DYNAMICAL SYSTEMS METHOD (DSM) FOR UNBOUNDED OPERATORS

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ABSTRACT. Let L be an unbounded linear operator in a real Hilbert space H , a generator of a C_0 semigroup, and let $g : H \rightarrow H$ be a C_{loc}^2 nonlinear map. The DSM (dynamical systems method) for solving equation $F(v) := Lv + g(v) = 0$ consists of solving the Cauchy problem $\dot{u} = \Phi(t, u)$, $u(0) = u_0$, where Φ is a suitable operator, and proving that i) $\exists u(t) \quad \forall t > 0$, ii) $\exists u(\infty)$, and iii) $F(u(\infty)) = 0$.

Conditions on L and g are given which allow one to choose Φ such that i), ii), and iii) hold.

1. INTRODUCTION

Let H be a real Hilbert space, let L be a linear, densely defined in H , closed operator, a generator of a C_0 semigroup (see [2]), and let $g : H \rightarrow H$ be a nonlinear C_{loc}^2 map, i.e.,

$$(1.1) \quad \sup_{u \in B(u_0, R)} \|g^{(j)}(u)\| \leq M_j(R), \quad j = 0, 1, 2, \quad B(u_0, R) := \{u : \|u - u_0\| \leq R\},$$

where $g^{(j)}$ is the Fréchet derivative of order j , $R > 0$ is some number, $M_j(R)$ are some positive constants, and $u_0 \in H$ is some element. In many applications the problems can be formulated as the following operator equation:

$$(1.2) \quad F(v) := Lv + g(v) = 0.$$

We want to study this equation by the dynamical systems method (DSM), which also allows one to develop numerical methods for solving equation (1.2). The DSM for solving equation (1.2) consists of solving the problem

$$(1.3) \quad \dot{u} = \Phi(t, u), \quad u(0) = u_0,$$

where $\dot{u} := \frac{du}{dt}$, and $\Phi(t, u)$ is a nonlinear operator chosen so that problem (1.3) has a unique global solution which stabilizes at infinity to the solution of equation (1.2):

$$(1.4) \quad \text{i) } \exists u(t) \forall t > 0, \quad \text{ii) } \exists u(\infty), \quad \text{iii) } F(u(\infty)) = 0.$$

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In [2] the DSM has been studied and justified for $F \in C_{loc}^2$ and

$$(1.5) \quad \sup_{u \in B(u_0, R)} \|[F'(u)]^{-1}\| \leq m_1;$$

for monotone $F \in C_{loc}^2$; for monotone hemicontinuous defined on all of H operators F ; for non-monotone $F \in C_{loc}^2$ such that there exists a y such that $F(y) = 0$ and the operator $A := F'(y)$ maps any ball $B(0, r)$ centered at the origin and of sufficiently small radius $r > 0$ into a set which has a nonempty intersection with $B(0, R)$; and for $F \in C_{loc}^2$ satisfying a spectral condition: $\|(F'(u) + \varepsilon)^{-1}\| \leq (c\varepsilon)^{-1}$, $0 < c \leq 1$, in which case F is replaced by $F + \varepsilon I$ in (1.3), and then ε is taken to zero.

In this paper the DSM is justified for a class of nonlinear unbounded operators of the type $L + g$, where L is a generator of a C_0 semigroup, $g \in C_{loc}^2$, and some suitable additional assumptions are made.

Which assumptions are suitable? A simple example is

$$(1.6) \quad \|L^{-1}\| \leq m.$$

If (1.6) holds, then (1.2) is equivalent to

$$(1.7) \quad f(u) := u + L^{-1}g(u) = 0, \quad f \in C_{loc}^2.$$

Assume

$$(1.8) \quad \sup_{u \in B(u_0, R)} \|[I + L^{-1}g'(u)]^{-1}\| \leq m_1.$$

This assumption, holds, e.g., if $L^{-1}g'(u)$ is a compact operator in H for any $u \in B(u_0, R)$ and the operator $I + L^{-1}g'(u)$ is injective.

Our first result is the following theorem.

Theorem 1.1. *Assume (1.6), (1.8), and let $\Phi := -[I + L^{-1}g'(u)]^{-1}[u + L^{-1}g(u)]$. If*

$$(1.9) \quad \|u_0 + L^{-1}g(u_0)\| m_1 \leq R,$$

then equation (1.2) has a unique solution $v \in B(u_0, R)$, the conclusions i), ii), and iii) hold for (1.3), and $u(\infty) = v$.

