} f(y_n, s) = 0.$$

Proof. Since $-C_1 \subset C$ (the notation for C_1 is as in Theorem 3), we get from Theorem 3 that

$$0 \leq \inf_{x \in C_0} \sup_{s \in S} f(x, s) \leq \sup_{t \in T} \lim_n f(x_n, t) = 0,$$

where $C_0 = \text{conv}\{x_1, x_2, \dots\}$. Therefore the desired sequence $(y_n) \subset C_0$ exists. $\square$

Corollary 2. *Let S be a compact topological space and X a Banach space. Let (φ_n) be a bounded sequence in the Banach space $C(S, X)$ of continuous functions $\varphi : S \rightarrow X$. Let $T \subset S$ be such that, for every $\varphi \in C(S, X)$, there exists $t \in T$ satisfying*

$$\|\varphi\| = \|\varphi(t)\|_X.$$

If the sequence (φ_n) converges to zero pointwise on T , then there exists a sequence (ψ_n) of convex combinations of (φ_n) which is uniformly convergent to zero on S (that is, $\|\psi_n\| \rightarrow 0$).

Proof. The result immediately follows from Corollary 1 if one recalls that

$$\|\varphi\| = \sup_{s \in S} \|\varphi(s)\|_X, \quad \varphi \in C(S, X),$$

and sets

$$f(\varphi, s) = \|\varphi(s)\|_X, \quad \varphi \in C, s \in S.$$

□

Remark 3. The particular case of Corollary 2 with $T = S$ was deduced from Theorem 2 in [DF, Proposition]. The particular case of Corollary 2 with $X = \mathbb{R}$ is immediate from [S1, Corollary 9].

Let S be a convex subset of a linear space. Recall that a point t of S is an *extreme point* of S if $t = \frac{1}{2}(s_1 + s_2)$ and $s_1, s_2 \in S$ imply that $t = s_1 = s_2$. Recall also that a function φ from S to a linear space is *affine* if $\varphi(\lambda s_1 + (1 - \lambda)s_2) = \lambda\varphi(s_1) + (1 - \lambda)\varphi(s_2)$ for all $s_1, s_2 \in S$ and all $\lambda \in (0, 1)$.

Corollary 3. *Let S be a compact convex subset of a Hausdorff locally convex topological vector space and let T be the set of extreme points of S . Let X be a Banach space. If a bounded sequence of affine functions $(\varphi_n) \subset C(S, X)$ converges to zero pointwise on T , then there exists a sequence (ψ_n) of convex combinations of (φ_n) which is uniformly convergent to zero on S .*

Proof. Put $f(\varphi, s) = \|\varphi(s)\|_X$ for all affine functions $\varphi \in C(S, X)$ and all $s \in S$. The claim is immediate from Corollary 1 and Bauer's maximum principle: if $g : S \rightarrow \mathbb{R}$ is a continuous convex function (we take $g(s) = \|\varphi(s)\|_X$ for affine functions $\varphi \in C(S, X)$), then there exists $t \in T$ such that $g(t) = \sup\{g(s) : s \in S\}$. □

We conclude with a corollary of Theorem 3 that can be efficiently applied in situations encountered for instance in key results of [L, Theorem 4.2] or [O3, Lemma 2.1]. Let $L(X, Y)$ denote the Banach space of bounded linear operators from a Banach space X to a Banach space Y . Recall that the functionals $x \otimes y^* \in (L(X, Y))^*$, where $x \in X$ and $y^* \in Y^*$, are defined by $(x \otimes y^*)(A) = y^*(Ax)$, $A \in L(X, Y)$.

Corollary 4. *Let X and Y be Banach spaces and let (A_n) be a bounded sequence in $L(X, Y)$. If*

$$\limsup_n \operatorname{Re} g(A_n) \leq \lambda,$$

for some $\lambda \geq 0$ and for all functionals g contained in the weak closure of the subset $\{x \otimes y^* : x \in X, y^* \in Y^*, \|x\| = \|y^*\| = 1\}$ of $(L(X, Y))^*$, then there exists a sequence (B_n) of convex combinations of (A_n) such that*

$$\lim_n \|B_n\| \leq \lambda.$$

Proof. For applying Theorem 3, let C denote the closed absolutely convex hull of (A_n) in $L(X, Y)$. Further, let S be the closed unit ball of $(L(X, Y))^*$ and let T be the weak* closure of $\{x \otimes y^* : x \in X, y^* \in Y^*, \|x\| = \|y^*\| = 1\}$. Note that $T \subset S$ and, by Alaoglu's theorem, T is weak* compact.

Define $f : C \times S \rightarrow \mathbb{R}$ by

$$f(A, g) = \operatorname{Re} g(A), \quad A \in C \subset L(X, Y), \quad g \in S \subset (L(X, Y))^*.$$

Then conditions (3) and (4) are clearly satisfied. Moreover, for any $A \in C$ (in fact, for any $A \in L(X, Y)$), using the weak* compactness of T , we have

$$\begin{aligned} \sup_{g \in S} f(A, g) &= \|A\| = \sup \{ \operatorname{Re}(x \otimes y^*)(A) : x \in X, y^* \in Y^*, \\ &\quad \|x\| = \|y^*\| = 1 \} = \max_{g \in T} f(A, g). \end{aligned}$$

Since $-C_1 \subset C$ (the notation for C_1 is as in Theorem 3), we get from Theorem 3 that

$$\inf_{A \in C_0} \|A\| \leq \sup_{g \in T} \limsup_n \operatorname{Re} g(A_n) \leq \lambda,$$

where $C_0 = \operatorname{conv}\{A_1, A_2, \dots\}$. To obtain the desired sequence, choose $(B_n) \subset C_0$ so that

$$\lim_n \|B_n\| = \inf_{A \in C_0} \|A\|.$$

□

Remark 4. The particular case of Corollary 4 with $\lambda = 0$ follows from the generalization of Rainwater's theorem due to Simons [S1, Corollary 11].

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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, TARTU UNIVERSITY, J. LIIVI 2, EE-50409
TARTU, ESTONIA

E-mail address: kersti.kivisoo@mail.ee

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, TARTU UNIVERSITY, J. LIIVI 2, EE-50409
TARTU, ESTONIA

E-mail address: eveoja@math.ut.ee