HANKEL-TYPE OPERATORS ON THE SPACE OF BOUNDED HARMONIC FUNCTIONS

KEIJI IZUCHI AND SHŪICHI OHNO

(Communicated by Joseph A. Ball)

Dedicated to Professor Kazuyuki Tsurumi on the occasion of his 70th birthday

ABSTRACT. We study Hankel-type operators on the space of bounded harmonic functions on the open unit disk. These operators are related to tight uniform algebras, the Dunford-Pettis property, and Bourgain algebras.

1. INTRODUCTION

Let X be a commutative Banach algebra and let Y be a closed subspace. We define a Hankel-type operator $S_g : Y \rightarrow X/Y$ with symbol $g \in X$ by $S_g f = gf + Y$ for all $f \in Y$. These operators are a natural generalization of classical Hankel operators on the Hardy space H^2 . The conditions for S_g to be compact, weakly compact and completely continuous have been investigated in various function spaces. The problem of whether all S_g are weakly compact on a uniform algebra is related to a tight algebra, and the problem of complete continuity appeared in the Dunford-Pettis property. The latter introduced a notion of Bourgain algebras which have been actively researched in analytic and harmonic function spaces on the open unit disk. Recently, Dudziak, Gamelin, and Gorkin [1] studied Hankel-type operators on analytic function spaces. But the problem of the compactness remains open in the case of the bounded harmonic function space on the open unit disk $\mathbb{D}$. In this paper we study this problem. See [3] and [5] as surveys for convenience.

Let $L^\infty(\mathbb{D})$ be the set of all bounded measurable functions on $\mathbb{D}$. Then $L^\infty(\mathbb{D})$ is a Banach algebra with the essential supremum norm $\|f\|_\infty = \text{ess. sup}\{|f(z)|; z \in \mathbb{D}\}$. Let $h^\infty := h^\infty(\mathbb{D})$ be the set of all bounded harmonic functions on $\mathbb{D}$. It follows that h^∞ is a closed subspace of $L^\infty(\mathbb{D})$. For $g \in L^\infty(\mathbb{D})$, we define the linear operator $S_g : h^\infty \rightarrow L^\infty(\mathbb{D})/h^\infty$ by $S_g f = gf + h^\infty$ for $f \in h^\infty$. Trivially $S_g : h^\infty \rightarrow L^\infty(\mathbb{D})/h^\infty$ is a bounded linear operator. Recall that S_g is said to be *compact* if S_g maps every bounded set into a relatively compact one, and that S_g is said to be *completely continuous* if S_g maps every weakly convergent sequence into a norm convergent one. In general, every compact operator is completely continuous.

Received by the editors September 15, 2004 and, in revised form, November 29, 2004.
 2000 *Mathematics Subject Classification*. Primary 47B38, 30D55.

Key words and phrases. Hankel-type operators, Banach space of bounded harmonic functions.

But the converse is not always true. We define the following sets:

$$\begin{aligned} h_c^\infty &= \{g \in L^\infty(\mathbb{D}) : S_g \text{ is compact}\}, \\ h_{cc}^\infty &= \{g \in L^\infty(\mathbb{D}) : S_g \text{ is completely continuous}\}. \end{aligned}$$

Note that $h_c^\infty \subset h_{cc}^\infty$. Although h^∞ is not an algebra, these are closed subalgebras of $L^\infty(\mathbb{D})$. Especially h_{cc}^∞ is called the Bourgain algebra of h^∞ .

The Bourgain algebra h_{cc}^∞ is characterized by Stroethoff, Yale, and the first author [4]. Let $C(\partial\mathbb{D})$ and $L^\infty(\partial\mathbb{D})$ be the spaces of continuous and bounded measurable functions on the unit circle $\partial\mathbb{D}$, respectively. For a function $f \in L^\infty(\partial\mathbb{D})$, we denote by $\hat{f}$ the Poisson integral of f on $\mathbb{D}$, that is,

