PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 3, Pages 653–663
S 0002-9939(05)08003-2
Article electronically published on July 19, 2005

THE GELFAND-KIRILLOV DIMENSION OF QUADRATIC ALGEBRAS SATISFYING THE CYCLIC CONDITION

FERRAN CEDÓ, ERIC JESPERS, AND JAN OKNIŃSKI

(Communicated by Martin Lorenz)

Abstract. We consider algebras over a field K presented by generators $x_1, \ldots, x_n$ and subject to $\binom{n}{2}$ square-free relations of the form $x_i x_j = x_k x_l$ with every monomial $x_i x_j, i \neq j$, appearing in one of the relations. It is shown that for $n > 1$ the Gelfand-Kirillov dimension of such an algebra is at least two if the algebra satisfies the so-called cyclic condition. It is known that this dimension is an integer not exceeding n. For $n \geq 4$, we construct a family of examples of Gelfand-Kirillov dimension two. We prove that an algebra with the cyclic condition with generators $x_1, \ldots, x_n$ has Gelfand-Kirillov dimension n if and only if it is of I-type, and this occurs if and only if the multiplicative submonoid generated by $x_1, \ldots, x_n$ is cancellative.

1. Introduction

In [4] Gateva-Ivanova and Van den Bergh studied the structure of monoids of left I-type and their algebras. These monoids originate from the work of Tate and Van den Bergh on homological properties of Sklyanin algebras [8]. It was shown in [4] that a monoid of left I-type has a presentation with generators $x_1, \ldots, x_n$ and $\binom{n}{2}$ relations of the form $x_i x_j = x_k x_l$ such that every monomial $x_i x_j$ with $1 \leq i, j \leq n$ appears at most once in one of the relations. Moreover, such monoids yield set-theoretical solutions of the quantum Yang-Baxter equation, and the corresponding monoid algebras share many properties with commutative polynomial algebras. In particular, they are noetherian domains of finite global dimension, satisfy a polynomial identity, are Koszul, Auslander-Gorenstein, Cohen-Macaulay and have Gelfand-Kirillov dimension n. In [6] the monoids of left I-type are characterized as natural submonoids of semidirect products of the free abelian monoid of rank n and the symmetric group of degree n. As a consequence, it is proved that a monoid is of left I-type if and only if it is of right I-type, [6, Corollary 2.3].

A monoid S is said to be of *skew type* if it has a presentation with $n \geq 2$ generators $x_1, \ldots, x_n$ and $\binom{n}{2}$ square-free relations of the form $x_i x_j = x_k x_l$ with every monomial $x_i x_j, i \neq j$, appearing in one of the relations (see [3], where a systematic study of these monoids and their algebras was initiated). Recall that

---

Received by the editors March 24, 2004 and, in revised form, October 19, 2004.

2000 *Mathematics Subject Classification.* Primary 16P90, 16S36, 16S15, 20M25; Secondary 16P40, 20M05, 20F05.

This work was supported in part by the Flemish-Polish bilateral agreement BIL 01/31 and KBN research grant 2P03A 033 25 (Poland), the MCyT-Spain and FEDER through grant BFM2002-01390, and by the Generalitat de Catalunya (Grup de Recerca consolidat 2001SGR00171).

a monoid $S = \langle x_1, \ldots, x_n\rangle$ of skew type is said to be *right (respectively left) non-degenerate* if for every $1 \leq i, k \leq n$ there exist $1 \leq j, l \leq n$ so that $x_i x_j = x_k x_l$ ($x_l x_k = x_j x_i$ respectively). Furthermore S is said to satisfy the *cyclic condition* if for every relation $x_i x_j = x_k x_l$ one also has a relation $x_i x_k = x_r x_l$ for some r (see [3, Lemma 2.1]). The latter is a powerful combinatorial condition that has already proved crucial in the study of monoids of I-type, their algebras and corresponding torsion-free groups, [4, 6]. The cyclic condition is symmetric, [3, Proposition 2.1]. Hence it is easy to see that it implies left and right non-degeneracy. It was shown in [3] that for monoids S satisfying the cyclic condition we have $1 \leq \mathrm{GK}(K[S]) \leq n$, where $\mathrm{GK}(K[S])$ denotes the Gelfand-Kirillov dimension of the monoid algebra $K[S]$. Furthermore there exist non-degenerate monoids S of skew type on 4^m generators (for any positive m) so that $\mathrm{GK}(K[S]) = 1$, [1].

In this paper we prove that $\mathrm{GK}(K[S]) \geq 2$ for any monoid S of skew type that satisfies the cyclic condition. For any $n \geq 4$ we construct examples of such monoids on n generators with $\mathrm{GK}(K[S]) = 2$. Furthermore we show that $\mathrm{GK}(K[S]) = n$ if and only if S is of I-type, and this occurs if and only if S is cancellative.

2. The Gelfand-Kirillov dimension

Let $S = \langle x_1, x_2, \ldots, x_n\rangle$ be a monoid of skew type that satisfies the cyclic condition.

Let $F = \langle y_1, y_2, \ldots, y_n\rangle$ be the free monoid of rank n and let $\pi\colon F \to S$ be the natural epimorphism, that is $\pi(y_i) = x_i$ for $i = 1, \ldots, n$. Let $x \in S$. We say that a word $w \in F$ represents x if $\pi(w) = x$.

