BUBBLING PHENOMENA FOR FOURTH-ORDER FOUR-DIMENSIONAL PDES WITH EXPONENTIAL GROWTH

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(Communicated by Jozef Dodziuk)

ABSTRACT. We are concerned in this paper with the bubbling phenomenon for nonlinear fourth-order four-dimensional PDE's. The operators in the equations are perturbations of the bi-Laplacian. The nonlinearity is of exponential growth. Such equations arise naturally in statistical physics and geometry. As a consequence of our theorem we get a priori bounds for solutions of our equations.

We are concerned in this paper with understanding the bubbling phenomenon for fourth-order four-dimensional PDE's of exponential growth. Such equations arise naturally in statistical physics and in geometry (see [7] and [9]). In what follows, we let (M, g) be a smooth compact Riemannian 4-manifold without boundary. We also let $(b_\varepsilon)_{\varepsilon>0}$ and $(f_\varepsilon)_{\varepsilon>0}$ be sequences of smooth functions on M , and we let $(A_\varepsilon)_{\varepsilon>0}$ be a sequence of smooth $(2, 0)$ -symmetric tensor fields. We assume that (b_ε) , (f_ε) and (A_ε) converge as $\varepsilon \rightarrow 0$ in the C^k -topologies, k a positive integer, to limiting objects of the same nature, b_0 , f_0 and A_0 . Then we consider sequences $(u_\varepsilon)_{\varepsilon>0}$ of solutions of

$$(1) \quad \Delta_g^2 u_\varepsilon + R_\varepsilon(x, du_\varepsilon) = f_\varepsilon(x) e^{u_\varepsilon},$$

where $\Delta_g = -\operatorname{div}_g(\nabla \cdot)$ is the Laplace-Beltrami operator and

$$(2) \quad R_\varepsilon(x, du) = -\operatorname{div}_g(A_\varepsilon du) + b_\varepsilon.$$

Following standard terminology, we say that the u_ε 's blow up if $u_\varepsilon(x_\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$ for a sequence (x_ε) of points in M . We let

$$(3) \quad L_0 = \Delta_g^2 u - \operatorname{div}_g(A_0 du)$$

be the limit operator in (1). At last, we let G be the Green function of L_0 . The Green function is unique up to a constant when the kernel of L_0 consists only of constants. We write G as

$$G(x, y) = \frac{1}{8\pi^2} \ln \frac{1}{d_g(x, y)} + \beta(x, y)$$

Received by the editors September 29, 2004.

2000 *Mathematics Subject Classification.* Primary 58E30, 58J05, 35J35.

Key words and phrases. Concentration estimates, fourth-order equations, compactness.

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for $(x, y) \in M \times M \setminus D$, with $D = \{(x, x), x \in M\}$ the diagonal in $M \times M$, where $\beta \in C^1(M \times M)$. We let φ be the function given by

$$\varphi(x) = \int_M G(x, y) b_0(y) dv_g(y).$$

For u a function on M we let

$$\bar{u} = \frac{1}{Vol_g(M)} \int_M u dv_g$$

be the mean value of u , where $Vol_g(M)$ is the volume of M with respect to g . Our theorem is stated as follows:

Theorem 1. *Let (M, g) be a smooth compact Riemannian manifold of dimension 4 without boundary. Let (u_ε) be a blowing-up sequence of solutions of (1). Assume that the kernel of L_0 consists only of constants and that f_0 is a positive function on M . Then*

$$\int_M b_0 dv_g = 64\pi^2 N$$

for some $N \in \mathbb{N}^*$. Moreover there exists a finite subset $S \subset M$, consisting of N points x_i , $i = 1, \dots, N$, such that

$$u_\varepsilon - \bar{u}_\varepsilon \rightarrow 64\pi^2 \sum_{i=1}^N G(x_i, \cdot) - \varphi$$

in $C_{loc}^4(M \setminus S)$. At last, we have that

$$64\pi^2 \nabla_y \beta(x_i, x_i) + 64\pi^2 \sum_{j \neq i} \nabla_x G(x_i, x_j) - \nabla \varphi(x_i) = -\frac{\nabla f_0(x_i)}{f_0(x_i)}$$

for all $i = 1, \dots, N$.

The proof of Theorem 1 comes with strong pointwise estimates on the u_ε 's and the observation that concentration points are isolated (we refer to section 1 for details). This should be compared to the more intricate situation of Yamabe-type equations for which concentration points are not necessarily isolated (see [3, 4, 5, 6]). Independently, as is easily checked, a priori C^4 -bounds on sequences of solutions follow from the above theorem when $\int_M b_0 dv_g \notin 64\pi^2 \mathbb{N}$. This includes compactness of the geometric Paneitz equation with arbitrary prescribed Q -curvature (we refer to the nice surveys [1] and [2] for material on the Q -curvature). Such a priori C^4 -bounds should be regarded as a first step towards a Morse theory for the equations we consider in this paper. We refer to [11], where this question was handled in the case of the Yamabe equation.