Remark 1.1. If L is not boundedly invertible, i.e., (1.6) fails, then one can use the following assumption (A).

Assumption A. *There exists a sector $S = \{z : 0 < |z| \leq a, |\arg z - \pi| \leq \delta\}$, which consists of regular points of L . Here $a > 0$ and $\delta > 0$ are arbitrary small fixed numbers.*

If Assumption (A) holds, then

$$(1.10) \quad \|(L + \varepsilon)^{-1}\| \leq \frac{1}{\varepsilon \sin(\delta)}.$$

This estimate holds in particular if $L = L^* \geq 0$, and in this case $\sin(\delta) = 1$.

The following theorem is our next result:

Theorem 1.2. *Assume that $L^* = L \geq 0$ is a densely defined linear operator, $g \in C_{loc}^2$, $g'(u) \geq 0 \forall u \in H$, equation (1.2) is solvable and v is its (unique) minimal-norm solution. Define $\Phi(u) = -[I + (L + \varepsilon)^{-1}g'(u)]^{-1}[u + (L + \varepsilon)^{-1}g(u)]$, $\varepsilon = \text{const} > 0$. Assume that (1.8) holds with $L_\varepsilon := L + \varepsilon I$ replacing L , and $m_1 = m_1(\varepsilon) > 0$. Then problem (1.3) has a unique global solution $u_\varepsilon(t)$, there*

exists $u_\varepsilon(\infty) := v_\varepsilon$, and $F(v_\varepsilon) + \varepsilon v_\varepsilon := Lv_\varepsilon + g(v_\varepsilon) + \varepsilon v_\varepsilon = 0$. Moreover, there exists the limit $\lim_{\varepsilon \rightarrow 0} v_\varepsilon = v$, which is the unique minimal-norm solution to (1.2).

In Section 2 proofs are given.

2. PROOFS

Proof of Theorem 1.1. If $\Phi = -[I + L^{-1}g'(u)]^{-1}[u + L^{-1}g(u)]$ and

$$p(t) := \|u + L^{-1}g(u)\|,$$

then $p\dot{p} = -p^2$. Thus

$$(2.1) \quad p(t) = p(0)e^{-t}.$$

From (2.1), (1.8) and (1.3) one gets

$$(2.2) \quad \|\dot{u}\| \leq m_1 p(0)e^{-t}, \quad p(0) = \|u_0 + L^{-1}g(u_0)\|.$$

Inequality (2.2) implies the global existence of $u(t)$, the existence of $u(\infty) := \lim_{t \rightarrow \infty} u(t)$, and the estimates

$$(2.3) \quad \|u(t) - u(\infty)\| \leq m_1 p(0)e^{-t}, \quad \|u(t) - u_0\| \leq m_1 p(0).$$

If (1.9) holds, then (2.3) implies $\|u(t) - u_0\| \leq R$, so the trajectory $u(t)$ stays in the ball $B(u_0, R)$, that is, $u(t) \in B(u_0, R) \forall t \geq 0$. Passing to the limit $t \rightarrow \infty$ in equation (1.3) yields

$$(2.4) \quad 0 = -[I + L^{-1}g'(u(\infty))]^{-1}[u(\infty) + L^{-1}g(u(\infty))].$$

Thus $u(\infty) := v$ solves the equation $v + L^{-1}g(v) = 0$, so v solves (1.2), and therefore i), ii) and iii) hold. Theorem 1.1 is proved. $\square$

Proof of Theorem 1.2. If $L^* = L \geq 0$ in H and $g'(u) \geq 0$, then, for any $\varepsilon > 0$, Theorem 1.1 yields the existence of a unique solution v_ε to equation (1.2) with L replaced by $L + \varepsilon I$. This solution $v_\varepsilon = u_\varepsilon(\infty)$, where $u_\varepsilon(t)$ is the solution to (1.3) with

$$\Phi = -[I + (L + \varepsilon)^{-1}g'(u)]^{-1}[u + (L + \varepsilon)^{-1}g(u)].$$

Let us prove that $\lim_{\varepsilon \rightarrow 0} v_\varepsilon = v$, where v solves (1.2). We do not assume that (1.2) has a unique solution.

