INVERTIBILITY OF LINEAR COMBINATIONS OF TWO IDEMPOTENTS

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(Communicated by Joseph A. Ball)

ABSTRACT. Let P and Q be two idempotents on a Hilbert space. In this note, we prove that the invertibility of the linear combination $\lambda_1 P + \lambda_2 Q$ is independent of the choice of λ_i , $i = 1, 2$, if $\lambda_1 \lambda_2 \neq 0$ and $\lambda_1 + \lambda_2 \neq 0$.

Let $\mathcal{H}$ be a Hilbert space, and let all bounded linear operators on $\mathcal{H}$ be denoted by $\mathcal{B}(\mathcal{H})$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be idempotent if $P^2 = P$. The set $\mathcal{P}$ of all idempotents in $\mathcal{B}(\mathcal{H})$ is invariant under similarity; that is, if $P \in \mathcal{P}$ and $S \in \mathcal{B}(\mathcal{H})$ is an invertible operator, then $S^{-1}PS$ is still an idempotent since $(S^{-1}PS)^2 = S^{-1}PSS^{-1}PS = S^{-1}P^2S = S^{-1}PS$. An idempotent P is called an orthogonal projection if $P^2 = P = P^*$, where P^* is the adjoint of P . Moreover, for an idempotent $P \in \mathcal{P}$, there exists an invertible operator $U \in \mathcal{B}(\mathcal{H})$ such that $U^{-1}PU$ is an orthogonal projection. In fact, if $P \in \mathcal{P}$, then P can be written in the form of

$$P = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp$, where $\mathcal{R}(M)$ denotes the range of the operator M . In this case, we have

$$\begin{pmatrix} I & P_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{P} = \begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix}$ is invertible and $\tilde{P}^{-1} = \begin{pmatrix} I & P_1 \\ 0 & I \end{pmatrix}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $(Ax, x) \geq 0$ for all $x \in \mathcal{H}$. If A is positive, then $A^{\frac{1}{2}}$ denotes the positive square root of A .

In recent years, a number of researchers have considered questions concerning the idempotents and linear combinations of idempotents (see [1]–[8]). Particularly, some researchers pay much attention to the study of linear combinations of two idempotents ([1], [5]). For example, if P_i , $i = 1, 2$, are idempotents in the finite-dimensional space C^n , J. K. Baksalary and O. M. Baksalary ([1]) have proved that the nonsingularity of $P_1 + P_2$ is equivalent to the nonsingularity of any linear combination $c_1 P_1 + c_2 P_2$, where $c_1 + c_2 \neq 0$. In the present note, we will study the

Received by the editors June 19, 2004 and, in revised form, December 20, 2004.

2000 *Mathematics Subject Classification*. Primary 47A05, 47L07.

Key words and phrases. Idempotent, invertibility, linear combination of two idempotents.

This research was partially supported by the National Natural Science Foundation of China (19771056).

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invertibility of linear combinations of two idempotents on an infinite-dimensional Hilbert space. We obtain the main result, which is similar to [1], but the idea of the proof is different from [1].

Theorem 1. *Let P and Q in $\mathcal{B}(\mathcal{H})$ be two idempotents. If λ_1 and λ_2 are nonzero complex numbers and $\lambda_1 + \lambda_2 \neq 0$, then the invertibility of $\lambda_1 P + \lambda_2 Q$ is independent of the choice of λ_i , $i = 1, 2$.*

To prove Theorem 1, we need some lemmas which are well known, so the proofs are omitted.

Lemma 2. *Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be a bounded linear operator on $\mathcal{H} \oplus \mathcal{K}$. Then A is a positive operator if and only if $A_{11} \geq 0$, $A_{22} \geq 0$, $A_{12} = A_{21}^*$ and there exists a contraction D from $\mathcal{K}$ into $\mathcal{H}$ such that*

$$A = \begin{pmatrix} A_{11} & A_{11}^{\frac{1}{2}} D A_{22}^{\frac{1}{2}} \\ A_{22}^{\frac{1}{2}} D^* A_{11}^{\frac{1}{2}} & A_{22} \end{pmatrix}.$$

Lemma 3. *Let $A \in \mathcal{B}(\mathcal{H})$ be invertible and $\tilde{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$. Then $\tilde{A}$ is invertible if and only if $D - C A^{-1} B$ is invertible.*

Lemma 4. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then the following statements hold:*

- (1) $\mathcal{R}(A) \subseteq \mathcal{R}(A^{\frac{1}{2}})$ and $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^{\frac{1}{2}})}$, where $\overline{K}$ denotes the closure of K ;
- (2) $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$;
- (3) $\mathcal{R}(A) = \mathcal{H}$ if and only if A is invertible.