1. PROOF OF THEOREM 1

Let us assume that we have a sequence (u_ε) of smooth solutions of

$$(4) \quad L_\varepsilon u_\varepsilon + b_\varepsilon(x) = f_\varepsilon(x) e^{u_\varepsilon},$$

where $L_\varepsilon = \Delta_g^2 - \operatorname{div}_g(A_\varepsilon d \cdot)$. Since we assumed that $\operatorname{Ker} L_0 = \{\text{constants}\}$, it is clear that $\operatorname{Ker} L_\varepsilon = \{\text{constants}\}$ for all $\varepsilon > 0$ small enough. Thus, if the sequence (u_ε) is bounded from above, it follows from standard elliptic theory that (u_ε) is

uniformly bounded in $C^4(M)$ except if $\int_M b_0 dv_g = 0$. This clarifies the remarks after the theorem. From now on, we assume that the u_ε 's blow up, i.e. that

$$(5) \quad \max_M u_\varepsilon \rightarrow +\infty \text{ as } \varepsilon \rightarrow 0.$$

Before starting the proof of Theorem 1, we note that, integrating equation (4),

$$(6) \quad \int_M f_\varepsilon e^{u_\varepsilon} dv_g = \int_M b_\varepsilon dv_g = \int_M b_0 dv_g + o(1).$$

We divide the proof into several steps. The first step is as follows:

STEP 1 - Assume that (5) holds. Then there exist $N \in \mathbb{N}^*$ and N sequences $(x_{i,\varepsilon})$ of converging points in M such that, after passing to a subsequence, the following assertions hold:

$$a) \quad \frac{d_g(x_{i,\varepsilon}, x_{j,\varepsilon})}{\mu_{i,\varepsilon}} \rightarrow +\infty \text{ as } \varepsilon \rightarrow 0 \text{ for all } i, j = 1, \dots, N, i \neq j, \text{ where}$$

$$f_\varepsilon(x_{i,\varepsilon}) \mu_{i,\varepsilon}^4 e^{u_\varepsilon(x_{i,\varepsilon})} = 1.$$

b) We have that

$$v_{i,\varepsilon}(x) = u_\varepsilon(\exp_{x_{i,\varepsilon}}(\mu_{i,\varepsilon}x)) - u_\varepsilon(x_{i,\varepsilon}) \rightarrow V_0(x) = -4 \ln \left(1 + \frac{|x|^2}{8\sqrt{6}} \right)$$

in $C_{loc}^4(\mathbb{R}^4)$ as $\varepsilon \rightarrow 0$ for all $i = 1, \dots, N$.

c) For all $i = 1, \dots, N$, we have that

$$\lim_{R \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{B_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon} dv_g = 64\pi^2.$$

d) At last, there exists $C > 0$ such that

$$\left(\inf_{i=1, \dots, N} d_g(x_{i,\varepsilon}, x)^4 \right) e^{u_\varepsilon(x)} \leq C$$

for all $\varepsilon > 0$ and all $x \in M$.

Proof of Step 1. We briefly sketch the proof below and we refer to [10] for the details. We let $x_\varepsilon \in M$ be such that $u_\varepsilon(x_\varepsilon) = \max_M u_\varepsilon$. By (5), $u_\varepsilon(x_\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. We let $\mu_\varepsilon > 0$ be defined by

$$(7) \quad f_\varepsilon(x_\varepsilon) \mu_\varepsilon^4 e^{u_\varepsilon(x_\varepsilon)} = 1$$

so that $\mu_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. For $x \in B_0(\delta\mu_\varepsilon^{-1})$, the Euclidean ball of center 0 and radius $\delta\mu_\varepsilon^{-1}$, $\delta > 0$ small fixed, we let

$$(8) \quad \begin{aligned} v_\varepsilon(x) &= u_\varepsilon(\exp_{x_\varepsilon}(\mu_\varepsilon x)) - u_\varepsilon(x_\varepsilon), \\ g_\varepsilon(x) &= (\exp_{x_\varepsilon}^* g)(\mu_\varepsilon x), \tilde{A}_\varepsilon(x) = (\exp_{x_\varepsilon}^* A_\varepsilon)(\mu_\varepsilon x), \\ \tilde{b}_\varepsilon(x) &= b_\varepsilon(\exp_{x_\varepsilon}(\mu_\varepsilon x)) \text{ and } \tilde{f}_\varepsilon(x) = f_\varepsilon(\exp_{x_\varepsilon}(\mu_\varepsilon x)). \end{aligned}$$

We then have that

$$(9) \quad \Delta_{g_\varepsilon}^2 v_\varepsilon - \mu_\varepsilon^2 \operatorname{div}_{g_\varepsilon} (\tilde{A}_\varepsilon dv_\varepsilon) + \mu_\varepsilon^4 \tilde{b}_\varepsilon = \frac{\tilde{f}_\varepsilon}{f_\varepsilon(x_\varepsilon)} e^{v_\varepsilon}$$

in $B_0(\delta\mu_\varepsilon^{-1})$. We write with the Green representation formula that

$$u_\varepsilon(x) - \bar{u}_\varepsilon = \int_M G_\varepsilon(x, y) L_\varepsilon u_\varepsilon(y) dv_g(y)$$

for all $x \in M$, where G_ε is the Green function of L_ε . Using equation (4) and differentiating the above with respect to x , we obtain for $k = 1, 2, 3$ that

$$\begin{aligned} |\nabla^k u_\varepsilon|_g(x) &\leq \int_M |\nabla_x^k G_\varepsilon(x, y)|_g \left| f_\varepsilon(y) e^{u_\varepsilon(y)} - b_\varepsilon(y) \right| dv_g(y) \\ &\leq \int_M |\nabla_x^k G_\varepsilon(x, y)|_g f_\varepsilon(y) e^{u_\varepsilon(y)} dv_g(y) + O(1) \end{aligned}$$

