HARDY'S INEQUALITY ON EXTERIOR DOMAINS

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ABSTRACT. Let Ω be a smooth exterior domain in $\mathbb{R}^N$ and $1 < p < \infty$. We prove that when $p \neq N$, Hardy's L^p inequality is valid on $\mathcal{D}_0^{1,p}(\Omega)$.

1. INTRODUCTION

We assume that

(H) Ω is an unbounded domain of $\mathbb{R}^N$ with a compact boundary of class $\mathcal{C}^2$.

We denote by $\delta(x)$ the function

$$\delta(x) = \delta_\Omega(x) = \text{dist}(x, \partial\Omega).$$

By definition $\mathcal{D}_0^{1,p}(\Omega)$ is the closure of the space of test functions $\mathcal{D}(\Omega)$ with respect to the norm $\|u\| = \|\nabla u\|_p$. Hardy's inequality on $\mathcal{D}_0^{1,p}(\Omega)$,

$$\int_\Omega \left| \frac{u}{\delta} \right|^p dx \leq c_p(\Omega) \int_\Omega |\nabla u|^p dx,$$

holds if and only if

$$0 < \mu_p(\Omega) = \inf \left\{ \int_\Omega |\nabla u|^p dx : u \in \mathcal{D}_0^{1,p}(\Omega), \int_\Omega \left| \frac{u}{\delta} \right|^p dx = 1 \right\}.$$

In this case, the value $\mu_p(\Omega)^{-1}$ is the best constant $c_p(\Omega)$ in the Hardy's inequality. Moreover the existence of a minimizer for $\mu_p(\Omega)$ implies that $\mu_p(\Omega) > 0$.

It was proved by F. Colin in [2] that if $N \geq 3$ and $\mu_2(\Omega) < 1/4$, then $\mu_2(\Omega)$ is achieved. We consider the general case $N \geq 2$ and $1 < p < \infty$.

By inequality (4.1) in [3],

$$\mu_p(\Omega) \leq c_{p,N} = \min \left\{ c_p, \left| \frac{p-N}{p} \right|^p \right\},$$

where $c_p = \left(\frac{p-1}{p} \right)^p$. Our main result is

Theorem 1. *Under assumption (H), let $1 < p < \infty$. If $\mu_p(\Omega) < c_{p,N}$, then $\mu_p(\Omega)$ is achieved. In particular $\mu_p(\Omega) > 0$ if and only if $p \neq N$.*

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Remark. We conjecture that, if $\mu_p(\Omega)$ is achieved, then $\mu_p(\Omega) < c_{p,N}$. It is proved in [3] that, for a bounded domain of class $\mathcal{C}^2$, $\mu_p(\Omega)$ is achieved if and only if $\mu_p(\Omega) < c_p$.

2. PROOF OF THEOREM 1

Let $(u_n) \subset \mathcal{D}_0^{1,p}(\Omega)$ be a minimizing sequence:

$$\int_{\Omega} |\nabla u_n|^p dx \rightarrow \mu_p(\Omega), \quad \int_{\Omega} \left| \frac{u_n}{\delta} \right|^p dx = 1.$$

By going if necessary to a subsequence, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u, & \text{in } \mathcal{D}_0^{1,p}(\Omega), \\ u_n &\rightarrow u, & \text{in } L_{\text{loc}}^p(\Omega), \\ u_n &\rightarrow u, & \text{a.e. on } \Omega. \end{aligned}$$

For $0 < \beta \ll 1$, we define

$$\Omega_{\beta} = \{x \in \Omega : \delta(x) < \beta \text{ or } |x| > \beta^{-1}\}.$$

The following quantities, defined by F. Colin in [2], when $p = 2$, take into account simultaneously concentration at boundary and concentration at infinity:

$$\mu_0 = \lim_{\beta \rightarrow 0^+} \overline{\lim}_{n \rightarrow \infty} \left(\int_{\Omega_{\beta}} |\nabla u_n|^p dx \right), \quad \nu_0 = \lim_{\beta \rightarrow 0^+} \overline{\lim}_{n \rightarrow \infty} \left(\int_{\Omega_{\beta}} \left| \frac{u_n}{\delta} \right|^p dx \right).$$

Lemma 1. *Under the above assumptions:*

- a) $\int_{\Omega} \left| \frac{u}{\delta} \right|^p dx + \nu_0 = 1,$
- b) $\int_{\Omega} |\nabla u|^p dx + \mu_0 \leq \mu_p(\Omega),$
- c) $c_{p,N} \nu_0 \leq \mu_0.$

Proof. a) Since

$$\int_{\Omega_{\beta}} \left| \frac{u_n}{\delta} \right|^p dx + \int_{\Omega \setminus \Omega_{\beta}} \left| \frac{u_n}{\delta} \right|^p dx = 1,$$

we have

$$\lim_{n \rightarrow \infty} \int_{\Omega_{\beta}} \left| \frac{u_n}{\delta} \right|^p dx + \int_{\Omega \setminus \Omega_{\beta}} \left| \frac{u}{\delta} \right|^p dx = 1,$$

and taking the limit as $\beta \rightarrow 0^+$, we obtain a).

b) Since

$$\int_{\Omega_{\beta}} |\nabla u_n|^p dx + \int_{\Omega \setminus \Omega_{\beta}} |\nabla u_n|^p dx = \mu_p(\Omega) + o(1),$$

we have

$$\overline{\lim}_{n \rightarrow \infty} \int_{\Omega_{\beta}} |\nabla u_n|^p dx + \underline{\lim}_{n \rightarrow \infty} \int_{\Omega \setminus \Omega_{\beta}} |\nabla u_n|^p dx \leq \mu_p(\Omega).$$