It is known that two words $w, w' \in F$ represent the same element $x \in S$ if and only if there exists a finite sequence of words

$$w = w_0,\ w_1,\ w_2,\ \ldots,\ w_m = w'$$

such that w_i is obtained from w_{i-1} ($i = 1, \ldots, m$) by substituting a subword $y_j y_k$ by $y_p y_q$, where $x_j x_k = x_p x_q$ is a defining relation of S. In this case, we say that w_i is obtained from w_{i-1} by an S-relation.

Lemma 2.1. *If $x_{i_1} x_{j_1} = x_{j_2} x_{i_2}$ is a defining relation of S, then there exist positive integers r, s such that $r + s \leq n$ and the submonoid $\langle x_{i_1}^r, x_{j_1}^s\rangle$ is free abelian of rank 2.*

Proof. By [3, Proposition 2.1], since S satisfies the cyclic condition, there exist positive integers r, s and $s + r$ different integers

$$i_1, i_2, \ldots, i_s, j_1, j_2, \ldots, j_r \in \{1, 2, \ldots, n\}$$

such that

$$\begin{array}{llll}
x_{i_1} x_{j_1} = x_{j_2} x_{i_2}, & x_{i_2} x_{j_1} = x_{j_2} x_{i_3}, & \ldots, & x_{i_s} x_{j_1} = x_{j_2} x_{i_1},\\
x_{i_1} x_{j_2} = x_{j_3} x_{i_2}, & x_{i_2} x_{j_2} = x_{j_3} x_{i_3}, & \ldots, & x_{i_s} x_{j_2} = x_{j_3} x_{i_1},\\
\vdots & \vdots & & \vdots\\
x_{i_1} x_{j_r} = x_{j_1} x_{i_2}, & x_{i_2} x_{j_r} = x_{j_1} x_{i_3}, & \ldots, & x_{i_s} x_{j_r} = x_{j_1} x_{i_1}.
\end{array}$$

From the relations in the first column we have

$$x_{i_1}^r x_{j_1} = x_{j_1} x_{i_2}^r.$$

Similarly, from the other columns, we obtain

$$x_{i_2}^r x_{j_1} = x_{j_1} x_{i_3}^r, \quad \ldots, \quad x_{i_s}^r x_{j_1} = x_{j_1} x_{i_1}^r.$$

Hence $x_{i_1}^r x_{j_1}^s = x_{j_1}^s x_{i_1}^r$, and thus the submonoid $\langle x_{i_1}^r, x_{j_1}^s \rangle$ is abelian. Note that the only words that represent $x_{i_1}^m$ and $x_{j_1}^m$ are $y_{i_1}^m$ and $y_{j_1}^m$ respectively. Let p, q be positive integers and $x = x_{i_1}^{rp} x_{j_1}^{sq}$. We claim that any word $w \in F$ that represents x is of the form

$$w = y_{i_{l_1}}^{n_1} y_{j_{k_1}}^{m_1} y_{i_{l_2}}^{n_2} y_{j_{k_2}}^{m_2} \dots y_{i_{l_{g-1}}}^{n_{g-1}} y_{j_{k_{g-1}}}^{m_{g-1}} y_{i_{l_g}}^{n_g}, \tag{1}$$

where g is an integer greater than 1; n_1, n_g are non-negative integers; $l_1 = l_g = 1$, and $n_2, n_3, \dots, n_{g-1}$, $m_1, m_2, \dots, m_{g-1}$ are positive integers such that

(*i*) $n_1 + k_1 \equiv k_{g-1} - n_g \equiv 1 \pmod r$ and $l_{t+1} - l_t \equiv m_t \pmod s$ for all $1 \le t \le g-1$;

(*ii*) if $g > 2$, then $k_u - k_{u+1} \equiv n_{u+1} \pmod r$ for all $1 \le u \le g-2$;

(*iii*) $n_1 + n_2 + \cdots + n_g = rp$ and $m_1 + m_2 + \cdots + m_{g-1} = sq$.

Note that the word $y_{i_1}^{rp} y_{j_1}^{sq}$ represents x and satisfies conditions (*i*), (*ii*) and (*iii*). Therefore, in order to prove the claim, it is sufficient to see that given any word w of the form (1) that satisfies conditions (*i*), (*ii*) and (*iii*), all the words obtained from w by an S-relation also satisfy conditions (*i*), (*ii*) and (*iii*). Suppose that $g = 2$. In this case

$$w = y_{i_1}^{n_1} y_{j_{k_1}}^{m_1} y_{i_1}^{n_2},$$

with $m_1 = sq > 0$, $n_1 + n_2 = rp$ and $n_1 + k_1 \equiv k_1 - n_2 \equiv 1 \pmod r$. If $n_1 > 0$, we can obtain by an S-relation (the relation $x_{i_1} x_{j_{k_1}} = x_{j_{k_1+1}} x_{i_2}$), the word

$$w' = y_{i_1}^{n_1 - 1} y_{j_{k_1+1}} y_{i_2} y_{j_{k_1}}^{m_1 - 1} y_{i_1}^{n_2},$$

where $k_1 + 1$ is taken modulo r in the set $\{1, \dots, r\}$, and it is easy to see that w' satisfies conditions (*i*), (*ii*) and (*iii*). If $n_2 > 0$, we can obtain by an S-relation the word

$$w' = y_{i_1}^{n_1} y_{j_{k_1}}^{m_1 - 1} y_{i_s} y_{j_{k_1 - 1}} y_{i_1}^{n_2 - 1},$$

where $k_1 - 1$ is taken modulo r in the set $\{1, \dots, r\}$, and it is easy to see that w' satisfies the conditions (*i*), (*ii*) and (*iii*). Similarly, it is straightforward to prove that, if $g > 2$, all the words w' obtained from w by an S-relation satisfy conditions (*i*), (*ii*) and (*iii*). Now condition (*iii*) implies that the submonoid $\langle x_{i_1}^r, x_{j_1}^s \rangle$ is free abelian of rank 2. □

As a direct consequence of Lemma 2.1 we get the following result.