since $b_\varepsilon \rightarrow b_0$ in $C^0(M)$ as $\varepsilon \rightarrow 0$. Let $y_\varepsilon \in B_{x_\varepsilon}(R\mu_\varepsilon)$, $R > 0$ fixed. We write that

$$\begin{aligned} &\int_M |\nabla_x^k G(y_\varepsilon, y)|_g e^{u_\varepsilon(y)} dv_g(y) \\ &= O\left(\mu_\varepsilon^{-k} \int_{M \setminus B_{y_\varepsilon}(\mu_\varepsilon)} e^{u_\varepsilon} dv_g\right) + O\left(e^{u_\varepsilon(x_\varepsilon)} \int_{B_{y_\varepsilon}(\mu_\varepsilon)} d_g(y_\varepsilon, y)^{-k} dv_g(y)\right) \\ &= O(\mu_\varepsilon^{-k}) \end{aligned}$$

thanks to the fact that $u_\varepsilon \leq u_\varepsilon(x_\varepsilon)$, to (7) and to standard estimates on the Green function (which are uniform in ε). Together with the definition (8) of v_ε , this gives that (v_ε) is uniformly bounded in $C^3(K)$ for all compact subsets K of $\mathbb{R}^4$. Standard elliptic theory then gives thanks to equation (9) that

$$(10) \quad \lim_{\varepsilon \rightarrow 0} v_\varepsilon = V_0 \text{ in } C_{loc}^4(\mathbb{R}^4)$$

where V_0 is a solution of

$$(11) \quad \Delta_\xi^2 V_0 = e^{V_0}$$

in $\mathbb{R}^4$ satisfying $V_0(x) \leq V_0(0) = 0$ for all $x \in \mathbb{R}^4$. Moreover, since

$$\lim_{\varepsilon \rightarrow 0} \int_{B_{x_\varepsilon}(R\mu_\varepsilon)} f_\varepsilon e^{u_\varepsilon} dv_g = \int_{B_0(R)} e^{V_0} dx,$$

equation (6) implies that $e^{V_0} \in L^1(\mathbb{R}^4)$. From the classification of the solutions of equation (11) by Lin [8], we get that either

$$(12) \quad V_0(x) = -4 \ln \left(1 + \frac{|x|^2}{8\sqrt{6}} \right)$$

or there exists $a > 0$ such that

$$(13) \quad \Delta_\xi V_0 \geq a$$

in $\mathbb{R}^4$. Let us prove that we are in the first situation. For that purpose, we write with the Green representation formula and equation (4) that

$$\begin{aligned} &\int_{B_0(R)} |\Delta_{g_\varepsilon} v_\varepsilon|_{g_\varepsilon} dv_{g_\varepsilon} = \mu_\varepsilon^{-2} \int_{B_{x_\varepsilon}(R\mu_\varepsilon)} |\Delta_g u_\varepsilon|_g dv_g \\ &\leq C \mu_\varepsilon^{-2} \int_{x \in B_{x_\varepsilon}(R\mu_\varepsilon)} \int_{y \in M} |\Delta_{g,x} G_\varepsilon(x, y)|_g \left(e^{u_\varepsilon(y)} + 1 \right) dv_g(y) dv_g(x) \\ &\leq C \mu_\varepsilon^{-2} \int_{y \in M} \left(e^{u_\varepsilon(y)} + 1 \right) \left(\int_{x \in B_{x_\varepsilon}(R\mu_\varepsilon)} d_g(x, y)^{-2} dv_g(x) \right) dv_g(y) \\ &\leq CR^2 \end{aligned}$$

thanks to standard estimates on the Green function and to (6), where $C > 0$ denotes some constant independent of R and $\varepsilon > 0$. Letting $\varepsilon \rightarrow 0$, we get that

$$\int_{B_0(R)} |\Delta_\xi V_0|_\xi dx \leq CR^2$$

for all $R > 0$. This clearly eliminates the possibility (13). Then (12) must hold. It is then easily checked that

$$(14) \quad \lim_{R \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{B_{x_\varepsilon}(R\mu_\varepsilon)} f_\varepsilon e^{u_\varepsilon} dv_g = \int_{\mathbb{R}^4} e^{V_0} dx = 64\pi^2.$$

For $k \geq 1$, we say that $\mathcal{H}_k$ holds if there exist $(x_{i,\varepsilon})_{i=1,\dots,k}$ k converging sequences of points in M and $(\mu_{i,\varepsilon})_{i=1,\dots,k}$ k sequences of positive real numbers going to 0 as $\varepsilon \rightarrow 0$ such that $f_\varepsilon(x_{i,\varepsilon})\mu_{i,\varepsilon}^4 e^{u_\varepsilon(x_{i,\varepsilon})} = 1$ and such that, after passing to a subsequence, the following assertions hold:

$$(A_k^1) \quad \frac{d_g(x_{i,\varepsilon}, x_{j,\varepsilon})}{\mu_{i,\varepsilon}} \rightarrow +\infty \text{ as } \varepsilon \rightarrow 0 \text{ for all } i, j = 1, \dots, N, i \neq j.$$

$$(A_k^2) \quad \text{We have that}$$

$$v_{i,\varepsilon}(x) = u_\varepsilon(\exp_{x_{i,\varepsilon}}(\mu_{i,\varepsilon}x)) - u_\varepsilon(x_{i,\varepsilon}) \rightarrow V_0(x) = -4 \ln \left(1 + \frac{|x|^2}{8\sqrt{6}} \right)$$

in $C_{loc}^4(\mathbb{R}^4)$ as $\varepsilon \rightarrow 0$ for all $i = 1, \dots, N$.

