ℓ_p ($p > 2$) DOES NOT COARSELY EMBED INTO A HILBERT SPACE

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ABSTRACT. We show that a Banach space with a normalized symmetric basis behaving like that of ℓ_p ($p > 2$) cannot coarsely embed into a Hilbert space.

A (not necessarily continuous) map f between two metric spaces (X, d) and (Y, δ) is called a *coarse embedding* (see [G, 7.G]) if there exist two non-decreasing functions $\varphi_1 : [0, \infty) \rightarrow [0, \infty)$ and $\varphi_2 : [0, \infty) \rightarrow [0, \infty)$ such that

- (1) $\varphi_1(d(x, y)) \leq \delta(f(x), f(y)) \leq \varphi_2(d(x, y))$,
- (2) $\varphi_1(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Nowak [N], improving a theorem due to A. N. Dranishnikov, G. Gong, V. Laforgue, and G. Yu [DGLY], gave a characterization of coarse embeddability of general metric spaces into a Hilbert space using a result of Schoenberg on negative definite kernels. He used this characterization to show that the spaces $L_p(\mu)$ coarsely embed into a Hilbert space for $p < 2$. In this article, we show that ℓ_p does not coarsely embed into a Hilbert space when $p > 2$. It was already proved in [DGLY] that the Lipschitz universal space c_0 (see [A]) does not coarsely embed into a Hilbert space.

In its full generality, the statement of our result is as follows:

Theorem 1. *Suppose that a Banach space X has a normalized symmetric basis $(e_n)_n$ and that $\liminf_{n \rightarrow \infty} n^{-\frac{1}{2}} \left\| \sum_{i=1}^n e_i \right\| = 0$. Then X does not coarsely embed into a Hilbert space.*

In [Y], Yu proved that a discrete metric space with bounded geometry must satisfy the coarse geometric Novikov conjecture if it coarsely embeds into a Hilbert space, and in [KY] G. Kasparov and Yu proved that to get the same conclusion it is sufficient that the metric space coarsely embeds into a uniformly convex Banach space. Our theorem suggests that the result of [KY] cannot be deduced from the earlier theorem in [Y], but as yet there is no example of a discrete metric space with bounded geometry which coarsely embeds into ℓ_p for some $2 < p < \infty$ but not

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into ℓ_2 . (The reader should be warned that what we called a “coarse embedding” is called a “uniform embedding” in many places, including [DGLY], [KY], and [Y]. Following [N], we use the term coarse embedding to avoid confusion with the closely related notion of uniform embedding as it is used in non-linear Banach space theory [BL]; i.e., a bi-uniformly continuous mapping.)

Besides Schoenberg’s classical work [S] on positive definite functions, an important tool for proving the theorem is Theorem 5.2 in [AMM], which asserts that the hypothesis on X in the theorem implies that every symmetric continuous positive definite function on X is constant. We present the proof in five steps.

STEP 0: REDUCING TO THE α -HÖLDER CASE

Let $f : X \rightarrow H$ be a coarse embedding satisfying

- (1) $\varphi_1(\|x - y\|) \leq \|f(x) - f(y)\| \leq \varphi_2(\|x - y\|)$,
- (2) $\varphi_1(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Our first claim is that we do not lose generality by assuming that $\varphi_2(t) = t^\alpha$ with $0 < \alpha < \frac{1}{2}$.

To prove this claim, note first that $(x, y) \mapsto \|f(x) - f(y)\|^2$ is a negative definite kernel on X . This can be seen by direct computations (see [N, Proposition 3.1]). We refer the reader to [BL, Chapter 8] or [N, Section 2] for the definitions of negative definite kernels and negative definite functions.

So ([N, Lemma 4.2]), for any $0 < \alpha < 1$, the kernel $N(x, y) = \|f(x) - f(y)\|^{2\alpha}$ is also negative definite and satisfies $N(x, x) = 0$ (such a negative definite kernel is called normalized).

As a result, a theorem of Schoenberg ([S] and [BL, Chapter 8]) allows us to find a Hilbert space H_α and a function $f_\alpha : X \rightarrow H_\alpha$ such that $N(x, y) = \|f_\alpha(x) - f_\alpha(y)\|^2$.