Corollary 2.2. *Let $S = \langle x_1, x_2, \dots, x_n \rangle$ be a monoid of skew type that satisfies the cyclic condition. Let $m = (n-1)!$. Then the submonoid $A = \langle x_1^m, \dots, x_n^m \rangle$ is commutative.* □

Theorem 2.3. *Let $S = \langle x_1, x_2, \dots, x_n \rangle$ (with $n > 1$) be a monoid of skew type that satisfies the cyclic condition. Let $A = \langle x_1^m, \dots, x_n^m \rangle$, where $m = (n-1)!$. If K is a field, then the Gelfand-Kirillov dimension of the monoid algebra $K[S]$ is an integer such that $2 \le \mathrm{GK}(K[S]) = \mathrm{GK}(K[A]) \le n$. Moreover, $\mathrm{GK}(K[S])$ is equal to the maximal rank k of a free abelian submonoid of the form $\langle x_{i_1}^m, \dots, x_{i_k}^m \rangle \subseteq S$.*

Proof. By [3, Theorem 4.5] and the comment after [3, Proposition 2.4], $K[S]$ is a finite left and right module over the commutative subring $K[A]$, where $A = \langle x_1^p, \dots, x_n^p \rangle$ for some $p \ge 1$. The proof actually shows that we may take $p = (n-1)!$. Hence $\mathrm{GK}(K[S]) = \mathrm{GK}(K[A]) \le n$ and it is an integer. By Lemma 2.1, we have that $2 \le \mathrm{GK}(K[S])$. Let P be a prime ideal of $K[A]$. Then the image A_P of

A in $K[A]/P$ is a 0-cancellative monoid. Let $C = \langle z_{i_1}^m, \ldots, z_{i_r}^m \rangle \subseteq A_P$ be a free abelian submonoid of maximal rank that is generated by certain images z_i^m of the elements x_i^m. Then $B = \langle x_{i_1}^m, \ldots, x_{i_r}^m \rangle \subseteq A$ is free abelian of rank r. It is easy to see that the group G_P of quotients of the cancellative semigroup of nonzero elements of A_P is of rank r, whence $\mathrm{GK}(K[A]/P) \leq \mathrm{GK}(K[A_P]) \leq \mathrm{GK}(K[G_P]) = r \leq k$. Therefore $\mathrm{GK}(A) \leq k$, since the Gelfand-Kirillov and the classical Krull dimensions coincide on finitely generated commutative algebras, [7, Theorem 4.5]. The result follows. □

3. Examples of dimension two

For $n \geq 4$, let $T^{(n)}$ be the monoid of skew type generated by $x_1, \ldots, x_n$ with defining relations

$$\begin{array}{l} x_1x_2 = x_3x_1, \quad \ldots, \quad x_1x_{n-2} = x_{n-1}x_1, \quad x_1x_{n-1} = x_2x_1, \\ x_nx_1 = x_{n-1}x_n, \quad x_nx_{n-1} = x_1x_n, \\ x_ix_{i+1} = x_{i+2}x_i, \quad \ldots, \quad x_ix_{n-1} = x_nx_i, \quad x_ix_n = x_{i+1}x_i, \end{array}$$

for all $2 \leq i \leq n-2$. Note that $T^{(n)}$ satisfies the cyclic condition.

Lemma 3.1. *Let ρ be the least cancellative congruence on $T^{(n)}$. If $n > 4$, then $x_2x_1x_2 = x_nx_1x_2$ and $x_1 \,\rho\, x_2 \,\rho\, \ldots \,\rho\, x_n$.*

Proof. By using the defining relations, we have

$$x_2x_1x_2 = x_1x_{n-1}x_2 = x_1x_2x_{n-2} = x_3x_1x_{n-2} = x_3x_{n-1}x_1 = x_nx_3x_1 = x_nx_1x_2.$$

Since $x_2x_1x_2 = x_nx_1x_2$, it follows that $x_2 \,\rho\, x_n$. Now the relations

$$x_2x_3 = x_4x_2, \quad \ldots, \quad x_2x_{n-1} = x_nx_2$$

imply that $x_2 \,\rho\, x_3 \,\rho\, \ldots \,\rho\, x_n$. Since $x_nx_1 = x_{n-1}x_n$, we also get

$$x_1 \,\rho\, x_2 \,\rho\, \ldots \,\rho\, x_n.$$

□

Let T_n' be the subset of $T^{(n)}$ of all elements right divisible by all generators of $T^{(n)}$. Since $T^{(n)}$ is left non-degenerate, T_n' is an ideal of $T^{(n)}$; see [3].