$$(A_k^3) \quad \text{For all } i = 1, \dots, N, \text{ we have that}$$

$$\lim_{R \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{B_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon} dv_g = 64\pi^2.$$

Clearly, with what we said above, $\mathcal{H}_1$ holds. Now we let $k \geq 1$ and assume that $\mathcal{H}_k$ holds. We also assume that

$$(15) \quad \sup_M R_{k,\varepsilon}(x)^4 e^{u_\varepsilon(x)} \rightarrow +\infty \text{ as } \varepsilon \rightarrow 0,$$

where

$$R_{k,\varepsilon}(x) = \min_{i=1,\dots,k} d_g(x_{i,\varepsilon}, x).$$

We prove in the following that, in this situation, $\mathcal{H}_{k+1}$ holds. For that purpose, we let $x_{k+1,\varepsilon} \in M$ be such that

$$(16) \quad R_{k,\varepsilon}(x_{k+1,\varepsilon})^4 e^{u_\varepsilon(x_{k+1,\varepsilon})} = \sup_M R_{k,\varepsilon}(x)^4 e^{u_\varepsilon(x)}$$

and we set

$$\mu_{k+1,\varepsilon} = \left(\frac{1}{f_\varepsilon(x_{k+1,\varepsilon}) e^{u_\varepsilon(x_{k+1,\varepsilon})}} \right)^{\frac{1}{4}}.$$

Since M is compact, (15) implies that $\mu_{k+1,\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$ and that

$$(17) \quad \frac{d_g(x_{i,\varepsilon}, x_{k+1,\varepsilon})}{\mu_{k+1,\varepsilon}} \rightarrow +\infty \text{ as } \varepsilon \rightarrow 0$$

for all $i = 1, \dots, k$. Thanks to (A_k^2) , it is also easily checked that $\frac{d_g(x_{i,\varepsilon}, x_{k+1,\varepsilon})}{\mu_{i,\varepsilon}} \rightarrow +\infty$ as $\varepsilon \rightarrow 0$ for all $i = 1, \dots, k$ so that (A_{k+1}^1) holds. It follows from (16) and (17) that

$$\lim_{\varepsilon \rightarrow 0} \sup_{z \in B_{x_{k+1,\varepsilon}}(R\mu_{k+1,\varepsilon})} (u_\varepsilon(z) - u_\varepsilon(x_{k+1,\varepsilon})) = 0.$$

Mimicking what we did above thanks to the Green representation formula, one then proves that, after passing to a subsequence,

$$u_\varepsilon \left(\exp_{x_{k+1,\varepsilon}} (\mu_{k+1,\varepsilon} x) \right) - u_\varepsilon (x_{k+1,\varepsilon}) \rightarrow V_0(x)$$

in $C_{loc}^4(\mathbb{R}^4)$ as $\varepsilon \rightarrow 0$. And, as a consequence,

$$\lim_{R \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{B_{x_{k+1,\varepsilon}}(R\mu_{k+1,\varepsilon})} f_\varepsilon e^{u_\varepsilon} dv_g = 64\pi^2.$$

Recollecting the information above, one gets that $\mathcal{H}_{k+1}$ holds. Since (A_k^1) and (A_k^3) of $\mathcal{H}_k$ imply that

$$\int_M f_\varepsilon e^{u_\varepsilon} dv_g \geq 64\pi^2 k + o(1),$$

we easily get, thanks to (6), that there exists a maximal k , $1 \leq k \leq \frac{1}{64\pi^2} \int_M b_0 dv_g$, such that $\mathcal{H}_k$ holds. Arriving to this maximal k , we get that (15) cannot hold. Writing $k = N$, we have finished the proof of Step 1. $\square$

STEP 2 - For $k = 1, 2, 3$, there exists $C_k > 0$ such that

$$R_\varepsilon(x)^k |\nabla^k u_\varepsilon|_g(x) \leq C_k$$

for all $x \in M$ and all $\varepsilon > 0$. Here,

$$R_\varepsilon(x) = \inf_{i=1,\dots,N} d_g(x_{i,\varepsilon}, x)$$

where the $x_{i,\varepsilon}$'s are as in Step 1.

Proof of Step 2. We use again the Green representation for u_ε that we differentiate. We let $x_\varepsilon \in M$ be such that $x_\varepsilon \neq x_{i,\varepsilon}$ for all $i = 1, \dots, N$. Note that, for $x_\varepsilon = x_{i,\varepsilon}$, the estimates of the proposition are obvious. We write, thanks to standard estimates on the Green function, that

$$|\nabla^k u_\varepsilon|_g(x_\varepsilon) = O\left(\int_M \frac{1}{d_g(x_\varepsilon, y)^k} e^{u_\varepsilon(y)} dv_g(y)\right) + O(1).$$