$$\hat{f}(z) = \int_0^{2\pi} f(e^{i\theta}) P_z(e^{i\theta}) d\theta / 2\pi,$$

where P_z is the Poisson kernel of $z \in \mathbb{D}$. Then $\hat{f} \in h^\infty$. For any nonempty subset B of $L^\infty(\partial\mathbb{D})$, we write $\hat{B} = \{\hat{f} : f \in B\}$. It is known that f in h^∞ has a boundary function f^* on $\partial\mathbb{D}$ and $\widehat{f^*} = f$ on $\mathbb{D}$, so that $h^\infty = \widehat{L^\infty(\partial\mathbb{D})}$. Let $H^\infty(\partial\mathbb{D})$ be the space of boundary functions of bounded analytic functions on $\mathbb{D}$. The algebra QC of bounded quasi-continuous functions on $\partial\mathbb{D}$ is given by

$$QC = (H^\infty(\partial\mathbb{D}) + C(\partial\mathbb{D})) \cap \overline{(H^\infty(\partial\mathbb{D}) + C(\partial\mathbb{D}))}.$$

Refer to [2] and [6] for more information. For $0 < r < 1$, we write $\mathbb{D}_r = \{|z| \leq r\}$. Put

$$V = \{g \in L^\infty(\mathbb{D}) : \|g\chi_{\mathbb{D} \setminus \mathbb{D}_r}\|_\infty \rightarrow 0 \text{ as } r \rightarrow 1^-\}.$$

The equality $h_{cc}^\infty = \widehat{QC} + V$ was given as Corollary 3 in [4]. In this paper, we show that

Theorem 1.1. $h_c^\infty = h_{cc}^\infty = \widehat{QC} + V$.

2. PROOF OF THEOREM 1.1

To prove Theorem 1.1, we use the following facts which can easily be checked:

- (i) $g \in h_c^\infty$ if and only if $\bar{g} \in h_c^\infty$,
- (ii) S_g is compact on h^∞ provided that for every sequence $\{f_n\}_n$ in h^∞ satisfying $\|f_n\|_\infty \leq 1$ and $f_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$, then there is a subsequence $\{f_{n_j}\}_j$ of $\{f_n\}_n$ such that $\|S_g f_{n_j}\|_\infty \rightarrow 0$ as $j \rightarrow \infty$.

Proof of Theorem 1.1. First, we show that

$$(2.1) \quad C(\overline{\mathbb{D}}) + V \subset h_c^\infty \subset h_{cc}^\infty = \widehat{QC} + V,$$

where $C(\overline{\mathbb{D}})$ is the space of continuous functions on the closed unit disk $\overline{\mathbb{D}}$. It is trivial that $V \subset h_c^\infty$. Since h_c^∞ is a closed subalgebra of $L^\infty(\mathbb{D})$, by (i) it is sufficient to prove that $z \in h_c^\infty$. Suppose that $f_n \in h^\infty$, $\|f_n\|_\infty \leq 1$ and $f_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$. We shall prove that $\|S_z f_{n_j}\|_\infty \rightarrow 0$ for some subsequence $\{f_{n_j}\}_j$ of $\{f_n\}_n$.

Write $f_n = u_n + iv_n$, where u_n and v_n are real-valued harmonic functions on $\mathbb{D}$. Denote by $^*u_n, ^*v_n$ the harmonic conjugate functions of u_n, v_n , with $^*u_n(0) = ^*v(0) = 0$, respectively. Set

$$g_n = \frac{1}{2}(u_n + i^*u_n + i(v_n + i^*v_n)), \quad h_n = \frac{1}{2}(u_n + i^*u_n - i(v_n + i^*v_n)).$$

Then $f_n = g_n + \overline{h_n}$, and it is known that $g_n, h_n \in H^2$, where H^2 is the Hardy space on $\mathbb{D}$; see Chapter III of Garnett's book [2]. Identifying functions with boundary functions, and denoting by $\|\cdot\|_2$ the square integral norm on $\partial\mathbb{D}$, we know that

$$\|u_n\|_2 \leq \|u_n\|_2 \leq \|f_n\|_2 \leq \|f_n\|_\infty \leq 1, \quad \|v_n\|_2 \leq \|v_n\|_2 \leq 1.$$

Hence

$$(2.2) \quad \sup_n \|g_n\|_{H^2} \leq 2, \quad \sup_n \|h_n\|_{H^2} \leq 2,$$

where $\|\cdot\|_{H^2}$ is the Hardy space norm. By a normal family argument, there exist subsequences (which we do not relabel) of $\{g_n\}, \{h_n\}$ such that $g_n \rightarrow g, h_n \rightarrow h$ uniformly on compact subsets of $\mathbb{D}$ for some analytic functions g and h . Since $f_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ and $f_n = g_n + \overline{h_n}$, $g = -\overline{h}$ on $\mathbb{D}$. Since g and h are analytic, $g = -\overline{h} = \alpha$ for some constant α with $|\alpha| \leq 2$.