On the other hand, since X , being a normed space, is (metrically) convex, the original function $f : X \rightarrow H$ is Lipschitz for large distances. In fact, using the upper bound $\varphi_2(\cdot)$, we can use the same proof as in [BL, proof of Proposition 1.11]. Namely, assume $\|x - y\| \geq 1$ and let n be the smallest integer greater than or equal to $\|x - y\|$; breaking the segment $[x, y]$ into n pieces of length less than or equal to 1, we get $\|f(x) - f(y)\| \leq n\varphi_2(1) \leq 2\varphi_2(1)\|x - y\|$. Consequently, without loss of generality, we can assume by rescaling that we have the following for $\|x - y\| \geq 1$:

$$\|f(x) - f(y)\|_H \leq \|x - y\|$$

and

$$(\varphi_1(\|x - y\|))^\alpha \leq \|f_\alpha(x) - f_\alpha(y)\|_{H_\alpha} \leq \|x - y\|^\alpha.$$

Now, let $\mathcal{N}$ be a 1-net in X (i.e. $\mathcal{N}$ is a maximal 1-separated subset of X). The restriction of f_α to $\mathcal{N}$ is α -Hölder, so if $0 < \alpha < \frac{1}{2}$, then we can extend f_α to an α -Hölder map $\widetilde{f}_\alpha$ defined on the whole of X (see [WW, last statement of Theorem 19.1]):

$$\begin{aligned} \widetilde{f}_\alpha : X &\rightarrow H_\alpha, \\ \forall x \in \mathcal{N}, \quad \widetilde{f}_\alpha(x) &= f_\alpha(x), \end{aligned}$$

and

$$\forall x, y \in X, \quad \|f_\alpha(x) - f_\alpha(y)\|_{H_\alpha} \leq \|x - y\|^\alpha.$$

Now, write $\widetilde{\varphi}_1(t) = \inf\{\|\widetilde{f}_\alpha(x) - \widetilde{f}_\alpha(y)\|, \|x - y\| \geq t\}$, and observe that $\widetilde{\varphi}_1(t) \rightarrow \infty$ as $t \rightarrow \infty$.

This finishes the proof of our reduction to the case where f is α -Hölder and thus uniformly continuous. So from now on we will assume that our coarse embedding is a map $f : X \rightarrow H$ satisfying the following for all $x, y \in X$:

$$\varphi_1(\|x - y\|) \leq \|f(x) - f(y)\| \leq \|x - y\|^\alpha$$

where $\varphi_1(t) \rightarrow \infty$ as $t \rightarrow \infty$.

STEP 1

Set $N(x, y) = \|f(x) - f(y)\|^2$. Then N is a normalized (i.e. $N(x, x) = 0$) negative definite kernel on X (see [N, Proposition 3.1]). Now if we write $\phi_1(t) = (\varphi_1(t))^2$ and $\phi_2(t) = t^{2\alpha}$, then N satisfies:
$$\begin{cases} \phi_1(\|x - y\|) \leq N(x, y) \leq \phi_2(\|x - y\|), \\ \phi_1(t) \rightarrow \infty \text{ as } t \rightarrow \infty. \end{cases}$$

STEP 2

The argument in this step comes from [AMM, Lemma 3.5].

Let μ be an invariant mean on the bounded functions on X (see e.g. [BL] for the definition of invariant means). Define:

$$g(x) = \int_X N(y + x, y) d\mu(y).$$

Then we have the following for g :

- g is well defined because the map $y \mapsto N(y + x, y)$ is bounded for each $x \in X$.
- $g(0) = \int_X N(y, y) d\mu(y) = 0$.
- For scalars $(c_i)_{1 \leq i \leq n}$ satisfying $\sum_{i=1}^n c_i = 0$, we have:

$$\begin{aligned} \sum_{i,j=1}^n c_i c_j g(x_i - x_j) &= \sum_{i,j} c_i c_j \int_X N(y + x_i - x_j, y) d\mu(y) \\ &= \sum_{i,j=1}^n c_i c_j \int_X N(y + x_i, y + x_j) d\mu(y) \\ &= \int_X \left(\sum_{i,j=1}^n c_i c_j N(y + x_i, y + x_j) \right) d\mu(y) \\ &= \int_X (\leq 0) d\mu(y) \\ &\leq 0. \end{aligned}$$

This is because μ is translation invariant, and N is negative definite. This shows that g is a negative definite function on X .

- Finally, since $\int_X d\mu(y) = 1$, we have:

$$\phi_1(\|x\|) \leq g(x) \leq \phi_2(\|x\|).$$

In summary, we have found a negative definite function g on X which satisfies $g(0) = 0$ and $\phi_1(\|x\|) \leq g(x) \leq \phi_2(\|x\|)$, where $\phi_1(t) \rightarrow \infty$ as $t \rightarrow \infty$.

STEP 3

Let $(e_n)_n$ be the normalized symmetric basis for X . This means that for any choice of signs $(\theta_n)_n \in \{-1, +1\}$ and any choice of permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$,

$$\left\| \sum_n \theta_n a_n e_{\sigma(n)} \right\|_X = \left\| \sum_n a_n e_n \right\|_X.$$

The purpose of this step is to show that the negative definite function g we found in the previous step can be chosen to be symmetric, i.e. to satisfy for any choice of signs $(\theta_n)_n \in \{-1, +1\}$ and any choice of permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ the equality:

$$g\left(\sum_n \theta_n a_n e_{\sigma(n)}\right) = g\left(\sum_n a_n e_n\right).$$

For $x = \sum_{n=1}^{\infty} x_n e_n \in X$, define $g_m(x)$ to be the average of $g\left(\sum_{n=1}^{\infty} \theta_n x_n e_{\sigma(n)}\right)$ over all choices of signs θ and permutations σ with the restrictions that $\theta_n = 1$ for $n > m$ and $\sigma(n) = n$ for $n > m$.