For $i = 1, \dots, N$, we let

$$\Omega_{i,\varepsilon} = \{y \in M, R_\varepsilon(y) = d_g(x_{i,\varepsilon}, y)\}$$

and we write that

$$\begin{aligned} & \int_{\Omega_{i,\varepsilon}} \frac{1}{d_g(x_\varepsilon, y)^k} e^{u_\varepsilon(y)} dv_g(y) \\ &= O\left(\frac{1}{d_g(x_\varepsilon, x_{i,\varepsilon})^k} \int_{\Omega_{i,\varepsilon} \cap B_{x_{i,\varepsilon}}\left(\frac{d_g(x_\varepsilon, x_{i,\varepsilon})}{2}\right)} e^{u_\varepsilon} dv_g\right) \\ & \quad + O\left(\int_{\Omega_{i,\varepsilon} \setminus B_{x_{i,\varepsilon}}\left(\frac{d_g(x_{i,\varepsilon}, x_\varepsilon)}{2}\right)} \frac{1}{d_g(x_\varepsilon, y)^k} \frac{1}{d_g(y, x_{i,\varepsilon})^4} dv_g(y)\right) \\ &= O\left(\frac{1}{d_g(x_\varepsilon, x_{i,\varepsilon})^k}\right) \end{aligned}$$

thanks to assertion d) of Step 1, to (6) and to some straightforward computations. Step 2 clearly follows. $\square$

STEP 3 - For any $1 \leq \nu < 2$, there exists $\delta_\nu > 0$ and $C_\nu > 0$ such that

$$\mu_{i,\varepsilon}^{4(1-\nu)} d_g(x_{i,\varepsilon}, x)^{4\nu} e^{u_\varepsilon(x)} \leq C_\nu$$

for all $i = 1, \dots, N$, all $\varepsilon > 0$ and all $x \in B_{x_{i,\varepsilon}}(\delta_\nu)$ where $x_{i,\varepsilon}$ and $\mu_{i,\varepsilon}$ are as in Step 1. In particular, we have that

$$d_g(x_{i,\varepsilon}, x_{j,\varepsilon}) \geq \delta_0$$

for all $i, j \in \{1, \dots, N\}$, $i \neq j$, where $\delta_0 > 0$ is independent of ε and i, j . At last, this implies that $\bar{u}_\varepsilon \rightarrow -\infty$ as $\varepsilon \rightarrow 0$.

Proof of Step 3. Fix $1 \leq \nu < 2$. We set for $i = 1, \dots, N$

$$(18) \quad R_{i,\varepsilon} = \min_{j \neq i} d_g(x_{i,\varepsilon}, x_{j,\varepsilon})$$

and we take some $i \in \{1, \dots, N\}$ such that there exists $\theta > 0$ such that

$$(19) \quad R_{i,\varepsilon} \leq \theta R_{j,\varepsilon}$$

for all $j \in \{1, \dots, N\}$. We set

$$(20) \quad \varphi_{i,\varepsilon}(r) = r^{4\nu} \exp \left((Vol_g(\partial B_{x_{i,\varepsilon}}(r)))^{-1} \int_{\partial B_{x_{i,\varepsilon}}(r)} u_\varepsilon d\sigma_g \right)$$

for $0 \leq r < \inf_j d_g(x_{i,\varepsilon}, x_{j,\varepsilon})$. A simple consequence of assertion b) of Step 1 is that

$$(21) \quad \varphi'_{i,\varepsilon}(R\mu_{i,\varepsilon}) < 0$$

for $\varepsilon > 0$ small and all $R \geq R_\nu$ where $R_\nu^2 = \frac{16\sqrt{6}\nu}{2-\nu}$. We define $r_{i,\varepsilon}$ by

$$(22) \quad r_{i,\varepsilon} = \inf \left\{ R_\nu \mu_{i,\varepsilon} \leq r \leq \frac{R_{i,\varepsilon}}{2} \text{ s.t. } \varphi'_{i,\varepsilon}(r) < 0 \text{ in } [R_\nu \mu_{i,\varepsilon}, r] \right\}.$$

Note that, by (21), we have that

$$(23) \quad \frac{r_{i,\varepsilon}}{\mu_{i,\varepsilon}} \rightarrow +\infty \text{ as } \varepsilon \rightarrow 0.$$

Let us assume that

$$(24) \quad r_{i,\varepsilon} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

For $x \in B_0(\delta r_{i,\varepsilon}^{-1})$, $\delta > 0$ small fixed, we set

$$(25) \quad v_{i,\varepsilon}(x) = u_\varepsilon(\exp_{x_{i,\varepsilon}}(r_{i,\varepsilon}x)) - C_{i,\varepsilon},$$

where

$$(26) \quad C_{i,\varepsilon} = (Vol_g(\partial B_{x_{i,\varepsilon}}(r_{i,\varepsilon})))^{-1} \int_{\partial B_{x_{i,\varepsilon}}(r_{i,\varepsilon})} u_\varepsilon d\sigma_g.$$

We also set, for $j \in \mathcal{S}_i = \{j \neq i \text{ s.t. } d_g(x_{i,\varepsilon}, x_{j,\varepsilon}) = O(r_{i,\varepsilon})\}$,

$$(27) \quad \tilde{x}_{j,\varepsilon} = r_{i,\varepsilon}^{-1} \exp_{x_{i,\varepsilon}}^{-1}(x_{j,\varepsilon}) \text{ and } \tilde{x}_j = \lim_{\varepsilon \rightarrow 0} \tilde{x}_{j,\varepsilon},$$

after passing to a subsequence, if necessary. Note that, thanks to (18), to (22) and to the choice of i we made (see (19)), we have that $|\tilde{x}_j| \geq 2$ for all $j \in \mathcal{S}_i$ and that $|\tilde{x}_j - \tilde{x}_k| \geq \frac{2}{\theta}$ for all $j, k \in \mathcal{S}_i$, $j \neq k$. By equation (4), we have that