Putting

$$k_n = zg_n + \overline{\left(\frac{h_n - h_n(0)}{z}\right)} - \alpha z,$$

then k_n is harmonic on $\mathbb{D}$ and

$$(2.3) \quad k_n = z(g_n + \overline{h_n}) - z\overline{h_n} + \overline{\left(\frac{h_n - h_n(0)}{z}\right)} - \alpha z.$$

Since $f_n \in h^\infty$, $z(g_n + \overline{h_n}) - \alpha z = zf_n - \alpha z \in L^\infty(\mathbb{D})$. It is clear that $z\overline{h_n} - \overline{((h_n - h_n(0))/z)}$ is bounded near the origin of $\mathbb{D}$. By (2.2), we have

$$(2.4) \quad |h_n(z)| \leq \|h_n\|_{H^2} \frac{1}{\sqrt{1-|z|^2}} \leq \frac{2}{\sqrt{1-|z|^2}} \quad \text{for } z \in \mathbb{D}.$$

Since

$$\left| z\overline{h_n(z)} - \overline{\left(\frac{h_n(z) - h_n(0)}{z}\right)} \right| \leq \frac{|h_n(z)|(1-|z|^2)}{|z|} + \frac{|h_n(0)|}{|z|},$$

$z\overline{h_n} - \overline{((h_n - h_n(0))/z)}$ is bounded on $\mathbb{D}$. Thus we get $k_n \in h^\infty$.

To show that $\|zf_n - k_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, let ϵ be an arbitrary small positive number and $\delta = \sqrt{1-\epsilon^2}$. By (2.3), we have

$$\|zf_n - k_n\|_\infty = \|z(\overline{h_n} + \overline{\alpha}) - \overline{\left(\frac{h_n + \overline{\alpha} - (h_n(0) + \overline{\alpha})}{z}\right)}\|_\infty.$$

Since $h_n \rightarrow -\overline{\alpha}$ uniformly on $\mathbb{D}_\delta$,

$$(2.5) \quad \sup_{z \in \mathbb{D}_\delta} \left| z(\overline{h_n(z)} + \overline{\alpha}) - \overline{\left(\frac{h_n(z) + \overline{\alpha} - (h_n(0) + \overline{\alpha})}{z}\right)} \right| \rightarrow 0$$

as $n \rightarrow \infty$. For $|z| > \delta$, we have

$$\begin{aligned} & \left| z(\overline{h_n(z)} + \overline{\alpha}) - \overline{\left(\frac{h_n(z) + \overline{\alpha} - (h_n(0) + \overline{\alpha})}{z}\right)} \right| \\ &= \left| \frac{1-|z|^2}{\overline{z}} (\overline{h_n(z)} + \overline{\alpha}) - \frac{\overline{h_n(0)} + \overline{\alpha}}{\overline{z}} \right| \\ &\leq \frac{\epsilon \|h_n + \overline{\alpha}\|_{H^2}}{\delta} + \frac{|h_n(0) + \overline{\alpha}|}{\delta} \quad \text{by (2.4)} \\ &< \frac{4\epsilon}{\delta} + \frac{|h_n(0) + \overline{\alpha}|}{\delta} \quad \text{by (2.2) and } |\alpha| \leq 2. \end{aligned}$$

Combining with (2.5),

$$\limsup_{n \rightarrow \infty} \|zf_n - k_n\|_\infty \leq \frac{4\epsilon}{\delta} = \frac{4\epsilon}{\sqrt{1-\epsilon^2}}.$$

Since ϵ is arbitrary,

$$\lim_{n \rightarrow \infty} \|S_z f_n\|_\infty \leq \lim_{n \rightarrow \infty} \|zf_n - k_n\|_\infty = 0.$$