Lemma 3.2. *Consider $z = x_2x_1x_2 \in T^{(n)}$. Then $z \in T_n'$.*

Proof. For $n = 4$ we have

$$z = x_2x_1x_2 = x_2x_3x_1 = x_4x_2x_1 = x_4x_1x_3 = x_3x_4x_3 = x_3x_1x_4 \in T_n'.$$

Suppose that $n > 4$. By Lemma 3.1, $z = x_nx_1x_2$ and thus

$$\begin{aligned} z &= x_nx_1x_2 = x_{n-1}x_nx_2 = x_{n-1}x_2x_{n-1} = x_2x_{n-2}x_{n-1} \\ &= x_2x_nx_{n-2} = x_3x_2x_{n-2} = x_3x_{n-1}x_2 = x_nx_3x_2 = x_nx_2x_n \\ &= x_2x_{n-1}x_n = x_2x_nx_1 = x_3x_2x_1 = x_3x_1x_{n-1} = x_1x_2x_{n-1} \\ &= x_1x_nx_2 = x_nx_{n-1}x_2 = x_nx_2x_{n-2} = x_2x_{n-1}x_{n-2}. \end{aligned}$$

We claim that $z = x_2x_{i+1}x_i$ for all $n-2 \geq i \geq 3$. We prove this by induction. If $n = 5$ the claim is proved. Suppose that $n > 5$ and that we know $z = x_2x_{i+1}x_i$ for some $4 \le i \le n-2$. Then

$$\begin{aligned} z &= x_2x_{i+1}x_i = x_2x_ix_n = x_{i+1}x_2x_n = x_{i+1}x_3x_2 \\ &= x_3x_ix_2 = x_3x_2x_{i-1} = x_2x_nx_{i-1} = x_2x_{i-1}x_{n-1} \\ &= x_ix_2x_{n-1} = x_ix_nx_2 = x_{i+1}x_ix_2 = x_{i+1}x_2x_{i-1} = x_2x_ix_{i-1}, \end{aligned}$$

which proves the inductive claim. It follows that $z \in T^{(n)}x_i$, for all $3 \le i \le n-2$. Since $z = x_2x_1x_2 = x_2x_3x_1 = x_{n-1}x_2x_{n-1} = x_nx_2x_n$, we have that $z \in T_n'$. $\square$

Let $m = (n-1)!$. By Corollary 2.2, the submonoid $A = \langle x_1^m, \ldots, x_n^m \rangle$ of $T^{(n)}$ is commutative.

Lemma 3.3. *If $n > 4$, then $x_k^{2m}x_j^{2m}x_i^{2m} \in T_n'$ for all $1 \le i < j < k \le n$.*

Proof. Note that from the relations

$$x_1x_2 = x_3x_1, \quad \ldots, \quad x_1x_{n-2} = x_{n-1}x_1, \quad x_1x_{n-1} = x_2x_1,$$

it follows that

$$x_1^jx_i = x_{j+i}x_1^j, \quad x_1^{n-2}x_i = x_ix_1^{n-2}, \tag{2}$$

for all $2 \le i \le n-1$ and $1 \le j < n-i$. From the relations

$$x_nx_1 = x_{n-1}x_n, \quad x_nx_{n-1} = x_1x_n, \tag{3}$$

it follows that

$$x_n^2x_1 = x_1x_n^2 \quad \text{and} \quad x_n^2x_{n-1} = x_{n-1}x_n^2. \tag{4}$$

For each $2 \le i \le n-2$, the relations

$$x_ix_{i+1} = x_{i+2}x_i, \quad \ldots, \quad x_ix_{n-1} = x_nx_i, \quad x_ix_n = x_{i+1}x_i$$

imply that

$$x_i^{n-i}x_j = x_jx_i^{n-i}, \tag{5}$$

for all $2 \le i < j \le n$.

Case 1. $1 < i < j < k \le n$. In this case it is easy to see that

$$x_kx_j^{k-j-1} = x_j^{k-j-1}x_{j+1} \tag{6}$$

and

$$x_jx_i^{j-i+1} = x_i^{j-i+1}x_{n-1}. \tag{7}$$

Then we have

$$\begin{aligned}
x_k^{2m}x_j^{2m}x_i^{2m} &= x_j^{k-j-1}x_{j+1}^{2m}x_j^{2m-k+j+1}x_i^{2m} \quad \text{(by (6))}\\
&= x_j^{k-j-1}x_{j+1}^{2m}x_j^{m}x_i^{2m}x_j^{m-k+j+1} \quad \text{(by (5))}\\
&= x_j^{k-j-1}x_j^{m}x_{j+1}^{2m}x_i^{2m}x_j^{m-k+j+1} \quad \text{(by (5))}\\
&= x_j^{k-j-1}x_i^{j-i+1}x_{n-1}^{m}x_n^{2m}x_i^{2m-j+i-1}x_j^{m-k+j+1} \quad \text{(by (7))}\\
&= x_j^{k-j-1}x_i^{j-i+1}x_nx_1^{m}x_n^{2m-1}x_i^{2m-j+i-1}x_j^{m-k+j+1} \quad \text{(by (3))}\\
&= x_j^{k-j-1}x_i^{j-i+1}x_nx_1^{m}x_n^{2}x_i^{m}x_n^{2m-3}x_i^{m-j+i-1}x_j^{m-k+j+1} \quad \text{(by (5))}\\
&= x_j^{k-j-1}x_i^{j-i+1}x_n^{3}x_1^{m}x_i^{m}x_n^{2m-3}x_i^{m-j+i-1}x_j^{m-k+j+1} \quad \text{(by (4))}\\
&= x_j^{k-j-1}x_i^{j-i+1}x_n^{3}x_1^{m-n+i}x_2^{m}x_1^{n-i}x_n^{2m-3}\\
&\quad \cdot x_i^{m-j+i-1}x_j^{m-k+j+1} \quad (\text{since } x_2x_1^{n-i} = x_1^{n-i}x_i)\\
&= x_j^{k-j-1}x_i^{j-i+1}x_nx_1^{m-n+i-1}x_n^{2}x_1x_2^{m}x_1^{n-i}x_n^{2m-3}\\
&\quad \cdot x_i^{m-j+i-1}x_j^{m-k+j+1} \quad \text{(by (4))}\\
&= (x_j^{k-j-1}x_i^{j-i+1}x_nx_1^{m-n+i-1}x_n)z(x_2^{m-1}x_1^{n-i}x_n^{2m-3}\\
&\quad \cdot x_i^{m-j+i-1}x_j^{m-k+j+1}).
\end{aligned}$$