It follows that for all such θ, σ , and for all $x = \sum_{n=1}^{\infty} x_n e_n \in X$,

$$g_m\left(\sum_{n=1}^{\infty} \theta_n x_n e_{\sigma(n)}\right) = g_m\left(\sum_{n=1}^{\infty} x_n e_n\right).$$

Moreover, we also have

$$\phi_1(\|x\|) \leq g_m(x) \leq \phi_2(\|x\|).$$

Next we show that the sequence $(g_m)_m$ is equicontinuous. To check this, let us first check the continuity of g :

$$\begin{aligned} |g(a) - g(b)| &\leq \int_X |N(y+a, y) - N(y+b, y)| d\mu(y) \\ &= \int_X \left| \|f(y+a) - f(y)\|^2 - \|f(y+b) - f(y)\|^2 \right| d\mu(y) \\ &= \int_X (\|f(y+a) - f(y)\| + \|f(y+b) - f(y)\|) \\ &\quad \cdot \left| \|f(y+a) - f(y)\| - \|f(y+b) - f(y)\| \right| d\mu(y) \\ &\leq \int_X (\|f(y+a) - f(y)\| + \|f(y+b) - f(y)\|) \\ &\quad \cdot \|f(y+a) - f(y+b)\| d\mu(y) \\ &\leq \int_X (\|a\|^\alpha + \|b\|^\alpha) \|a - b\|^\alpha d\mu(y). \end{aligned}$$

So $|g(a) - g(b)| \leq \|a - b\|^\alpha (\|a\|^\alpha + \|b\|^\alpha)$ and g is continuous.

Now for the equicontinuity of $(g_m)_m$:

$$\begin{aligned}
|g_m(a) - g_m(b)| &= \|\text{ave} \left(g \left(\sum \theta_n a_n e_{\sigma(n)} \right) - g \left(\sum \theta_n b_n e_{\sigma(n)} \right) \right) \| \\
&\leq \text{ave} \| g \left(\sum \theta_n a_n e_{\sigma(n)} \right) - g \left(\sum \theta_n b_n e_{\sigma(n)} \right) \| \\
&\leq \text{ave} \left(\left\| \sum \theta_n a_n e_{\sigma(n)} - \sum \theta_n b_n e_{\sigma(n)} \right\|^\alpha \right. \\
&\quad \cdot \left. \left(\left\| \sum \theta_n a_n e_{\sigma(n)} \right\|^\alpha + \left\| \sum \theta_n b_n e_{\sigma(n)} \right\|^\alpha \right) \right) \\
&= \text{ave} \left(\|a - b\|^\alpha (\|a\|^\alpha + \|b\|^\alpha) \right) \\
&= \|a - b\|^\alpha (\|a\|^\alpha + \|b\|^\alpha).
\end{aligned}$$

So by Ascoli's theorem [R, Chapter 7, Section 10], there is a subsequence $(g_{m_k})_k$ of $(g_m)_m$ which converges pointwise to a continuous function $\tilde{g}$. The property of the g_m 's implies that $\tilde{g}$ must necessarily be symmetric. We have that $\tilde{g}(0) = 0$, and that $\phi_1(\|x\|) \leq \tilde{g}(x) \leq \phi_2(\|x\|)$. Finally, as it is easily checked that the g_m 's are negative definite functions, it also follows easily that $\tilde{g}$ is a negative definite function.

STEP 4

There is a relation between negative and positive definite kernels as given by a result of Schoenberg [S]; see also [BL, Chapter 8]. This result states that a kernel N on X is negative definite if and only if e^{-tN} is positive definite for every $t > 0$.

Since $\liminf_{n \rightarrow \infty} \frac{\|e_1 + e_2 + \cdots + e_n\|}{\sqrt{n}} = 0$, and $\tilde{f} = e^{-\tilde{g}}$ is a symmetric continuous positive definite function on X , we conclude by a theorem of Aharoni, Maurey and Mityagin (see [AMM, Theorem 5.2]), that $\tilde{f}$ is constant.

On the other hand, $\tilde{f}(0) = e^{-\tilde{g}(0)} = 1$, while $0 \leq \tilde{f}(x) \leq e^{-\phi_1(\|x\|)} \rightarrow 0$ as $\|x\| \rightarrow \infty$. This gives a contradiction and finishes the proof. $\square$

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