Thus by (ii), $z \in h_c^\infty$, and it holds that

$$C(\overline{\mathbb{D}}) + V \subset h_c^\infty \subset h_{cc}^\infty = \widehat{QC} + V.$$

Next we show that for any $g \in QC$, $\hat{g} \in h_c^\infty$, that is, $S_{\hat{g}}$ is compact on h^∞ . To show this, note the equality $h^\infty = L^\infty(\partial\mathbb{D})$. Let $\{f_n\}_n$ be a sequence in h^∞ satisfying that $\|f_n\|_\infty \leq 1$ and $\hat{f}_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$. We have $\|S_{\hat{g}}\hat{f}_n + h^\infty\| \leq \|\hat{g}\hat{f}_n - \widehat{gf_n}\|_\infty$ and

$$|\hat{g}(z)\hat{f}_n(z) - \widehat{gf_n}(z)| \leq \int_0^{2\pi} |\hat{g}(z) - g(e^{i\theta})| |P_z(\theta)| d\theta / 2\pi.$$

Since $g \in QC$, we have

$$(2.6) \quad \sup_{r < |z| < 1} \sup_n |\hat{g}(z)\hat{f}_n(z) - \widehat{gf_n}(z)| \rightarrow 0 \quad \text{as } r \rightarrow 1;$$

see [2, 4, 6]. Then for any $\varepsilon > 0$, there exists $R, 0 < R < 1$, such that

$$(2.7) \quad \sup_{R < |z| < 1} \sup_n |\hat{g}(z)\hat{f}_n(z) - \widehat{gf_n}(z)| < \varepsilon.$$

On the other hand, by the assumption, $\sup_{z \in \mathbb{D}_R} |\hat{g}(z)\hat{f}_n(z)| \rightarrow 0$ as $n \rightarrow \infty$. Now we define $g_t(e^{i\theta}) = \hat{g}(te^{i\theta})$ for $0 < t < 1$. Then $g_t \in C(\partial\mathbb{D})$, $\hat{g}_t(z) = \hat{g}(tz)$ for $z \in \mathbb{D}$, and

$$\begin{aligned} \sup_{z \in \mathbb{D}_R} |\widehat{gf_n}(z) - \widehat{g_t f_n}(z)| &\leq \sup_{z \in \mathbb{D}_R} \int_0^{2\pi} |g(e^{i\theta}) - g_t(e^{i\theta})| |f_n(e^{i\theta})| |P_z(e^{i\theta})| d\theta / 2\pi \\ &\leq \frac{1+R}{1-R} \int_0^{2\pi} |g(e^{i\theta}) - g_t(e^{i\theta})| d\theta / 2\pi. \end{aligned}$$

By the Lebesgue dominated convergence theorem,

$$\lim_{t \rightarrow 1} \sup_n \sup_{z \in \mathbb{D}_R} |\widehat{gf_n}(z) - \widehat{g_t f_n}(z)| = 0.$$

Take a positive number t_0 such that $t_0 < 1$ and

$$(2.8) \quad \sup_n \sup_{z \in \mathbb{D}_R} |\widehat{gf_n}(z) - \widehat{g_{t_0} f_n}(z)| < \varepsilon.$$

By (2.1), $\hat{g}_{t_0} \in C(\overline{\mathbb{D}}) \subset h_c^\infty$, so that by considering a subsequence we may assume that $\|\hat{g}_{t_0}\hat{f}_n + h^\infty\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Then

$$(2.9) \quad \|\hat{g}_{t_0}\hat{f}_n - \widehat{g_{t_0} f_n} + h^\infty\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We want to prove that

$$(2.10) \quad \sup_{z \in \mathbb{D}_R} |\widehat{g_{t_0} f_n}(z)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let $\{h_n\}_n$ be a sequence in h^∞ satisfying

$$(2.11) \quad \|\hat{g}_{t_0}\hat{f}_n - \widehat{g_{t_0}f_n} + h_n\|_\infty = \|\hat{g}_{t_0}\hat{f}_n - \widehat{g_{t_0}f_n} + h^\infty\|_\infty.$$