By Lemma 3.2 we know that $z \in T'_n$. Since T'_n is an ideal of $T^{(n)}$, it follows that $x_k^{2m}x_j^{2m}x_i^{2m} \in T'_n$.

Case 2. $1 = i < j < k \leq n$ and $j < n-1$. Then we have

$$\begin{aligned}
x_k^{2m}x_j^{2m}x_1^{2m} &= x_j^{k-j-1}x_{j+1}^{2m}x_j^{2m-k+j+1}x_1^{2m} \quad \text{(by (6))}\\
&= x_j^{k-j-1}x_{j+1}^{2m}x_j^{m}x_1^{2m}x_j^{m-k+j+1} \quad \text{(by (2))}\\
&= x_j^{k-j-1}x_j^{m}x_{j+1}^{2m}x_1^{2m}x_j^{m-k+j+1} \quad \text{(by (5))}\\
&= x_j^{k-j-1}x_j^{m}x_{j+1}x_1^{2m}x_{j+1}^{2m-1}x_j^{m-k+j+1} \quad \text{(by (2))}\\
&= x_j^{k-j-1}x_j^{m}x_1^{j-2}x_3x_1^{2m-j+2}x_{j+1}^{2m-1}x_j^{m-k+j+1} \quad \text{(by (2))}\\
&= x_j^{k-j-1}x_1^{j-2}x_2^{m}x_3x_1^{2m-j+2}x_{j+1}^{2m-1}x_j^{m-k+j+1} \quad \text{(by (2))}\\
&= (x_j^{k-j-1}x_1^{j-2}x_2^{m-1})z(x_1^{2m-j+1}x_{j+1}^{2m-1}x_j^{m-k+j+1}).
\end{aligned}$$

By Lemma 3.2, $x_k^{2m}x_j^{2m}x_i^{2m} \in T'_n$ in this case.

Case 3. $i = 1$, $j = n-1$ and $k = n$. Then we have

$$\begin{aligned}
x_n^{2m}x_{n-1}^{2m}x_1^{2m} &= x_n^{2m}x_{n-1}x_1^{2m}x_{n-1}^{2m-1} \quad \text{(by (2))}\\
&= x_n^{2m}x_1^{n-3}x_2x_1^{2m-n+3}x_{n-1}^{2m-1} \quad \text{(by (2))}\\
&= x_n^{2m-1}x_{n-1}^{n-4}x_nx_1x_2x_1^{2m-n+3}x_{n-1}^{2m-1} \quad \text{(by (3))}\\
&= (x_n^{2m-1}x_{n-1}^{n-4})z(x_1^{2m-n+3}x_{n-1}^{2m-1}).
\end{aligned}$$

Again, by Lemma 3.2, $x_k^{2m}x_j^{2m}x_i^{2m} \in T'_n$ in this case.

Therefore $x_k^{2m}x_j^{2m}x_i^{2m} \in T'_n$ for all $1 \leq i < j < k \leq n$. $\square$

Theorem 3.4. *Let K be a field. Then* $\mathrm{GK}(K[T^{(n)}]) = 2$ *for all* $n \geq 4$.

Proof. For $n = 4$ the result follows from [5, Proposition 2.1], because $T^{(4)}$ coincides with the monoid $C^{(1)}$ of [5]. Suppose that $n > 4$. As above, let $m = (n-1)!$. From [3, Proposition 6.3] we know that $(T'_n)^q I(\rho) = 0$ for some q, where $I(\rho)$ is the ideal of $K[T^{(n)}]$ determined by the least cancellative congruence ρ on $T^{(n)}$. In particular, by Lemma 3.1, $x_k^m - x_j^m \in I(\rho)$ for all k, j. Therefore, from Lemma 3.3 it follows that

$$x_k^{2mq} x_j^{2mq} x_i^{2mq} (x_k^m - x_j^m) = 0,$$

for all $1 \leq i < j < k \leq n$, which implies that x_k^m, x_j^m, x_i^m do not generate a free abelian semigroup. Therefore Theorem 2.3 implies that $\mathrm{GK}(K[T^{(n)}]) = 2$. $\square$

Corollary 3.5. *For any integers $n \geq 4$ and $2 \leq j \leq n$, there exists a monoid $M = \langle x_1, x_2, \ldots, x_n \rangle$ of skew type, satisfying the cyclic condition and such that $\mathrm{GK}(K[M]) = j$ for any field K.*

Proof. If $j = n$, then the free abelian monoid of rank n, $M = \mathrm{FaM}_n$, satisfies the conditions.

