ISOLATED INVARIANT CURVES OF A FOLIATION

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ABSTRACT. We bound the equisingularity type of the set of isolated separatrices of a holomorphic foliation $\mathcal{F}$ of $(\mathbb{C}^2, 0)$ in terms of the Milnor number of $\mathcal{F}$. This result gives a bound for the degree of an algebraic invariant curve $C \subset \mathbb{P}_{\mathbb{C}}^2$ of a foliation $\mathcal{G}$ of $\mathbb{P}_{\mathbb{C}}^2$ in terms of the degree of $\mathcal{G}$, provided that all the branches of C are isolated separatrices.

1. INTRODUCTION

In this paper we bound the degree of any algebraic invariant curve C of a holomorphic singular foliation $\mathcal{F}$ of $\mathbb{P}_{\mathbb{C}}^2$ in terms of the degree of $\mathcal{F}$, under the assumption that C is an *isolated separatrix* at any singular point of $\mathcal{F}$. Let us note that, in general, one cannot bound the degree of an algebraic invariant curve and hence it is not possible to give a direct answer to the Poincaré Problem [15]. To see this it is enough to consider the dicritical foliation $pydx - qxdy = 0$.

The isolated separatrices are those that are not “inside” a dicritical component after a minimal reduction of the singularities of $\mathcal{F}$ (see [5]). To bound their degree has consequences in counting the algebraic cycles of a real vector field in the frame of the 16th Hilbert Problem: the cycles will be of two types, the ones corresponding to isolated separatrices (we could bound the number of these cycles) and the ones corresponding to non-isolated separatrices. These last ones will produce a kind of Poincaré map in the corresponding dicritical component of the minimal reduction of $\mathcal{F}$, that could be useful to control their number. We also think that this work will shed light on the comprehension of the structure of pencils of curves (rational first integrals) and, more generally, logarithmic foliations in terms of the isolated separatrices.

We proceed by a local method. First, we bound the number of isolated branches, the multiplicities and the length of the reduction of singularities of the isolated separatrices in terms of the Milnor number. Second, we recover the arguments in [8] and in [3] to get a bound for the degree.

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2. LOCAL INVARIANTS AND BLOWING-UPS

Let M be an analytic complex manifold of dimension two and let $\mathcal{F}$ be a singular holomorphic foliation on M . Given a point $p \in M$, we denote by $\nu_p(\mathcal{F})$ the algebraic multiplicity of $\mathcal{F}$ at p and by $\mu_p(\mathcal{F})$ the Milnor number of $\mathcal{F}$ at p . Consider the blowing-up $\pi_1 : M_1 \rightarrow (M, p)$ of p , the exceptional divisor $E_1 = \pi_1^{-1}(p)$, and let $\mathcal{F}_1$ be the strict transform of $\mathcal{F}$ by π_1 . We say that π_1 is a *non-dicritical* blowing-up if E_1 is an invariant curve for $\mathcal{F}_1$, otherwise π_1 is a *dicritical* blowing-up. Then we have that (see [13])

$$\begin{aligned} (1) \quad \mu_p(\mathcal{F}) &= \nu_p^2(\mathcal{F}) - (\nu_p(\mathcal{F}) + 1) + \sum_{p' \in E_1} \mu_{p'}(\mathcal{F}_1) && \text{if } \pi_1 \text{ is non-dicritical,} \\ (2) \quad \mu_p(\mathcal{F}) &= (\nu_p(\mathcal{F}) + 1)^2 - (\nu_p(\mathcal{F}) + 2) + \sum_{p' \in E_1} \mu_{p'}(\mathcal{F}_1) && \text{if } \pi_1 \text{ is dicritical.} \end{aligned}$$

Assume that $\mathcal{F}$ is defined by the vector field X in a neighbourhood of p and let B be an irreducible curve at p which is invariant by $\mathcal{F}$. The *Gómez-Mont-Seade-Verjovsky index* $Z_p(\mathcal{F}, B)$ was introduced in [11]; when B is a smooth curve, the index $Z_p(\mathcal{F}, B)$ is just the order at p of the restriction of X to B . Moreover, its behavior under blowing-up is given by the following formula:

$$\begin{aligned} (3) \quad Z_p(\mathcal{F}, B) &= Z_{p_1}(\mathcal{F}_1, B_1) + \nu_p(B)(\nu_p(\mathcal{F}) - \nu_p(B)) && \text{if } \pi_1 \text{ is non-dicritical,} \\ (4) \quad Z_p(\mathcal{F}, B) &= Z_{p_1}(\mathcal{F}_1, B_1) + \nu_p(B)(\nu_p(\mathcal{F}) + 1 - \nu_p(B)) && \text{if } \pi_1 \text{ is dicritical} \end{aligned}$$