Since $g_t \in C(\partial\mathbb{D}) \subset QC$, by (2.6)

$$\|\hat{g}_{t_0}\hat{f}_n - \widehat{g_{t_0}f_n} + h_n\|_\infty \geq \limsup_{|z| \rightarrow 1} |h_n(z)| = \|h_n\|_\infty.$$

By (2.9) $\|h_n\|_\infty \rightarrow 0$, and by (2.11)

$$(2.12) \quad \|\hat{g}_{t_0}\hat{f}_n - \widehat{g_{t_0}f_n}\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We have

$$\|\hat{g}_{t_0}\hat{f}_n - \widehat{g_{t_0}f_n}\|_\infty \geq \sup_{z \in \mathbb{D}_R} |\widehat{g_{t_0}f_n}(z) - \hat{g}_{t_0}(z)\hat{f}_n(z)|.$$

By the assumption, $\sup_{z \in \mathbb{D}_R} |\hat{g}_{t_0}(z)\hat{f}_n(z)| \rightarrow 0$ as $n \rightarrow \infty$. Then by (2.12), we get (2.10).

Combining (2.8) and (2.10), we have

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{D}_R} |\widehat{gf_n}(z)| \leq \epsilon.$$

This shows that $\sup_{z \in \mathbb{D}_R} |\widehat{gf_n}(z)| \rightarrow 0$ as $n \rightarrow \infty$. Since $\sup_{z \in \mathbb{D}_R} |\hat{g}(z)\hat{f}_n(z)| \rightarrow 0$ by our assumption, we get

$$\sup_{z \in \mathbb{D}_R} |\hat{g}(z)\hat{f}_n(z) - \widehat{gf_n}(z)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Combining with (2.7), we have

$$\limsup_{n \rightarrow \infty} \|\hat{g}\hat{f}_n - \widehat{gf_n}\|_\infty \leq \epsilon.$$

Consequently we obtain

$$\|S_{\hat{g}}\hat{f}_n + h^\infty\| \leq \|\hat{g}\hat{f}_n - \widehat{gf_n}\|_\infty \rightarrow 0,$$

that is, $S_{\hat{g}}$ is compact and so $\widehat{QC} \subset h_c^\infty$. We complete the proof. $\square$

ACKNOWLEDGMENTS

The first author was partially supported by Grant-in-Aid for Scientific Research (No. 16340037), Ministry of Education, Science and Culture. The second author was partially supported by Grant-in-Aid for Scientific Research (No. 15540181), Ministry of Education, Science and Culture. The second author also thanks Professor K. Izuchi for his hospitality while he visited Department of Mathematics, Niigata University, where the work for part of this paper was done.

REFERENCES

1. J. Dudziak, T. W. Gamelin and P. Gorkin, *Hankel operators on bounded analytic functions*, *Trans. Amer. Math. Soc.* **352** (2000), 363–377. MR1473437 (2000c:47049)
2. J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981. MR0628971 (83g:30037)
3. P. Gorkin, *Hankel-type operators, Bourgain algebras, and uniform algebras*, *Holomorphic Spaces (Berkeley, CA, 1995)*, 121–134, Math. Sci. Res. Inst. Publ. **33**, Cambridge Univ. Press, Cambridge, 1998. MR1630647 (99g:47052)
4. K. Izuchi, K. Stroethoff and K. Yale, *Bourgain algebras of spaces of harmonic functions*, *Michigan Math. J.* **41** (1994), 309–321. MR1278437 (95d:46051)

5. S. Saccone, *Tight uniform algebras, Holomorphic Spaces (Berkeley, CA, 1995)*, 135–154, Math. Sci. Res. Inst. Publ. **33**, Cambridge Univ. Press, Cambridge, 1998. MR1630648 (2000a:46085)
6. D. Sarason, *Functions of vanishing mean oscillation*, *Trans. Amer. Math. Soc.* **207** (1975), 391–405. MR0377518 (51:13690)

DEPARTMENT OF MATHEMATICS, NIIGATA UNIVERSITY, NIIGATA, 950-2181, JAPAN

E-mail address: `izuchi@math.sc.niigata-u.ac.jp`

NIPPON INSTITUTE OF TECHNOLOGY, MIYASHIRO, MINAMI-SAITAMA 345-8501, JAPAN

E-mail address: `ohno@nit.ac.jp`