Let $v_\varepsilon - v := w$. Then $Lw + \varepsilon v_\varepsilon + g(v_\varepsilon) - g(v) = 0$, so, using the assumptions $L \geq 0$ and $g'(u) \geq 0$, one gets $\varepsilon(v_\varepsilon, v_\varepsilon - v) \leq 0$, $\|v_\varepsilon\|^2 \leq \|v_\varepsilon\|\|v\|$, and $\|v_\varepsilon\| \leq \|v\|$, $\forall \varepsilon > 0$. Thus $v_\varepsilon \rightharpoonup v_0$ and $Lv_\varepsilon + g(v_\varepsilon) \rightarrow 0$, where $\rightharpoonup$ stands for the weak convergence in H and the convergent subsequence is again denoted v_ε .

In the above argument the element v can be an arbitrary element in the set $N_F := \{v : Lv + g(v) = 0\}$. Thus, we have proved that $\|v_0\| \leq \|v\|$ for all $v \in N_F$.

Let us prove that $L(v_0) + g(v_0) = 0$, i.e., $v_0 \in N_F$.

Assume first that $v_0 \in D(L)$. We prove this assumption later. The monotonicity of $L + g$ yields

$$(2.5) \quad (L(v_\varepsilon) + g(v_\varepsilon) + \varepsilon v_\varepsilon - L(v_0 - tz) - g(v_0 - tz) - \varepsilon(v_0 - tz), v_\varepsilon - v_0 + tz) \geq 0,$$

where $t > 0$, and $z \in D(L)$ is arbitrary. Let $\varepsilon \rightarrow 0$ in (2.5). Then, using $v_\varepsilon \rightharpoonup v_0$ and $Lv_\varepsilon + g(v_\varepsilon) \rightarrow 0$, one gets

$$(2.6) \quad (-L(v_0 - tz) - g(v_0 - tz), z) \geq 0 \quad \forall z \in D(L).$$

Let $t \rightarrow 0$ in (2.6). Then $(Lv_0 + g(v_0), z) \leq 0 \quad \forall z \in D(L)$. Since $D(L)$ is dense in H , it follows that $Lv_0 + g(v_0) = 0$, so $v_0 \in N_F$.

We have proved above that $\|v_0\| \leq \|v\|$ for any $v \in N_F$. Because N_F is a closed and convex set, as we prove below, and H is a uniformly convex space, there is a unique element $v \in N_F$ with minimal norm. Therefore, it follows that $v_0 = v$, where v is the minimal-norm solution to (1.2) and v_0 is the weak limit of v_ε .

Let us prove the strong convergence $v_\varepsilon \rightarrow v$.

We know that $v_\varepsilon \rightharpoonup v$. The inequality $\|v_\varepsilon\| \leq \|v\|$ implies

$$\|v\| \leq \liminf_{\varepsilon \rightarrow 0} \|v_\varepsilon\| \leq \limsup_{\varepsilon \rightarrow 0} \|v_\varepsilon\| \leq \|v\|.$$

Therefore $\lim_{\varepsilon \rightarrow 0} \|v_\varepsilon\| = \|v\|$. Consequently one gets

$$\lim_{\varepsilon \rightarrow 0} \|v_\varepsilon - v\|^2 = \lim_{\varepsilon \rightarrow 0} [\|v_\varepsilon\|^2 + \|v\|^2 - 2\Re(v_\varepsilon, v)] \leq 2[\|v\|^2 - (v, v)] = 0.$$

Thus, $v_\varepsilon \rightarrow v$, as claimed.

Since g is continuous, it follows that $g(v_\varepsilon) \rightarrow g(v)$.

The equation

$$(2.7) \quad Lv_\varepsilon + \varepsilon v_\varepsilon + g(v_\varepsilon) = 0,$$

the inequality $\|v_\varepsilon\| \leq \|v\|$, and the relation $g(v_\varepsilon) \rightarrow g(v)$ imply $Lv_\varepsilon \rightarrow \eta := -g(v)$.

Let us prove that $v_0 \in D(L)$.

Because $v_0 = v$, it is sufficient to check that $v \in D(L)$. Since L is selfadjoint and $Lv_\varepsilon \rightarrow \eta$, one has

$$(\eta, \psi) = \lim_{\varepsilon \rightarrow 0} (Lv_\varepsilon, \psi) = \lim_{\varepsilon \rightarrow 0} (v_\varepsilon, L\psi) = (v, L\psi) \quad \forall \psi \in D(L).$$

Thus $v \in D(L)$ and $Lv = \eta$.