Suppose that $j = n - 1$. By [5], there exists a monoid A of skew type with 4 generators that satisfies the cyclic condition such that, for any field K, $\mathrm{GK}(K[A]) = 3$. Let $M = A \times \mathrm{FaM}_{n-4}$. Then it is easy to see that M is a monoid of skew type with n generators that satisfies the cyclic condition. Since $K[M]$ is the polynomial algebra over $K[A]$ with $n - 4$ indeterminates, by [7, Example 3.6], $\mathrm{GK}(K[M]) = 3 + (n-4) = n - 1$.

Suppose that $j \leq n - 2$. Let $M = T^{(n-j+2)} \times \mathrm{FaM}_{j-2}$. It is easy to see that M is a monoid of skew type with n generators that satisfies the cyclic condition. Since $K[M]$ is the polynomial algebra over $K[T^{(n-j+2)}]$ with $j-2$ indeterminates, by [7, Example 3.6], $\mathrm{GK}(K[M]) = \mathrm{GK}(K[T^{(n-j+2)}]) + (j-2)$. By Theorem 3.4, $\mathrm{GK}(K[M]) = j$. $\square$

4. I-TYPE MONOIDS

Let FaM_n be the multiplicative free abelian monoid of rank n with basis $u_1, \ldots, u_n$. Recall that a monoid S generated by $x_1, \ldots, x_n$ is said to be of left I-type if there exists a bijection (called a left I-structure)

$$v \colon \mathrm{FaM}_n \to S$$

such that

$$v(1) = 1 \quad \text{and} \quad \{v(u_1 a), \ldots, v(u_n a)\} = \{x_1 v(a), \ldots, x_n v(a)\}$$

for all $a \in \mathrm{FaM}_n$. As mentioned in the introduction, it is proved in [6] that a monoid S is of left I-type if and only if it is of right I-type. So we call a monoid of left or right I-type simply a monoid of I-type.

Let $S = \langle x_1, x_2, \ldots, x_n \rangle$ be a monoid of skew type. Let $X = \{x_1, x_2, \ldots, x_n\}$. As in [6], we define the associated bijective map $r \colon X \times X \to X \times X$ by

$$r(x_i, x_j) = (x_k, x_l)$$

if $x_i x_j = x_k x_l$ is a defining relation of S, and $r(x_i, x_i) = (x_i, x_i)$. For each $x \in X$, we also denote by $f_x \colon X \to X$ and $g_x \colon X \to X$ the mappings defined by $f_x(x_i) = p_1(r(x, x_i))$ and $g_x(x_i) = p_2(r(x_i, x))$, where p_1 and p_2 denote the projections onto the first and second component respectively. So $r(x_i, x_j) = (f_{x_i}(x_j), g_{x_j}(x_i))$. Suppose that S is right non-degenerate. So f_x is bijective for all $x \in X$. We denote by $\sigma_i \in \mathrm{Sym}_n$ the permutation defined by $f_{x_i}(x_j) = x_{\sigma_i(j)}$.

The next result is a partial generalization of Proposition 2.2(c) of [2].

Theorem 4.1. *Let $S = \langle x_1, x_2, \dots, x_n\rangle$ be a right non-degenerate monoid of skew type. Then the following conditions are equivalent.*

(i) *S is of I-type.*
(ii) *$\sigma_i \circ \sigma_{\sigma_i^{-1}(j)} = \sigma_j \circ \sigma_{\sigma_j^{-1}(i)}$ for all i, j.*
(iii) *For every defining relation $x_i x_j = x_k x_l$ of S we have $\sigma_i \circ \sigma_j = \sigma_k \circ \sigma_l$.*

Proof. We denote by $r_i \colon X^3 \to X^3$, for $i = 1, 2$, the mappings defined by $r_1 = r \times id_X$ and $r_2 = id_X \times r$. Then

$$\begin{aligned}(r_1 \circ r_2 \circ r_1)(x_i, x_j, x_k) &= (r_1 \circ r_2)(x_{\sigma_i(j)}, x_{\sigma^{-1}_{\sigma_i(j)}(i)}, x_k)\\ &= r_1(x_{\sigma_i(j)}, x_{\sigma_{\sigma^{-1}_{\sigma_i(j)}(i)}(k)}, x_{\sigma^{-1}_{\sigma_{\sigma^{-1}_{\sigma_i(j)}(i)}(k)}(\sigma^{-1}_{\sigma_i(j)}(i))}) \end{aligned} \tag{8}$$

and

$$\begin{aligned}(r_2 \circ r_1 \circ r_2)(x_i, x_j, x_k) &= (r_2 \circ r_1)(x_i, x_{\sigma_j(k)}, x_{\sigma^{-1}_{\sigma_j(k)}(j)})\\ &= r_2(x_{\sigma_i(\sigma_j(k))}, x_{\sigma^{-1}_{\sigma_i(\sigma_j(k))}(i)}, x_{\sigma^{-1}_{\sigma_j(k)}(j)}). \end{aligned} \tag{9}$$

Recall from [6, Corollary 3.1] that S is of I-type if and only if r yields a solution of the quantum Yang-Baxter equation, that is $r_1 \circ r_2 \circ r_1 = r_2 \circ r_1 \circ r_2$. Therefore, if S is of I-type, then by (8) and (9), we have

$$\sigma_{\sigma_i(j)}(\sigma_{\sigma^{-1}_{\sigma_i(j)}(i)}(k)) = \sigma_i(\sigma_j(k)),$$

for all i, j, k. Thus

$$\sigma_{\sigma_i(j)} \circ \sigma_{\sigma^{-1}_{\sigma_i(j)}(i)} = \sigma_i \circ \sigma_j, \tag{10}$$

for all i, j. By putting $j' = \sigma_i(j)$, we can write (10) as

$$\sigma_{j'} \circ \sigma_{\sigma_{j'}^{-1}(i)} = \sigma_i \circ \sigma_{\sigma_i^{-1}(j')},$$

for all i, j'. Therefore (ii) is a consequence of (i).