(see [3, 4]), where B_1 is the strict transform of B by π_1 , we put $p_1 = B_1 \cap E_1$ and $\nu_p(B)$ is the multiplicity of B at p . The index $Z_p(\mathcal{F}, B)$ has also a sense for a curve with several branches, by the formula

$$(5) \quad Z_p(\mathcal{F}, B' \cup B'') = Z_p(\mathcal{F}, B') + Z_p(\mathcal{F}, B'') - 2(B', B'')_p.$$

Finally, in the case of a compact curve C , the sum $Z(\mathcal{F}, C)$ of the local indices at the singular points of the foliation has many global interpretations as indicated, for instance, in [3, 4, 11]. The most interesting for us in this paper is

$$(6) \quad Z(\mathcal{F}, C) = k(d + 2 - k),$$

(see [3], Prop. 4, or [4], Chap. 2, Prop. 3) where k, d are the respective degrees of $C, \mathcal{F}$ and the ambient space is the projective plane $\mathbb{P}_{\mathbb{C}}^2$.

Assume now that B is a smooth curve at p which is not invariant by $\mathcal{F}$. Consider local coordinates (x, y) at p such that $B = (y = 0)$ and $X = a(x, y)\partial/\partial x + b(x, y)\partial/\partial y$. The *order of tangency of $\mathcal{F}$ with B at p* is given by $\eta_p(\mathcal{F}, B) = \text{ord}_x(b(x, 0))$ (see [5]). With the same notation as above, we have that

$$\begin{aligned} (7) \quad \eta_{p_1}(\mathcal{F}_1, B_1) &= \eta_p(\mathcal{F}, B) - \nu_p(\mathcal{F}) && \text{if } \pi_1 \text{ is non-dicritical,} \\ (8) \quad \eta_{p_1}(\mathcal{F}_1, B_1) &= \eta_p(\mathcal{F}, B) - (\nu_p(\mathcal{F}) + 1) && \text{if } \pi_1 \text{ is dicritical.} \end{aligned}$$

Moreover, if the blowing-up π_1 of p is dicritical, we obtain that ([5])

$$(9) \quad \nu_p(\mathcal{F}) - 1 = \sum_{p' \in E_1} \eta_{p'}(\mathcal{F}_1, E_1).$$

3. EQUISINGULARITY TYPES OF THE ISOLATED SEPARATRICES

Consider a singular foliation $\mathcal{F}$ of $(\mathbb{C}^2, 0)$ and a normal crossing divisor D in $(\mathbb{C}^2, 0)$. We denote by $Sing(\mathcal{F}, D)$ the *adapted singular locus*, i.e., the singular points of $\mathcal{F}$ and the points where $\mathcal{F}$ and D do not have normal crossings. We say that a point $p \in Sing(\mathcal{F}, D)$ is *simple adapted to D* if p is a simple singularity of $\mathcal{F}$, the divisor $D \neq \emptyset$ and all the components of D through p are non-dicritical ones. The *minimal reduction of singularities* [16] of the pair $(\mathcal{F}, D)$ is the morphism

$$\pi : M \rightarrow (\mathbb{C}^2, 0),$$

a composition of the minimal number of blowing-ups of points such that any singularity of $Sing(\mathcal{F}', E)$ is simple adapted to E , where $\mathcal{F}'$ is the strict transform of $\mathcal{F}$ by π and $E = \pi^{-1}(D \cup \{0\})$ (see [6, 7]).

We say that an irreducible separatrix B of $\mathcal{F}$ is an *isolated separatrix* of $(\mathcal{F}, D)$ if the strict transform B' of B by π cuts the exceptional divisor $E = \pi^{-1}(D \cup \{0\})$ in a non-dicritical component (see [5]). In the case $D = \emptyset$, we say just isolated separatrix of $\mathcal{F}$.

Remark 1. In [3], Brunella introduces the concept of *non-dicritical separatrices*, that is, separatrices whose reduction of singularities does not meet a dicritical component. This concept is different from the one of isolated separatrices as the following example shows. Take the (logarithmic) foliation defined locally by

$$(10) \quad d \left(\frac{(y^2 - x^3)((y - x)^2 - x^3)}{x^4} \right) = 0.$$

It is a foliation of multiplicity 4 with exactly three isolated separatrices which are three (2,3)-cusps having different tangent lines. The first blowing-up is dicritical, hence the irreducible non-dicritical separatrices are exactly the (infinitely many) non-singular ones.

