GLOBAL SOLUTIONS TO SPECIAL LAGRANGIAN EQUATIONS

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ABSTRACT. We show that any global solution to the special Lagrangian equations with the phase larger than a critical value must be quadratic.

1. INTRODUCTION

In this note, we show that any global solution u in $\mathbb{R}^n$ to the special Lagrangian equation

$$(1.1) \quad \sum_{i=1}^n \arctan \lambda_i = c$$

with phase $|c| > \frac{\pi}{2}(n-2)$ must be a quadratic polynomial, which states the λ_i 's are the eigenvalues of the Hessian D^2u . Recall the Bernstein theorem, where any global solution to the minimal surface equation $\operatorname{div} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0$ in $\mathbb{R}^7$ must be a linear function.

Equation (1.1) stems from the special Lagrangian geometry [HL]. The Lagrangian graph $(x, \nabla u(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the phase or the argument of the complex number $(1 + \sqrt{-1}\lambda_1) \cdots (1 + \sqrt{-1}\lambda_n)$ is constant c , that is, u satisfies equation (1.1), and it is special if and only if $(x, \nabla u(x))$ is a minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ [HL, Theorem 2.3, Proposition 2.17]. To be precise, we state

Theorem 1.1. *Let u be a smooth solution in $\mathbb{R}^n$ to (1.1) with $|c| > \frac{\pi}{2}(n-2)$. Then u is quadratic.*

Fu [F] proved Theorem 1.1 in the two-dimensional case. Indeed (1.1) with $c = \frac{\pi}{2}$ in the 2-d case also has the Monge-Ampère form $\det D^2u = 1$, and Jörgens already showed Theorem 1.1 in this special case earlier on (cf. [N]).

Other Bernstein-Liouville type results concerning (1.1) are in order. Borishenko [B] showed that any convex solution with linear growth to (1.1) with $c = k\pi$ is linear. The author [Y] proved that any convex solution to (1.1) must be quadratic. For $c = k\pi$ in $n = 3$ case, (1.1) has another form

$$(1.2) \quad \Delta u = \det D^2u.$$

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It was proved in [BCGJ] that any strictly convex solution to (1.2) with quadratic growth must be quadratic. Under the assumption that the Hessian is bounded and $\lambda_i \lambda_j \geq -\frac{3}{2}$, it was also shown in [TW] that any global solution to (1.1) is quadratic.

The heuristic idea of the proof of Theorem 1.1 is to find a better graph representation of $(x, \nabla u(x))$ so that the Hessian of the new potential is bounded, and the new potential function satisfies a convex uniformly elliptic equation. By Krylov-Evan's [K], [E] interior Hölder estimates on the Hessian, we draw the conclusion.

As there are nontrivial global harmonic solutions to (1.1) with $c = 0$ in the case $n = 2$, we guess (1.1) with $c = \frac{\pi}{2}(n-2)$ also has nontrivial global solutions in the higher-dimensional case. Observe that in the case $n = 3$ and $c = \frac{\pi}{2}$, (1.1) also takes the interesting form $\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = 1$.

2. PROOF

Step 1. We first find a better graph representation of M through Lewy rotation (cf. [N]) so that the Hessian of the potential function is bounded. By symmetry we only consider the case $c > \frac{\pi}{2}(n-2)$. Let $\sum_{i=1}^n \theta_i = \frac{\pi}{2}(n-2) + \delta$, where $\theta_i = \arctan \lambda_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\delta \in (0, \pi)$. Note that

$$(2.1) \quad -\frac{\pi}{2} + \frac{(n-1)}{n}\delta < \theta_i - \frac{\delta}{n} < \frac{\pi}{2} - \frac{\delta}{n}.$$

The first inequality follows from $\frac{\pi}{2}(n-2) + \delta < \theta_i + \frac{\pi}{2}(n-1)$. We rotate the $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ coordinate system to $(\bar{x}, \bar{y})$ by $\frac{\delta}{n}$, namely, $\bar{x} = x \cos \frac{\delta}{n} + y \sin \frac{\delta}{n}$, $\bar{y} = -x \sin \frac{\delta}{n} + y \cos \frac{\delta}{n}$. In terms of complex variables $z = x + \sqrt{-1}y$, that is, we identify $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$, the rotation takes the form $\bar{z} = e^{-\sqrt{-1}\delta/n}z$. Then M has a new parametrization

$$\begin{cases} \bar{x} = x \cos \frac{\delta}{n} + \nabla u(x) \sin \frac{\delta}{n}, \\ \bar{y} = -x \sin \frac{\delta}{n} + \nabla u(x) \cos \frac{\delta}{n}. \end{cases}$$