$$(28) \quad \Delta_{g_{i,\varepsilon}}^2 v_{i,\varepsilon} - r_{i,\varepsilon}^2 \operatorname{div}_{g_{i,\varepsilon}}(A_{i,\varepsilon} \nabla v_{i,\varepsilon}) + r_{i,\varepsilon}^4 b_{i,\varepsilon} = f_{i,\varepsilon} \varphi_{i,\varepsilon}(r_{i,\varepsilon}) r_{i,\varepsilon}^{4(1-\nu)} e^{v_{i,\varepsilon}}$$

in $B_0(\delta r_{i,\varepsilon}^{-1})$, where

$$(29) \quad \begin{aligned} g_{i,\varepsilon}(x) &= \left(\exp_{x_{i,\varepsilon}}^* g \right) (r_{i,\varepsilon} x), \quad A_{i,\varepsilon}(x) = \left(\exp_{x_{i,\varepsilon}}^* A_\varepsilon \right) (r_{i,\varepsilon} x), \\ b_{i,\varepsilon}(x) &= b_\varepsilon \left(\exp_{x_{i,\varepsilon}} (r_{i,\varepsilon} x) \right) \quad \text{and} \quad f_{i,\varepsilon}(x) = f_\varepsilon \left(\exp_{x_{i,\varepsilon}} (r_{i,\varepsilon} x) \right). \end{aligned}$$

Thanks to Step 2, we know that $(v_{i,\varepsilon})$ is uniformly bounded in $C^3(K)$ for all compact subsets K of $\mathbb{R}^4 \setminus \{0, \tilde{x}_j\}_{j \in \mathcal{S}_i}$. Thanks to the definition (22) of $r_{i,\varepsilon}$ and to (23), we have that

$$\varphi_{i,\varepsilon}(r_{i,\varepsilon}) \leq \varphi_{i,\varepsilon}(R\mu_{i,\varepsilon})$$

for all $R > R_\nu$. Thanks to assertion b) of Step 1 and to (23), it is now rather easily checked that

$$\lim_{R \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \varphi_{i,\varepsilon}(R\mu_{i,\varepsilon}) r_{i,\varepsilon}^{4(1-\nu)} = 0$$

since $1 \leq \nu < 2$. Thus standard elliptic theory leads, thanks to (28) and (29), that, after passing to a subsequence,

$$(30) \quad v_{i,\varepsilon} \rightarrow H_i \quad \text{in } C_{loc}^4 \left(\mathbb{R}^4 \setminus \{0, \tilde{x}_j\}_{j \in \mathcal{S}_i} \right) \quad \text{as } \varepsilon \rightarrow 0,$$

where H_i satisfies

$$(31) \quad \Delta_\xi^2 H_i = 0 \quad \text{in } \mathbb{R}^4 \setminus \{0, \tilde{x}_j\}_{j \in \mathcal{S}_i}.$$

Moreover, thanks to Step 2, we have that, for $l = 1, 2, 3$,

$$(32) \quad R(x)^l |\nabla^l H_i(x)|_\xi \leq C_l \quad \text{in } \mathbb{R}^4 \setminus \{0, \tilde{x}_j\}_{j \in \mathcal{S}_i},$$

where

$$R(x) = \min \{ |x|; |x - \tilde{x}_j| \}_{j \in \mathcal{S}_i}.$$

Equation (32) easily allows us to prove that

$$(33) \quad H_i(x) = \alpha \ln \frac{1}{|x|} + \sum_{j \in \mathcal{S}_i} \alpha_j \ln \frac{1}{|x - \tilde{x}_j|} + \beta,$$

where α, β and the α_j 's are real numbers. Integrating equation (28) over $B_0(1)$ and passing to the limit as $\varepsilon \rightarrow 0$ thanks to (29), (30) and (33), we obtain that

$$\lim_{\varepsilon \rightarrow 0} \varphi_{i,\varepsilon}(r_{i,\varepsilon}) r_{i,\varepsilon}^{4(1-\nu)} \int_{B_0(1)} f_{i,\varepsilon} e^{v_{i,\varepsilon}} dv_{g_{i,\varepsilon}} = - \int_{\partial B_0(1)} \partial_\nu \Delta_\xi H_i d\sigma_\xi = 8\alpha\pi^2.$$

With a change of variable, we get that

$$\varphi_{i,\varepsilon}(r_{i,\varepsilon}) r_{i,\varepsilon}^{4(1-\nu)} \int_{B_0(1)} f_{i,\varepsilon} e^{v_{i,\varepsilon}} dv_{g_{i,\varepsilon}} = \int_{B_{x_{i,\varepsilon}}(r_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon} dv_g$$

so that

$$(34) \quad \lim_{\varepsilon \rightarrow 0} \int_{B_{x_{i,\varepsilon}}(r_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon} dv_g = 8\alpha\pi^2.$$