Theorem 2. *Let $\mathcal{F}$ be a singular foliation of $(\mathbb{C}^2, 0)$ and S the curve union of the isolated separatrices. Then $\mu_0(\mathcal{F}) + 1$ is a common upper bound for $r_0(S)$ and $n_0(S)$, where $r_0(S)$ is the number of irreducible components of S and $n_0(S)$ the minimal number of blowing-ups needed to desingularize S .*

In the proof of this theorem it is useful to consider the situation relative to a normal crossings divisor D . Thus, take a pair $(\mathcal{F}, D)$ in $(\mathbb{C}^2, 0)$ and consider the curve S given by the union of the isolated separatrices of $(\mathcal{F}, D)$. Denote by $n_0(\mathcal{F}, D)$ the minimal number of blowing-ups we need to desingularize $(\mathcal{F}, D)$ and $n_0(S, D)$ the minimal number of blowing-ups to desingularize (S, D) (that is, to get normal crossings between S and the total transform of D). Note that $n_0(S, D) \leq n_0(\mathcal{F}, D)$. Let $e_0(D)$ be the number of invariant components of D and $d_0(D)$ the number of dicritical components of D . Put

$$\eta = \begin{cases} 0 & \text{if } d_0(D) = 0, \\ \eta_0(\mathcal{F}, D_1) & \text{if } d_0(D) = 1 \text{ and } D_1 \text{ is the dicritical component,} \\ \eta_0(\mathcal{F}, D_1) + \eta_0(\mathcal{F}, D_2) & \text{if } d_0(D) = 2 \text{ and } D = D_1 \cup D_2. \end{cases}$$

We shall prove that

$$(11) \quad n_0(S, D) \leq \mu_0(\mathcal{F}) + \eta + 1 \quad \text{and} \quad r_0(S) \leq \mu_0(\mathcal{F}) + \eta + 1,$$

and the theorem follows taking $D = \emptyset$.

Let us first prove it under the assumption $\nu_0(\mathcal{F}) = 1$.

Assume that $D = \emptyset$. Let $\omega = a(x, y)dx + b(x, y)dy$ be a 1-form which defines $\mathcal{F}$ and denote by λ_1, λ_2 the eigenvalues of the vector field $X = b\partial/\partial x - a\partial/\partial y$. We have several possibilities:

Simple case: we have either $\lambda_1 \cdot \lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}_{\geq 0}$ or $\lambda_1 \neq 0$ and $\lambda_2 = 0$. Then the origin is a simple singularity and $\mathcal{F}$ has at most two isolated separatrices which are non-singular and transversal. We have $r_0(S) \leq 2$ and $n_0(S, D) = 1$.

Resonant case: $\lambda_1 \cdot \lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \in \mathbb{Q}_{\geq 0}$. If $\lambda_1/\lambda_2 = n \in \mathbb{N}$ (or similarly if $\lambda_2/\lambda_1 \in \mathbb{N}$), by the normal form of Dulac we can write $\omega = ydx - (nx + ay^n)dy$. If $a \neq 0$, we have only one separatrix given by $y = 0$; if $a = 0$, the foliation $\mathcal{F}$ is dicritical and $y = 0$ is the only isolated separatrix. When $\lambda_1/\lambda_2 \notin \mathbb{N} \cup 1/\mathbb{N}$, then $\mathcal{F}$ is linearizable: we can write $\omega = qydx - pxdy$ with $\lambda_1/\lambda_2 = p/q$ and $\mathcal{F}$ has only two isolated separatrices given by $xy = 0$. We have $r_0(S) \leq 2$ and $n_0(S, D) \leq 1$.

Nilpotent case: $\lambda_1 = \lambda_2 = 0$. The linear part of ω is equal to ydy . It is known [17, 12] that ω has a normal form of the type

$$\Omega^{n,p} = d(y^2 + x^n) + x^p U(x)dy$$

where $\mu_0(\mathcal{F}) = n - 1 \geq 2$, $p \geq 2$ and $U(x) \in \mathbb{C}\{x\}$, $U(0) \neq 0$. The number of isolated separatrices of $\mathcal{F}$ is at most 2. If $n < 2p$, then $\mathcal{F}$ has the same reduction of singularities as $d(y^2 + x^n)$ (see [10]). Then $n_0(S, D) \leq (n - 1)/2 + 2$ if n is odd and $n_0(S, D) \leq n/2$ if n is even. If $n \geq 2p$, after the results of [14, 1], we have that $n_0(S, D) \leq p$.