Hence, observing that $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\Omega \setminus \Omega_{\beta})$, we obtain

$$\overline{\lim}_{n \rightarrow \infty} \int_{\Omega_{\beta}} |\nabla u_n|^p dx + \int_{\Omega \setminus \Omega_{\beta}} |\nabla u|^p dx \leq \mu_p(\Omega).$$

Taking the limit as $\beta \rightarrow 0^+$, we obtain b).

c) Let $0 < \varepsilon < \frac{p-1}{p}$. There exists $\beta_0 > 0$ such that $\mathbb{R}^N \setminus \Omega \subset B(0, \beta_0^{-1})$,

$$|x| \geq \beta_0^{-1} \Rightarrow (1 - \varepsilon) \leq \frac{\delta(x)}{|x|},$$

and, for every $v \in \mathcal{D}_0^{1,p}(\Omega)$, $0 < \beta < \beta_0$,

$$\left[\frac{p-1}{p} - \varepsilon \right]^p \int_{\Omega'_\beta} \left| \frac{v}{\delta} \right|^p dx \leq \int_{\Omega'_\beta} |\nabla v|^p dx,$$

where $\Omega'_\beta = \{x \in \Omega : \delta(x) < \beta\}$ (see [1], [4]).

Let $\psi_\beta \in \mathcal{C}^\infty(\Omega)$ be such that $0 \leq \psi_\beta \leq 1$, $\psi_\beta = 1$ on $\Omega_{\beta/2}$ and $\psi_\beta = 0$ on $\Omega \setminus \Omega_\beta$. Since for every $w \in \mathcal{D}(\mathbb{R}^N \setminus \{0\})$

$$\left| \frac{p-N}{p} \right|^p \int_{\mathbb{R}^N} \frac{|w(x)|^p}{|x|^p} dx \leq \int_{\mathbb{R}^N} |\nabla w(x)|^p dx,$$

we have, for every $v \in \mathcal{D}_0^{1,p}(\Omega)$,

$$(1) \quad c(p, N, \varepsilon) \int_{\Omega} \left| \frac{\psi_\beta v}{\delta} \right|^p dx \leq \int_{\Omega} |\nabla(\psi_\beta v)|^p dx,$$

where

$$c(p, N, \varepsilon) = \min \left\{ (1 - \varepsilon)^p \left| \frac{p-N}{p} \right|^p, \left[\frac{p-1}{p} - \varepsilon \right]^p \right\}.$$

By using the inequality

$$||a + b|^p - |a|^p| \leq \eta |a|^p + c(\eta, p) |b|^p,$$

it is easy to verify, as in [5], that

$$(2) \quad \mu_0 = \lim_{\beta \rightarrow 0^+} \overline{\lim}_{n \rightarrow \infty} \int_{\Omega_\beta} |\nabla(u_n - u)|^p dx,$$

$$(3) \quad \nu_0 = \lim_{\beta \rightarrow 0^+} \overline{\lim}_{n \rightarrow \infty} \int_{\Omega_\beta} \left| \frac{u_n - u}{\delta} \right|^p dx,$$

$$(4) \quad \lim_{\beta \rightarrow 0^+} \overline{\lim}_{n \rightarrow \infty} \int_{\Omega} |\psi_\beta \nabla(u_n - u)|^p dx = \lim_{\beta \rightarrow 0^+} \overline{\lim}_{n \rightarrow \infty} \int_{\Omega} |\nabla[\psi_\beta(u_n - u)]|^p dx.$$

Since by (2) and (3)

$$\mu_0 = \lim_{\beta \rightarrow 0^+} \overline{\lim}_{n \rightarrow \infty} \int_{\Omega} \psi_\beta^p |\nabla(u_n - u)|^p dx, \nu_0 = \lim_{\beta \rightarrow 0^+} \overline{\lim}_{n \rightarrow \infty} \int_{\Omega} \psi_\beta^p \left| \frac{u_n - u}{\delta} \right|^p dx,$$

it follows from (1) and (4) that $c(p, N, \varepsilon) \nu_0 \leq \mu_0$. Since $\varepsilon > 0$ is arbitrary, $c_{p,N} \nu_0 \leq \mu_0$. $\square$

Proof of Theorem 1. Assume that $\mu_p(\Omega) < c_{p,N}$. Lemma 1 implies that

$$c_{p,N} \nu_0 + \mu_p(\Omega) \int_{\Omega} \left| \frac{u}{\delta} \right|^p dx \leq \mu_0 + \int_{\Omega} |\nabla u|^p dx \leq \mu_p(\Omega)$$

and $\nu_0 + \int_{\Omega} \left| \frac{u}{\delta} \right|^p dx = 1$. It follows that $\nu_0 = 0$. But then $\int_{\Omega} \left| \frac{u}{\delta} \right|^p dx = 1$ and u is a minimizer for $\mu_p(\Omega)$. $\square$

Concluding remark. Our approach is also applicable to some problems with a parameter like

$$\mu_{p,\lambda,\eta}(\Omega) = \inf_{u \in \mathcal{D}_0^{1,p}} \frac{\int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} \eta |u|^p dx}{\int_{\Omega} |u/\delta|^p dx}.$$

The case $p = 2$ and Ω bounded was treated in [1].

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