Step 2 with $k = 1$ together with the definitions of $R_{i,\varepsilon}$ and $r_{i,\varepsilon}$ gives the existence of some $C > 0$ such that for any $0 \leq r \leq 3/2$,

$$\left| u_\varepsilon \left(\exp_{x_{i,\varepsilon}} (r_{i,\varepsilon} x) \right) - u_\varepsilon \left(\exp_{x_{i,\varepsilon}} (r_{i,\varepsilon} y) \right) \right| \leq C$$

for all $x, y \in \mathbb{R}^4$ such that $|x| = |y| = r$. With point b) of Step 1, (22) and (23), we then get that for any $\eta > 0$, there exists $R_\eta > 0$ such that for any $R > R_\eta$, we have that

$$(35) \quad d_g(x, x_{i,\varepsilon})^{4\nu} e^{u_\varepsilon(x)} \leq \eta \mu_{i,\varepsilon}^{4(\nu-1)}$$

for all $x \in B_{x_{i,\varepsilon}}(r_{i,\varepsilon}) \setminus B_{x_{i,\varepsilon}}(R\mu_{i,\varepsilon})$. With point b) of Step 1 and (35), we get that

$$\lim_{\varepsilon \rightarrow 0} \int_{B_{x_{i,\varepsilon}}(r_{i,\varepsilon})} f_\varepsilon e^{u_\varepsilon} dv_g = 64\pi^2.$$

With (34), we obtain that $\alpha = 8$. Integrating on $B_{\tilde{x}_j}(\delta)$ for $\delta > 0$ small instead of $B_0(1)$, one proves in the same way that $\alpha_j \geq 8$ for all $j \in \mathcal{S}_i$. We let

$$\bar{H}_i(r) = \frac{1}{2\pi^2 r^3} \int_{\partial B_0(r)} H_i(x) d\sigma.$$

A simple computation gives

$$\frac{d}{dr} \left(r^{4\nu} e^{\bar{H}_i(r)} \right) = 4 \left(\nu - 2 - \left(\sum_{j \in \mathcal{S}_i} \frac{\alpha_j}{8|\tilde{x}_j|^2} \right) r^2 \right) r^{4\nu-1} e^{\bar{H}_i(r)}$$

for $r \in (0, \frac{3}{2})$. Since $\nu < 2$, we get in particular that

$$\frac{d}{dr} \left(r^{4\nu} e^{\bar{H}_i(r)} \right) (1) < 0.$$

This clearly proves that

$$(36) \quad r_{i,\varepsilon} = \frac{R_{i,\varepsilon}}{2}$$

for all i such that (19) holds. Thanks to (24), this in turn implies that $R_{i,\varepsilon} \rightarrow 0$ and that $\mathcal{S}_j \neq \emptyset$. Note that, for the moment, we have proved, with the help of Step 2 (see (35)), that the estimate of Step 3 holds if for any $i \in \{1, \dots, N\}$, we have that $R_{i,\varepsilon} \not\rightarrow 0$ as $\varepsilon \rightarrow 0$. Indeed, if this is the case, there exists some $\delta > 0$ such that $R_{j,\varepsilon} \geq \delta$ for all $j \in \{1, \dots, N\}$, and one can easily repeat the above arguments with any of the j 's in $\{1, \dots, N\}$. Thus, in order to end the proof of the step, it remains to prove that $R_{i,\varepsilon} \not\rightarrow 0$ as $\varepsilon \rightarrow 0$ for all $i \in \{1, \dots, N\}$. We let $i_0 \in \{1, \dots, N\}$ be such that, up to a subsequence,

$$R_{i_0,\varepsilon} = \min_{i=1,\dots,N} R_{i,\varepsilon}.$$

We assume by contradiction that

$$\lim_{\varepsilon \rightarrow 0} R_{i_0,\varepsilon} = 0.$$

Clearly (19) holds for $i = i_0$, and (36) holds. It then follows from the definition of $\mathcal{S}_{i_0}$ that for any $i \in \mathcal{S}_{i_0}$, there exists $C(i) > 0$ such that

$$R_{i,\varepsilon} \leq C(i) R_{j,\varepsilon}$$

for all $j \in \{1, \dots, N\}$. It follows that (19) holds for all $i \in \mathcal{S}_{i_0}$, and that the preceding analysis can be carried out. We pick up $i \in \mathcal{S}_{i_0}$ such that

$$d_g(x_{i,\varepsilon}, x_{i_0,\varepsilon}) \geq d_g(x_{j,\varepsilon}, x_{i_0,\varepsilon})$$

for all $j \in \mathcal{S}_{i_0}$ and all $\varepsilon > 0$. With (27), we get that $|\tilde{x}_{i_0}| \geq |\tilde{x}_j - \tilde{x}_{i_0}|$ for all $j \in \mathcal{S}_{i_0}$. Since $\mathcal{S}_i = (\mathcal{S}_{i_0} \setminus \{i\}) \cup \{i_0\}$, we have that

$$|\tilde{x}_{i_0}| \geq |\tilde{x}_j - \tilde{x}_{i_0}|$$

for all $j \in \mathcal{S}_i$. A consequence of this inequality is that

$$(37) \quad (\tilde{x}_{i_0}, \tilde{x}_j) > 0$$

for all $j \in \mathcal{S}_i$, where $(\cdot, \cdot)$ denotes the Euclidean scalar product. This amounts to assuming that all the $\tilde{x}_j$'s, $j \in \mathcal{S}_i$, lie in the same half-space where boundary contains 0. Let $0 < \delta < 1$. We write thanks to equation (28) that

$$\begin{aligned} & \int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta_{g_{i,\varepsilon}}^2 v_{i,\varepsilon} dv_{g_{i,\varepsilon}} - r_{i,\varepsilon}^2 \int_{B_0(\delta)} \nabla v_{i,\varepsilon} \operatorname{div}_{g_{i,\varepsilon}} (A_{i,\varepsilon} \nabla v_{i,\varepsilon}) dv_{g_{i,\varepsilon}} \\ &= \varphi_{i,\varepsilon}(r_{i,\varepsilon}) r_{i,\varepsilon}^{4(1-\nu)} \int_{B_0(\delta)} f_{i,\varepsilon} \nabla e^{v_{i,\varepsilon}} dv_{g_{i,\varepsilon}} - r_{i,\varepsilon}^4 \int_{B_0(\delta)} b_{i,\varepsilon} \nabla v_{i,\varepsilon} dv_{g_{i,\varepsilon}}. \end{aligned}$$