If $D \neq \emptyset$ and all its components are invariant ones, then $r_0(S) \leq 2 - e_0(D)$ and the same bounds as above work for $n_0(S, D)$. If $d_0(D) \geq 1$, it is easy to see that $n_0(S, D) \leq \mu_0(\mathcal{F}) + \eta + 1$ and $r_0(S) \leq 2 - e_0(D)$.

End of the proof of Theorem 2. Put $\mu = \mu_0(\mathcal{F})$ and $\nu = \nu_0(\mathcal{F})$. We prove the result by induction on μ . We consider several cases depending on the number of components of D and their dicriticalness or not. Assume $\mu = 0$ and hence the foliation $\mathcal{F}$ is non-singular. Let us consider all the possible situations:

- (i) $D = \emptyset$: we have that $r_0(S) = 1$ and $n_0(S, D) = 0$.
- (ii) $e_0(D) = 1$ and $d_0(D) = 0$: we have that $r_0(S) = 0$ and $n_0(S, D) = 0$.
- (iii) If $e_0(D) = 0$ and $d_0(D) = 1$, there exists at most one irreducible isolated separatrix S through the origin which is tangent to D . In this case $\eta_0(\mathcal{F}, D) \neq 0$ and $n_0(S, D) = \eta_0(\mathcal{F}, D) + 1$.
- (iv) $e_0(D) = 1$ and $d_0(D) = 1$: we have that $r_0(S) = 0$ and $n_0(S, D) = 0$.
- (v) $d_0(D) = 2$: we have that $r_0(S) = 1$ and $n_0(S, D) = 1 + \eta$ (note that the foliation could be tangent only to one of the components of D).

The case $\mu = 1$ is done, since in this situation $\nu = 1$.

Let us proceed by induction on μ , assume that $\mu \geq 2$ and also $\nu \geq 2$. Consider $\pi_1 : M_1 \rightarrow (\mathbb{C}^2, 0)$ the blowing-up of the origin and denote by $E_1 = \pi_1^{-1}(0)$ and $D' = \pi_1^{-1}(\{0\} \cup D)$. Let $\mathcal{F}_1$ be the strict transform of $\mathcal{F}$ by π_1 and let $Q_1, \dots, Q_t$ be the points in $Sing(\mathcal{F}_1, D')$. Note that $t \leq \nu - 1$ if π_1 is dicritical and $t \leq \nu + 1$ otherwise. Let S_i be the curve of isolated separatrices of the pair $(\mathcal{F}_1, D')$ at Q_i , for $i = 1, \dots, t$. Observe that

$$\begin{aligned} r_0(S) &= r_{Q_1}(S_1) + \dots + r_{Q_t}(S_t), \\ n_0(S, D) &= 1 + n_{Q_1}(S_1, D') + \dots + n_{Q_t}(S_t, D'). \end{aligned}$$

We consider all the situations according to the number of components of D as we have done above.

Assume first that $D = \emptyset$. If π_1 is a dicritical blowing-up, by formula (2), we have that $\mu_{Q_i}(\mathcal{F}_1) < \mu$ for each $i = 1, \dots, t$. Applying the induction hypothesis and equalities (2) and (9), we obtain that $n_0(S, D)$ and $r_0(S)$ are bounded by

$$1 + \sum_{i=1}^t (1 + \mu_{Q_i}(\mathcal{F}_1) + \eta_{Q_i}(\mathcal{F}_1, E_1)) \leq \mu - \nu^2 + \nu \leq \mu + 1.$$

Assume now that π_1 is a non-dicritical blowing-up. Since $\nu \geq 2$, by equation (1), we get that $\mu_{Q_i}(\mathcal{F}_1) < \mu$. By the induction hypothesis and formula (1), we get that $n_0(S, D)$ and $r_0(S)$ are bounded by