Suppose that

$$\sigma_i \circ \sigma_{\sigma_i^{-1}(j)} = \sigma_j \circ \sigma_{\sigma_j^{-1}(i)},$$

for all i, j. We will prove that r yields a solution of the quantum Yang-Baxter equation and thus S is of I-type. By (8) and (9), it is sufficient to prove the following equalities:

(a) $\sigma_{\sigma_i(j)}(\sigma_{\sigma^{-1}_{\sigma_i(j)}(i)}(k)) = \sigma_i(\sigma_j(k))$;
(b) $\sigma^{-1}_{\sigma_{\sigma_i(j)}(\sigma_{\sigma^{-1}_{\sigma_i(j)}(i)}(k))}(\sigma_i(j)) = \sigma_{\sigma^{-1}_{\sigma_i(\sigma_j(k))}(i)}(\sigma^{-1}_{\sigma_j(k)}(j))$;
(c) $\sigma^{-1}_{\sigma_{\sigma^{-1}_{\sigma_i(j)}(i)}(k)}(\sigma^{-1}_{\sigma_i(j)}(i)) = \sigma^{-1}_{\sigma_{\sigma^{-1}_{\sigma_i(\sigma_j(k))}(i)}(\sigma^{-1}_{\sigma_j(k)}(j))}(\sigma^{-1}_{\sigma_i(\sigma_j(k))}(i))$.

The equality (a) follows from

$$\sigma_{j'} \circ \sigma_{\sigma_{j'}^{-1}(i)} = \sigma_i \circ \sigma_{\sigma_i^{-1}(j')},$$

with $j' = \sigma_i(j)$. By (a), the equality (b) is equivalent to

$$\sigma^{-1}_{\sigma_i(\sigma_j(k))}(\sigma_i(j)) = \sigma_{\sigma^{-1}_{\sigma_i(\sigma_j(k))}(i)}(\sigma^{-1}_{\sigma_j(k)}(j)), \tag{11}$$

and the latter follows from our assumption

$$\sigma_l^{-1} \circ \sigma_i = \sigma_{\sigma_l^{-1}(i)} \circ \sigma^{-1}_{\sigma_i^{-1}(l)},$$

with $l = \sigma_i(\sigma_j(k))$. By (a), the equality (c) is equivalent to

$$\sigma^{-1}_{\sigma^{-1}_{\sigma_i(j)}(\sigma_i(\sigma_j(k)))}(\sigma^{-1}_{\sigma_i(j)}(i)) = \sigma^{-1}_{\sigma_{\sigma^{-1}_{\sigma_i(\sigma_j(k))}(i)}(\sigma^{-1}_{\sigma_j(k)}(j))}(\sigma^{-1}_{\sigma_i(\sigma_j(k))}(i)).$$

In view of (11), this equality is equivalent to

$$\sigma^{-1}_{\sigma^{-1}_{\sigma_i(j)}(\sigma_i(\sigma_j(k)))}(\sigma^{-1}_{\sigma_i(j)}(i)) = \sigma^{-1}_{\sigma^{-1}_{\sigma_i(\sigma_j(k))}(\sigma_i(j))}(\sigma^{-1}_{\sigma_i(\sigma_j(k))}(i)),$$

and the latter follows from our assumption

$$\sigma^{-1}_{\sigma_{j'}^{-1}(l)} \circ \sigma_{j'}^{-1} = \sigma^{-1}_{\sigma_l^{-1}(j')} \circ \sigma_l^{-1},$$

with $l = \sigma_i(\sigma_j(k))$ and $j' = \sigma_i(j)$. Hence r yields a solution of the quantum Yang-Baxter equation and (ii) implies (i).

Finally, notice that $x_i x_p = x_j x_q$ if and only if $\sigma_i(p) = j$ and $\sigma_j(q) = i$. The latter is equivalent to $\sigma_i^{-1}(j) = p$ and $\sigma_j^{-1}(i) = q$. Hence, saying that $\sigma_i \sigma_p = \sigma_j \sigma_q$ whenever $x_i x_p = x_j x_q$ is equivalent to saying that $\sigma_i \sigma_{\sigma_i^{-1}(j)} = \sigma_j \sigma_{\sigma_j^{-1}(i)}$. So conditions (ii) and (iii) are equivalent. This completes the proof. $\square$

5. The dimension n case

Let $S = \langle x_1, x_2, \dots, x_n \rangle$ be a monoid of skew type that satisfies the cyclic condition. In this section we study the second extreme case, namely the case where $\mathrm{GK}(K[S]) = n$ for any field K. As in Section 4, we define $\sigma_i \in \mathrm{Sym}_n$ by

$$\sigma_i(j) = \begin{cases} i & \text{if} \quad j = i, \\ k & \text{if} \quad x_i x_j = x_k x_l \quad \text{is a defining relation of } S. \end{cases}$$