Integrating by parts, using the estimates of Step 2, (6) and (30), one can easily estimate the different terms involved in this equation to arrive to

$$(38) \quad \int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta_{g_{i,\varepsilon}}^2 v_{i,\varepsilon} dv_{g_{i,\varepsilon}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Using the Cartan expansion of the metric in the exponential chart and the estimates on the derivatives of $v_{i,\varepsilon}$, some integrations by parts then lead with (30) to

$$\begin{aligned} \left(\int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta_{g_{i,\varepsilon}}^2 v_{i,\varepsilon} dv_{g_{i,\varepsilon}} \right)_k & \rightarrow - \int_{\partial B_0(\delta)} \partial_k H_i (\nabla \Delta_\xi H_i, \nu)_\xi d\sigma_\xi \\ & + \int_{\partial B_0(\delta)} \partial_{lk} H_i \nu^l \Delta_\xi H_i d\sigma_\xi \\ & + \frac{1}{2} \int_{\partial B_0(\delta)} (\Delta_\xi H_i)^2 \nu_k d\sigma_\xi \end{aligned}$$

as $\varepsilon \rightarrow 0$. We let

$$H_i(x) = 8 \ln \frac{1}{|x|} + G_i(x).$$

Simple computations then give that

$$\int_{B_0(\delta)} \nabla v_{i,\varepsilon} \Delta_{g_{i,\varepsilon}}^2 v_{i,\varepsilon} dv_\xi \rightarrow 64\pi^2 \nabla G_i(0)$$

as $\varepsilon \rightarrow 0$. Coming back to (38), we obtain that $\nabla G_i(0) = 0$, a contradiction with the choice of i we made in (37). This ends the proof of Step 3. Note that the fact that $\bar{u}_\varepsilon \rightarrow -\infty$ is a direct consequence of the estimate we just proved and of Step 2. $\square$

We are now in a position to conclude the proof of Theorem 1. Using the estimates of Step 3, it is easily checked that

$$\int_M f_\varepsilon e^{u_\varepsilon} dv_g \rightarrow 64\pi^2 N \text{ as } \varepsilon \rightarrow 0,$$

which gives the first assertion of the theorem thanks to (6). Since we already proved that $\bar{u}_\varepsilon \rightarrow -\infty$ as $\varepsilon \rightarrow 0$, it remains to prove the convergence of $u_\varepsilon - \bar{u}_\varepsilon$ outside the concentration points and to prove the last property of the theorem concerning the

location of concentration points. We let $\mathcal{S} = \{x_i\}_{i=1,\dots,N}$, where $x_i = \lim_{\varepsilon \rightarrow 0} x_{i,\varepsilon}$. We let $x_0 \in M \setminus \mathcal{S}$ and we write with the Green representation formula that

$$u_\varepsilon(x_0) - \bar{u}_\varepsilon = \int_M G_\varepsilon(x_0, y) \left(f_\varepsilon(y) e^{u_\varepsilon(y)} - b_\varepsilon(y) \right) dv_g(y),$$

where G_ε is the Green function of L_ε . It is then easy to compute an asymptotic expansion of the different terms involved to get that

$$(39) \quad u_\varepsilon(x_0) - \bar{u}_\varepsilon \rightarrow 64\pi^2 \sum_{i=1}^N G(x_0, x_i) - \int_M G(x_0, y) b_0(y) dv_g(y)$$

as $\varepsilon \rightarrow 0$, where G is the Green function of the limit operator L_0 . The convergence result in the theorem easily follows. The last part of the theorem is a consequence of a Pohozaev-type identity. More precisely, we write in the exponential chart around $x_i \in \mathcal{S}$ and for $\delta > 0$ small enough that

$$\int_{B_{x_i}(\delta)} (L_\varepsilon u_\varepsilon + b_\varepsilon) \nabla u_\varepsilon dv_g = \int_{B_{x_i}(\delta)} f_\varepsilon e^{u_\varepsilon} \nabla u_\varepsilon dv_g$$

thanks to equation (4). Integration by parts together with the dominated convergence theorem then lead to

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{B_{x_i}(\delta)} f_\varepsilon e^{u_\varepsilon} \nabla u_\varepsilon dv_g = -64\pi^2 \frac{\nabla f_0(x_i)}{f_0(x_i)}$$

thanks to Steps 1 to 3 and to (39). On the other hand, after integration by parts, using (39), rather long but easy computations lead to

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{B_{x_i}(\delta)} (L_\varepsilon u_\varepsilon + b_\varepsilon) \nabla u_\varepsilon dv_g = 64\pi^2 \nabla G_i(x_i),$$

where

$$G_i(x) = 64\pi^2 \beta(x_i, x) + 64\pi^2 \sum_{j \neq i}^N G(x, x_j) - \int_M G(x, y) b_0(y) dv_g(y)$$

with β the regular part of G . The last assertion of the theorem follows.

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