$$1 + \sum_{i=1}^t (1 + \mu_{Q_i}(\mathcal{F}_1)) \leq \mu - \nu^2 + 2\nu + 3.$$

Thus the result holds when $-\nu^2 + 2\nu + 3 \leq 1$, or equivalently, if $\nu > 2$. Consider now the case $\nu = 2$, this implies $\mu \geq 4$ and $t \leq 3$. If $t = 3$, then at each point Q_i the vector field defining $\mathcal{F}_1$ has a non-zero eigenvalue (see [13], p. 515). Thus $\nu_{Q_i}(\mathcal{F}_1) = 1$ and each singularity Q_i is either simple or resonant. By the study done for multiplicity one, we have that $r_{Q_i}(S_i) \leq 1$ and $n_{Q_i}(S_i, D') = 0$ and then $r_0(S) \leq 3$ and $n_0(S, D) = 1$. If $t = 2$, one of the points in $Sing(\mathcal{F}_1, E_1)$, say Q_1 , has a non-zero eigenvalue and then $r_{Q_1}(S_1) \leq 1$ and $n_{Q_1}(S_1, D') = 0$. At the other singular point Q_2 , by equation (1), we have that $\mu_{Q_2}(\mathcal{F}_1) < \mu - 1$, then applying the induction hypothesis we have that $r_{Q_2}(S_2), n_{Q_2}(S_2, D') \leq \mu_{Q_2}(\mathcal{F}_1) + 1$. This gives us that $r_0(S), n_0(S, D) \leq \mu + 1$. The last possibility is $t = 1$ and by (1), the only singular point Q_1 of $\mathcal{F}_1$ in E_1 has $\mu_{Q_1}(\mathcal{F}_1) = \mu - 1$, we can apply the induction hypothesis at Q_1 and we deduce the result.

With arguments similar to the ones above we prove the result for the other cases with $d_0(D) = 0$ and $e_0(D) = 1, 2$.

Now we study the cases which correspond to $e_0(D) = 0$ and $d_0(D) = 1, 2$. Denote by $d = d_0(D)$. If π_1 is a dicritical blowing-up, by equation (2), we can apply the induction hypothesis at each point Q_i . The use of equations (2), (8) and (9) gives that $n_0(S, D)$ and $r_0(S)$ are bounded by

$$\begin{aligned} 1 + \sum_{i=1}^t (1 + \mu_{Q_i}(\mathcal{F}_1) + \eta_{Q_i}(\mathcal{F}_1, E_1)) + \eta - d(\nu + 1) \\ \leq \mu + \eta - \nu^2 - (d - 1)\nu - d \leq \mu + \eta + 1. \end{aligned}$$

When π_1 is a non-dicritical blowing-up and $\nu \geq 2$, from the induction hypothesis and equations (1) and (7), we get the bound

$$1 + \sum_{i=1}^t (1 + \mu_{Q_i}(\mathcal{F}_1)) + \eta - d\nu \leq \mu + \eta - \nu^2 + (2 - d)\nu + 3$$

and hence $n_0(S, D), r_0(S) \leq \mu + \eta + 1$ since $\nu \geq 2$.

For the case $e_0(D) = d_0(D) = 1$ the calculations work like in the case $e_0(D) = 0$ and $d_0(D) = 1$. $\square$

Example 3. The bound given in Theorem 2 is attained as it is shown by the following examples. Consider the foliation $\mathcal{F}$ given by $d(y^2 - x^3) = 0$ which has Milnor number $\mu = 2$. The foliation $\mathcal{F}$ has only one isolated separatrix $S = (y^2 - x^3 = 0)$ and $n_0(S) = \mu + 1 = 3$.

Consider now the foliation $\mathcal{G}$ given by $ydy - xdx = 0$. The Milnor number μ of $\mathcal{G}$ is 1. The curve of isolated separatrices is $S = (y^2 - x^2 = 0)$, then $r_0(S) = \mu + 1 = 2$.

Given a plane curve S in $(\mathbb{C}^2, 0)$ and the minimal reduction of singularities $\pi : M \rightarrow (\mathbb{C}^2, 0)$ of S , we construct as usual the dual graph $G(S)$ in the following way. We associate a vertex to each irreducible component D of $E = \pi^{-1}(0)$ with a weight equal to the self-intersection of $D \subset M$. Two vertices are joined by an edge if and only if their associated divisors meet. Each irreducible component B of S is represented by an arrow joined to the only component D of E which meets the strict transform B' of B by π . This weighted graph $G(S)$ is equivalent to the data of the equisingularity type of the curve S (see [2]).