Let us finally check that N_F is closed and convex.

Assume $F(v_n) := Lv_n + g(v_n) = 0$, $v_n \rightarrow v$. Then $g(v_n) \rightarrow g(v)$ because g is continuous. Thus, $Lv_n \rightarrow \eta := -g(v)$. Since L is closed, the relations $v_n \rightarrow v$ and $Lv_n \rightarrow \eta$ imply $Lv = \eta = -g(v)$. So, $v \in N_F$, and consequently N_F is closed.

Assume that $0 < s < 1$, $v, w \in N_F$ and $\psi = sv + (1-s)w$. Let us show that $\psi \in N_F$. One has $w \in N_F$ if and only if

$$(2.8) \quad (F(z), z - w) \geq 0 \quad \forall z \in D(L).$$

Indeed, if $F(w) = 0$, then $F(z) = F(z) - F(w)$, and the above equation holds because F is a monotone operator. Conversely, if this equation holds, then take $z - w = t\eta$, $t = \text{const} > 0$, $\eta \in D(L)$ is arbitrary, and get $(F(w + t\eta), \eta) \geq 0$. Let $t \rightarrow 0$; then $(F(w), \eta) \geq 0 \quad \forall \eta \in D(L)$. Thus, since $\overline{D(L)} = H$, one gets $w \in N_F$. If $\psi = sv + (1-s)w$, and $v, w \in N_F$, then $(F(z), z - sv - (1-s)w) = s(F(z), z - v) + (1-s)(F(z), z - w) \geq 0$. Thus, N_F is convex.

Theorem 1.2 is proved. $\square$

Remark 2.1. Theorem 1.2 gives a theoretical framework for a study of nonlinear ill-posed operator equations $F(u) = f$, with $F(u) = Lu + g(u)$, where the operator $F'(u)$ is not boundedly invertible. In this short paper, we do not discuss the case when the data f are given with some error: f_δ is given in place of f , $\|f_\delta - f\| \leq \delta$. In this case one can use DSM for a stable solution of the equation $F(v) = f$ with noisy data f_δ , and one uses an algorithm for choosing stopping time t_δ such that $\lim_{\delta \rightarrow 0} \|u_\delta(t_\delta) - v\| = 0$, where $u_\delta(t)$ is the solution to (1.3) with $\Phi = \Phi_\delta$ chosen suitably (see [3]).

Remark 2.2. Under the assumptions of Theorem 1.1 one of the many possibilities to choose Φ in (1.3) so that the DSM can be justified, that is, (1.4) is valid, is the choice made in this theorem, the choice which can be interpreted as a continuous analog of the Newton's method. The classical Newton's method for solving an equation $G(u) = 0$ consists in calculating the sequence $u_{n+1} = u_n - [G'(u_n)]^{-1}G(u_n)$, $u_0 = u_0$, where the initial approximation u_0 is given, and one assumes that $\|[G'(u)]^{-1}\| \leq m_1$, which is an analog of condition (1.5). Under certain assumptions (see, e.g., [1]), one can prove that the limit $\lim_{n \rightarrow \infty} u_n = u$ solves the equation $G(u) = 0$. If one takes $G(u) := u + L^{-1}g(u)$, then $G'(u)w = [I + L^{-1}g'(u)]w$, and problem (1.3) with $\Phi = -[G'(u)]^{-1}G(u)$ is a continuous analog of the Newton's method. The assumption (1.9) can be written as $\|G(u_0)\|_{m_1} \leq R$. By Theorem 1.1, this assumption together with the assumption $\sup_{u \in B(u_0, R)} \|G^{(j)}(u)\| \leq M_j$, $j = 0, 1, 2$, $B(u_0, R) := \{u : \|u - u_0\| \leq R\}$, imply the existence of the unique solution to problem (1.3) with $\Phi = -[G'(u)]^{-1}G(u)$ and the validity of conclusions (1.4) for this solution. This means that the DSM is justified for the solution to the equation $G(u) = 0$ under the above assumptions, and the DSM can be viewed for this particular choice of Φ as a continuous analog of Newton's method. The basic condition $\|G(u_0)\|_{m_1} \leq R$ for the existence of a solution to equation $G(u) = 0$ is very simple, simpler than the classical conditions for the convergence of the discrete Newton's method (cf., e.g., [1]). Some numerical aspects of using DSM are discussed in [4]. Theoretical aspects of the DSM are discussed in [3] and [5].

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