CUNTZ-PIMNSER ALGEBRAS, COMPLETELY POSITIVE MAPS AND MORITA EQUIVALENCE

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ABSTRACT. Let P be a completely positive map on $M_n(\mathbb{C})$ and let E_P be the associated GNS- C^* -correspondence. We prove a result that implies, in particular, that the Cuntz-Pimsner algebra of E_P , $\mathcal{O}(E_P)$, is strongly Morita equivalent to the Cuntz algebra $\mathcal{O}_{d(P)}$, where $d(P)$ is the index of P .

By definition $E_P := M_n(\mathbb{C}) \otimes_P M_n(\mathbb{C})$ is the algebraic tensor product of $M_n(\mathbb{C})$ with $M_n(\mathbb{C})$ endowed with the $M_n(\mathbb{C})$ -valued inner product: $\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle = b_1^* P(a_1^* a_2) b_2$, $a_i \otimes b_i \in E_P$, $i = 1, 2$, and obvious left and right actions of $M_n(\mathbb{C})$ by matrix multiplication. Thus E_P is a C^* -correspondence over $M_n(\mathbb{C})$ in the sense of [2]. The index of P , $d(P)$, is defined in [4, Definition 4.9]. However, thanks to Kraus's formulation of Stinespring's theorem, we may write P in the form $P(a) = \sum_{r=1}^N t_r a t_r^*$, $a \in M_n(\mathbb{C})$, where $t_1, t_2, \dots, t_N$ are in $M_n(\mathbb{C})$. Then Proposition 4.11 of [4] asserts that $d(P)$ is the complex dimension of the linear span of the t_i 's. The principal result of this note is

Theorem. *The correspondence E_P is strongly Morita equivalent, in the sense of [3], to the Hilbert space $\mathbb{C}^{d(P)}$.*

Proof. Since $M_n(\mathbb{C})$ is strongly Morita equivalent to $\mathbb{C}$ via the column space $C_n(\mathbb{C})$, the correspondence E_P is strongly Morita equivalent to the correspondence over $\mathbb{C}$, $F_P := R_n(\mathbb{C}) \otimes_{M_n(\mathbb{C})} E_P \otimes_{M_n(\mathbb{C})} C_n(\mathbb{C})$, where $R_n(\mathbb{C})$ is the n -dimensional row space over $\mathbb{C}$. (See [1] for discussions of column and row spaces.) Thus F_P is a finite-dimensional Hilbert space over $\mathbb{C}$ and we need to calculate its dimension. Lemma 4.10 of [4] allows us to write $d(P)$ as $\dim_{\mathbb{C}}(e_{11} M_n(\mathbb{C}) \otimes_P M_n(\mathbb{C}) e_{11})$, where $\{e_{ij}\}$ are the matrix units in $M_n(\mathbb{C})$. Since the tensor products in the definition of F_P as $R_n(\mathbb{C}) \otimes_{M_n(\mathbb{C})} E_P \otimes_{M_n(\mathbb{C})} C_n(\mathbb{C})$ are balanced over $M_n(\mathbb{C})$, the space F_P is spanned by the elements $\varepsilon_{ij} := e_j \otimes I \otimes_P I \otimes e_i^*$, where $e_j = (0, \dots, 1, \dots, 0)$, with 1 in the j^{th} slot, and $i, j = 1, \dots, n$. On the other hand, the space $(e_{11} M_n(\mathbb{C}) \otimes_P M_n(\mathbb{C}) e_{11})$ is spanned by the matrices $e_{1i} \otimes_P e_{j1}$, $i, j = 1, \dots, n$. Consequently, to show that the

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two spaces F_P and $(e_{11}M_n(\mathbb{C}) \otimes_P M_n(\mathbb{C})e_{11})$ are isomorphic and so have the same dimension, it suffices to show that the inner products $\langle e_{1i} \otimes_P e_{j1}, e_{1k} \otimes_P e_{l1} \rangle$ and $\langle \varepsilon_{ij}, \varepsilon_{kl} \rangle$ are the same. To do this, we write the matrices t_r in Kraus's representation of P as $t_r = [t(r; i, j)]_{i,j=1}^n$, $r = 1, \dots, N$. Then, in F_P , we have

$$\begin{aligned} \langle \varepsilon_{ij}, \varepsilon_{kl} \rangle &= \langle e_i \otimes I \otimes_P I \otimes e_j^*, e_k \otimes I \otimes_P I \otimes e_l^* \rangle \\ &= e_j P(e_i^* e_k) e_l^* = \sum_{r=1}^N t(r; i, j) \overline{t(r; k, l)}. \end{aligned}$$

On the other hand, by [4, Lemma 4.10], the inner product in $(e_{11}M_n(\mathbb{C}) \otimes_P M_n(\mathbb{C})e_{11})$ is given by the formula $\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle = \text{tr}(b_1^* P(a_1^* a_2) b_2)$. So, we find that

$$\langle e_{1i} \otimes e_{j1}, e_{1k} \otimes e_{l1} \rangle = \text{tr}(e_{j1}^* P(e_{1i}^* e_{1k}) e_{l1}) = \sum_{r=1}^N t(r; i, j) \overline{t(r; k, l)},$$

also. □

- Corollary.** (1) *The tensor algebra over E_P in the sense of [2], $\mathcal{T}_+(E_P)$, is strongly Morita equivalent, in the sense of [1] to Popescu's noncommutative disc algebra [5] $\mathcal{A}_{d(P)}$ of the size $d(P)$.*
- (2) *The Toeplitz C^* -algebra of E_P is strongly Morita equivalent to the Toeplitz extension of Cuntz algebra $\mathcal{O}_{d(P)}$.*
- (3) *The Cuntz-Pimsner algebra $\mathcal{O}(E_P)$ is strongly Morita equivalent to the Cuntz algebra $\mathcal{O}_{d(P)}$.*

Proof. The first two assertions are immediate from the Theorem and [3, Theorem 3.2], once it is noted that the tensor algebra of $\mathbb{C}^{d(P)}$ is $\mathcal{A}_{d(P)}$. (See [2].) The third assertion follows from the Theorem and [3, Theorem 3.5]. □

Example. Let $P : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be the completely positive map that takes a matrix to its diagonal part, i.e., suppose that $P(a) = \sum_i e_{ii} a e_{ii}$, $a \in M_n(\mathbb{C})$. Evidently, the linear span of the e_{ii} is n -dimensional and so we conclude that $\mathcal{O}(E_P)$ is strongly Morita equivalent to $\mathcal{O}_n$.

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