Corollary 4. *Let $\mathcal{F}$ be a singular foliation of $(\mathbb{C}^2, 0)$ and let S be the curve of isolated separatrices of $\mathcal{F}$. Then the number of possible equisingularity types for S is bounded by a function of $\mu = \mu_0(\mathcal{F})$. In particular, the multiplicities $\nu_q(S)$ of S at each infinitely near point q of S are bounded by a function of μ .*

Proof. By Theorem 2, the number $r_0(S)$ of isolated separatrices of $\mathcal{F}$ and the minimal number $n_0(S)$ of blowing-ups to desingularize the curve S are bounded by $\mu + 1$. Then there exists a finite number of possible configurations for the weighted dual graph $G(S)$ of the minimal reduction of singularities of S . Each of these graphs determines one equisingularity type for the curve S . $\square$

Proposition 5. *There is a function $K : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ such that $-K(\mu)$ is a lower bound for the index $Z_0(\mathcal{F}, S)$, where $\mu = \mu_0(\mathcal{F})$.*

Proof. Consider the minimal reduction of singularities of S ,

$$M_n \xrightarrow{\pi_n} M_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} M_1 \xrightarrow{\pi_1} (\mathbb{C}^2, 0).$$

Denote by $\mathcal{F}_i$, S_i the strict transforms by $\pi_1 \circ \cdots \circ \pi_i$ of $\mathcal{F}$ and S . Let $E_i = (\pi_1 \circ \cdots \circ \pi_i)^{-1}(0)$ and take q any point in $S_n \cap E_n$. Since S_n is a nonsingular curve at q , we have that

$$Z_q(\mathcal{F}_n, S_n) \leq \mu_q(\mathcal{F}_n).$$

Using the formulas in Section 2 and the fact that $\nu_{p_i}(\mathcal{F}_i) \leq \nu_0(\mathcal{F}) + 1$ for each $p_i \in E_i$ (see [7]), we obtain that the Milnor number $\mu_q(\mathcal{F}_n)$ is bounded by $\mu_0(\mathcal{F})$ plus a constant which depends on the equisingularity type of S at 0. By Corollary 4, we have only a finite number of possible equisingularity types for S and hence for the multiplicities $\nu_{p_i}(S_i)$, depending on μ . Then, the index $Z_0(\mathcal{F}, S)$ can be calculated by blowing-down using the formulas (3), (4) and it can only take a finite number of values. $\square$

Remark 6. Following referee's comments, one can get a lower bound for the local indices just invoking the fact that they bound the "virtual local Euler characteristic". In fact, this last invariant depends only on the equisingularity type.

Remark 7. In Brunella's argument to bound the degree of a non-dicritical separatrix B it is used that the indices $Z_p(\mathcal{F}, B)$ are non-negative. One could ask if this result is still true for the case of isolated separatrices (referee's question). Actually, this is not true, since for the example in Remark 1, we have $Z_p(\mathcal{F}, S) = -6$, where S is the union of the three isolated separatrices.

4. THE DEGREE OF AN ISOLATED INVARIANT CURVE

Let $\mathcal{F}$ be a foliation of $\mathbb{P}_{\mathbb{C}}^2$ and let C be an invariant curve of $\mathcal{F}$. We say that C is an *isolated invariant curve* of $\mathcal{F}$ if at each singularity $p \in \mathbb{P}_{\mathbb{C}}^2$ of $\mathcal{F}$, the germ of C at p is a union of isolated separatrices of $\mathcal{F}$.

Theorem 8. *There exists a function $G : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ such that $G(d)$ is an upper bound for the degree of any isolated algebraic invariant curve C of a degree d foliation $\mathcal{F}$ of $\mathbb{P}_{\mathbb{C}}^2$.*

Proof. Consider a degree d foliation $\mathcal{F}$ of $\mathbb{P}_{\mathbb{C}}^2$ and let C be a degree k isolated algebraic invariant curve of $\mathcal{F}$. The foliation $\mathcal{F}$ has at most $d^2 + d + 1$ singular points, more precisely, the sum of the Milnor numbers at the singular points is exactly $d^2 + d + 1$. Thus there is only finitely many possible singular points and the possible Milnor numbers at them are also a finite number, depending on the degree d . Recall that the index $Z(\mathcal{F}, C)$ is the sum of the local indices $Z_p(\mathcal{F}, C)$ at the singular points $p \in \mathbb{P}_{\mathbb{C}}^2$ of $\mathcal{F}$. Hence, in view of Proposition 5, we get a negative lower bound $-K_g(d)$ for the global index $Z(\mathcal{F}, C)$ depending on d . By the formula (6), we have that

$$-K_g(d) \leq Z(\mathcal{F}, C) = (d + 2 - k)k.$$

We get that $G(d) = d + 2 + K_g(d) \geq k$. □

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