By (2.1), $M = (x, \nabla u(x))$ is still a graph over the whole $\bar{x}$ space $\mathbb{R}^n$. Further the rotation belongs to $U(n)$. Then M is also a special Lagrangian graph $(\bar{x}, \nabla \bar{u}(\bar{x}))$, where $\bar{u}$ is a smooth function [HL, p. 87, Proposition 2.17]. Let $\bar{\lambda}_i$ be the eigenvalues of $D^2 \bar{u}$. Then $\bar{\theta}_i = \arctan \bar{\lambda}_i = \theta_i - \frac{\delta}{n} \in (-\frac{\pi}{2} + \frac{(n-1)}{n}\delta, \frac{\pi}{2} - \frac{\delta}{n})$. That is,

$$|D^2 \bar{u}| \leq C(\delta).$$

Finally $\bar{u}$ satisfies the equation

$$(2.2) \quad \sum_{i=1}^n \arctan \bar{\lambda}_i = \frac{\pi}{2}(n-2).$$

Step 2. We proceed with the following lemma, which is Lemma 8.1 in [CNS] when n is even and $c = \frac{\pi}{2}(n-2)$.

Lemma 2.1. *Let $f(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^n \arctan \lambda_i$ and let $\Gamma = \{\lambda | f(\lambda) = c\}$ with $|c| \geq \frac{\pi}{2}(n-2)$. Then Γ is convex.*

Proof. We skip the case $n = 1$. By symmetry we just consider the case $c \geq 0$. Set $c = \frac{\pi}{2}(n-2) + \delta$ with $\delta \in [0, \pi)$. We may assume that $\theta_i = \arctan \lambda_i \geq 0$ for

$i = 1, \dots, n-1$. The normal of Γ is $\nabla f = (\cos^2 \theta_1, \dots, \cos^2 \theta_n)$. Let

$$A \triangleq -\frac{1}{2}D^2f = \begin{bmatrix} \tan \theta_1 \cos^4 \theta_1 & & \\ & \dots & \\ & & \tan \theta_n \cos^4 \theta_n \end{bmatrix}.$$

Take any tangent vector $T = (t_1, \dots, t_n) \in T_\lambda \Gamma$, that is,

$$\sum_{i=1}^n t_i \cos^2 \theta_i = 0.$$

We show that $A(T, T) \geq 0$.

Case a) $\theta_n \geq 0$. Certainly it is true.

Case b) $\theta_n < 0$. First we know that $\theta_i > 0$ for $i = 1, \dots, n-1$ and $\delta < \frac{\pi}{2}$. Next we have

$$A(T, T) = \sum_{i=1}^{n-1} \tan \theta_i \cos^4 \theta_i t_i^2 + \tan \theta_n \cos^4 \theta_n t_n^2.$$

Now we use the trick in [CNS, p. 299],

$$\begin{aligned} (-t_n \cos \theta_n)^2 &= \left(\sum_{i=1}^{n-1} t_i \cos^2 \theta_i \right)^2 \\ &\leq \left(\sum_{i=1}^{n-1} t_i^2 \cos^4 \theta_i \tan \theta_i \right) \left(\sum_{i=1}^{n-1} \cot \theta_i \right). \end{aligned}$$

Then

$$\tan \theta_n \cos^4 \theta_n t_n^2 \geq \left(\sum_{i=1}^{n-1} t_i^2 \cos^4 \theta_i \tan \theta_i \right) \left(\sum_{i=1}^{n-1} \cot \theta_i \right) \tan \theta_n$$

and

$$\begin{aligned} A(T, T) &\geq \left(\sum_{i=1}^{n-1} t_i^2 \cos^4 \theta_i \tan \theta_i \right) \left[1 + \left(\sum_{i=1}^{n-1} \cot \theta_i \right) \tan \theta_n \right] \\ &= \left(\sum_{i=1}^{n-1} t_i^2 \cos^4 \theta_i \tan \theta_i \right) \left(\sum_{i=1}^n \cot \theta_i \right) \tan \theta_n. \end{aligned}$$