Let $m = (n-1)!$. Since S satisfies the cyclic condition, for all i, j we have that

$$x_i x_j^m = x_{\sigma_i(j)}^m x_i. \tag{12}$$

Theorem 5.1. *Let $S = \langle x_1, x_2, \dots, x_n \rangle$ be a monoid of skew type that satisfies the cyclic condition. Let K be a field. Then the following conditions are equivalent:*

(i) $\mathrm{GK}(K[S]) = n$.
(ii) *S is of I-type.*
(iii) *S is cancellative.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $\mathrm{GK}(K[S]) = n$. Let $m = (n-1)!$. We know that $A = \langle x_1^m, \dots, x_n^m \rangle$ is abelian. Moreover $\mathrm{GK}(K[A]) = \mathrm{GK}(K[S]) = n$ by Theorem 2.3. This implies that A is a free abelian monoid of rank n. Indeed, otherwise the natural map $K[y_1, \dots, y_n] \to K[A]$ has a nontrivial kernel, whence the classical Krull dimension of $K[A]$ is smaller than n, while it is equal to the Gelfand-Kirillov dimension; see [7, Theorem 4.5].

Suppose that $x_i x_j = x_k x_l$ is a defining relation of S. Then for all $t \in \{1, \dots, n\}$ we have, by (12),

$$x_i x_j x_t^m = x_i x_{\sigma_j(t)}^m x_j = x_{\sigma_i(\sigma_j(t))}^m x_i x_j.$$

Also we have

$$x_k x_l x_t^m = x_k x_{\sigma_l(t)}^m x_l = x_{\sigma_k(\sigma_l(t))}^m x_k x_l.$$

Since

$$x_i x_j x_j^{m-1} x_i^{m-1} = x_i x_j^m x_i^{m-1} = x_k^m x_i x_i^{m-1} = x_k^m x_i^m,$$

multiplying the two previous equalities by $x_j^{m-1} x_i^{m-1}$ on the right, we get

$$x_{\sigma_i(\sigma_j(t))}^m x_k^m x_i^m = x_{\sigma_k(\sigma_l(t))}^m x_k^m x_i^m.$$

Since A is free abelian, this implies that

$$\sigma_i(\sigma_j(t)) = \sigma_k(\sigma_l(t)).$$

By Theorem 4.1, S is of I-type.

$(ii) \Rightarrow (i)$. The definition of a monoid of I-type implies that the growth function of S is the same as that of a free abelian monoid of rank n. Hence $\mathrm{GK}(K[S]) = n$.

$(ii) \Rightarrow (iii)$. This follows from [4, Corollary 1.5].

$(iii) \Rightarrow (ii)$. Suppose that $x_i x_j = x_k x_l$ is a defining relation of S. Then for all $t \in \{1, \ldots, n\}$, as in the proof of the implication $(i) \Rightarrow (ii)$ we get

$$x_{\sigma_i(\sigma_j(t))}^m x_i x_j = x_{\sigma_k(\sigma_l(t))}^m x_k x_l.$$

Since S is cancellative, this implies that

$$x_{\sigma_i(\sigma_j(t))}^m = x_{\sigma_k(\sigma_l(t))}^m.$$

By the form of the defining relations of S, it is then clear that

$$\sigma_i(\sigma_j(t)) = \sigma_k(\sigma_l(t)).$$

By Theorem 4.1, S is of I-type. $\square$

Corollary 5.2. *Let S be a monoid of skew type. Then S is of I-type if and only if S is cancellative and satisfies the cyclic condition.*

Proof. Suppose that $S = \langle x_1, \ldots, x_n \rangle$ is of I-type. By [4, Theorem 1.3], the associated map $r \colon X^2 \to X^2$, where $X = \{x_1, \ldots, x_n\}$, yields a solution of the quantum Yang-Baxter equation. By [6, Corollary 3.1], S satisfies the cyclic condition. Now the result follows from Theorem 5.1. $\square$

References

[1] F. Cedó, E. Jespers and J. Okniński, Semiprime quadratic algebras of Gelfand-Kirillov dimension one, J. Algebra Appl. 3(2004), 283-300. MR2096451

[2] P. Etingof, T. Schedler and A. Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equation, Duke Math. J. 100(1999), 169-209. MR1722951 (2001c:16076)

[3] T. Gateva–Ivanova, E. Jespers and J. Okniński, Quadratic algebras of skew type and the underlying semigroups, J. Algebra 270(2003), 635-659. MR2019633 (2004m:16039)

[4] T. Gateva–Ivanova and M. Van den Bergh, Semigroups of I-Type, J. Algebra 206(1998), 97-112. MR1637256 (99h:20090)

[5] E. Jespers and J. Okniński, Quadratic algebras of skew type satisfying the cyclic condition, Int. J. Algebra and Computation 14(2004), 479-498. MR2084381

[6] E. Jespers and J. Okniński, Monoids and groups of I-type, Algebras and Representation Theory, to appear.

[7] G.R. Krause and T.H. Lenagan, Growth of Algebras and Gelfand-Kirillov Dimension, Revised edition. Graduate Studies in Mathematics, 22. American Mathematical Society, Providence, RI, 2000. MR1721834 (2000j:16035)

[8] J. Tate and M. Van den Bergh, Homological properties of Sklyanin algebras, Invent. Math. 124(1996), 619–647. MR1369430 (98c:16057)

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain
E-mail address: cedo@mat.uab.es

Department of Mathematics, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium
E-mail address: efjesper@vub.ac.be

Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland
E-mail address: okninski@mimuw.edu.pl