Let $\alpha_i = \frac{\pi}{2} - \theta_i$. We have

$$\begin{aligned} \frac{\pi}{2} &< \pi - \delta = \alpha_1 + \dots + \alpha_n < \pi, \\ 0 &< \alpha_1, \dots, \alpha_{n-1} < \frac{\pi}{2} < \alpha_n, \end{aligned}$$

and

$$\sum_{i=1}^n \cot \theta_i = \sum_{i=1}^{n-1} \tan \alpha_i + \tan \alpha_n.$$

It follows that $\tan \alpha_n < 0$ and

$$\frac{\tan(\alpha_1 + \dots + \alpha_{n-1}) + \tan \alpha_n}{1 - \tan(\alpha_1 + \dots + \alpha_{n-1}) \tan \alpha_n} = \tan(\alpha_1 + \dots + \alpha_n) < 0.$$

Then $\tan(\alpha_1 + \cdots + \alpha_{n-1}) + \tan \alpha_n < 0$. Note that $\alpha_1 + \cdots + \alpha_{n-1} = \pi - \delta - \alpha_n < \frac{\pi}{2}$; we have

$$\begin{aligned} \tan(\alpha_1 + \cdots + \alpha_{n-1}) &\geq \tan \alpha_1 + \tan(\alpha_2 + \cdots + \alpha_{n-1}) \\ &\quad \dots \\ &\geq \tan \alpha_1 + \tan \alpha_2 + \cdots + \tan \alpha_{n-1}. \end{aligned}$$

So $\sum_{i=1}^n \cot \theta_i = \sum_{i=1}^n \tan \alpha_i < 0$ and $A(T, T) \geq 0$. Therefore Γ is convex (w.r.t. the normal ∇f). $\square$

Remark 1. The level set Γ is no longer convex or concave when $|c| < \frac{\pi}{2}(n-2)$.

Step 3. The final argument is standard. We now have global solution $\bar{u}$ with bounded Hessian on the convex level set Γ , more precisely a convex level set in the symmetric matrix space (cf. [CNS, p. 276]). In another word, $\bar{u}$ satisfies (2.2), which is now uniformly elliptic. By Krylov-Evans theorem ([K],[E])

$$[D^2 \bar{u}]_{C^\beta(B_r)} \leq C(n, \delta) \frac{\|D^2 \bar{u}\|_{L^\infty(B_{2r})}}{r^\beta} \leq \frac{C(n, \delta)}{r^\beta},$$

where $\beta = \beta(n, \delta) \in (0, 1)$. Let r go to $+\infty$; we see that $D^2 \bar{u}$ is a constant matrix. Thus $(\bar{x}, \nabla \bar{u})$ is a plane, and consequently u is quadratic.

REFERENCES

- [BCGJ] Bao, J.-G., Chen, J.-Y., Guan, B., and Ji, M., *Liouville property and regularity of a Hessian quotient equation*, Amer. J. Math. **125** (2003), 301–316. MR1963687 (2004b:35079)
- [B] Borisenko, A. A., *On a Liouville-type theorem for the equation of special Lagrangian submanifolds*, (Russian) Mat. Zametki **52** (1992), 22–25; English translation in Math. Notes **52** (1992), 1094–1096 (1993). MR1201944 (93k:53055)
- [CNS] Caffarelli, L. A., Nirenberg, L., and Spruck, J., *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian*, Acta Math. **155** (1985), 261–301. MR0806416 (87f:35098)
- [E] Evans, L. C., *Classical solutions of fully nonlinear, convex, second-order elliptic equations*, Comm. Pure Appl. Math. **35** (1982), no. 3, 333–363. MR0649348 (83g:35038)
- [F] Fu, L., *An analogue of Bernstein's theorem*, Houston J. Math. **24** (1998), 415–419. MR1686614 (2000c:53006)
- [HL] Harvey, R. and Lawson, H. B., Jr., *Calibrated geometry*, Acta Math. **148** (1982), 47–157. MR0666108 (85i:53058)
- [K] Krylov, N. V., *Boundedly nonhomogeneous elliptic and parabolic equations*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 487–523 in Russian; English translation in Math. USSR Izv. **20**(1983), 459–492. MR0661144 (84a:35091)
- [N] Nitsche, J. C. C., *Elementary proof of Bernstein's theorem on minimal surfaces*, Ann. of Math. **66** (1957), 543–544. MR0090833 (19:878f)
- [TW] Tsui, M.-P. and Wang, M.-T., *A Bernstein type result for special Lagrangian submanifolds*, Math. Res. Lett. **9** (2002), 529–535. MR1928873 (2003k:53059)
- [Y] Yuan, Y., *A Bernstein problem for special Lagrangian equations*, Invent. Math. **150** (2002), 117–125. MR1930884 (2003k:53060)

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