Proof. Let P and Q be two idempotents. By the discussion above, since $\lambda_1 P + \lambda_2 Q$ is invertible if and only if $\lambda_1 S^{-1} P S + \lambda_2 S^{-1} Q S$ is invertible, to consider the invertibility of $\lambda_1 P + \lambda_2 Q$, without loss of generality, we can assume that one of P and Q is an orthogonal projection. For example, assume that Q is an orthogonal projection. Of course, Q is a positive operator. In this case, by Lemma 2, P and Q have the following operator matrix forms:

$$P = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} Q_1 & Q_1^{\frac{1}{2}} D Q_2^{\frac{1}{2}} \\ Q_2^{\frac{1}{2}} D^* Q_1^{\frac{1}{2}} & Q_2 \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp$, where Q_1 and Q_2 are positive operators on $\mathcal{R}(P)$ and $\mathcal{R}(P)^\perp$, respectively, and D is a contraction operator from $\mathcal{R}(P)^\perp$ into $\mathcal{R}(P)$.

Suppose $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. If $\lambda_1 P + \lambda_2 Q$ is invertible, that is, the operator matrix

$$\lambda_1 P + \lambda_2 Q = \begin{pmatrix} \lambda_1 I + \lambda_2 Q_1 & \lambda_1 P_1 + \lambda_2 Q_1^{\frac{1}{2}} D Q_2^{\frac{1}{2}} \\ \lambda_2 Q_2^{\frac{1}{2}} D^* Q_1^{\frac{1}{2}} & \lambda_2 Q_2 \end{pmatrix}$$

is invertible, then $\mathcal{R}((\lambda_2 Q_2^{\frac{1}{2}} D^* Q_1^{\frac{1}{2}}, \lambda_2 Q_2)) = \mathcal{R}(P)^\perp$. By Lemma 4, $\mathcal{R}(Q_2) \subseteq \mathcal{R}(Q_2^{\frac{1}{2}})$ and observing that $\mathcal{R}((\lambda_2 Q_2^{\frac{1}{2}} D^* Q_1^{\frac{1}{2}}, \lambda_2 Q_2)) \subseteq \mathcal{R}(Q_2)^{\frac{1}{2}} \subseteq \mathcal{R}(P)^\perp$, then

$$\mathcal{R}(Q_2^{\frac{1}{2}}) = \mathcal{R}(P)^\perp.$$

By Lemma 4 again we have

$$\mathcal{R}(Q_2) = \mathcal{R}(P)^\perp.$$

This shows that Q_2 is invertible. In this case, by Lemma 3, $\lambda_1 P + \lambda_2 Q$ is invertible if and only if

$$\lambda_1 I + \lambda_2 Q_1 - (\lambda_1 P_1 Q_2^{-\frac{1}{2}} + \lambda_2 Q_1^{\frac{1}{2}} D) D^* Q_1^{\frac{1}{2}}$$

is invertible.

Since Q_1 is a positive contraction on $\mathcal{R}(P)$ and Q_2 is an invertible positive contraction on $\mathcal{R}(P)^\perp$, then Q_1 as an operator on $\mathcal{R}(P)$ and Q_2 as an operator on $\mathcal{R}(P)^\perp$ have the following operator matrix forms:

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q_{11} \end{pmatrix}, Q_2 = \begin{pmatrix} Q_{22} & 0 \\ 0 & I \end{pmatrix}$$

with respect to the space decomposition

$$\mathcal{R}(P) = \mathcal{N}(Q_1) \oplus \mathcal{N}(I - Q_1) \oplus (\mathcal{R}(P) \ominus (\mathcal{N}(Q_1) \oplus \mathcal{N}(I - Q_1)))$$

and the space decomposition

$$\mathcal{R}(P)^\perp = (\mathcal{R}(P)^\perp \ominus \mathcal{N}(I - Q_2)) \oplus \mathcal{N}(I - Q_2),$$

respectively.

Then denote $\mathcal{H}_0 = \mathcal{N}(Q_1)$, $\mathcal{H}_1 = \mathcal{N}(I - Q_1)$, $\mathcal{H}_2 = \mathcal{R}(P) \ominus (\mathcal{N}(Q_1) \oplus \mathcal{N}(I - Q_1))$, $\mathcal{H}_3 = \mathcal{R}(P)^\perp \ominus \mathcal{N}(I - Q_2)$ and $\mathcal{H}_4 = \mathcal{N}(I - Q_2)$, Q_{11} and $I - Q_{11}$ are injective positive contractions on $\mathcal{H}_2$ and Q_{22} is an invertible positive contraction on $\mathcal{H}_3$ with $1 \notin \sigma_p(Q_{22})$, where $\sigma_p(M)$ denotes the point spectrum of the operator M . In this case, P and Q have the following matrix representations:

$$(1) \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} & 0 \\ 0 & 0 & Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

and

$$(2) \quad P = \begin{pmatrix} I & 0 & 0 & P_{11} & P_{12} \\ 0 & I & 0 & P_{21} & P_{22} \\ 0 & 0 & I & P_{31} & P_{32} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = \bigoplus_{i=0}^4 \mathcal{H}_i$. If we let

$$Q_0 = \begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix},$$

then Q being an orthogonal projection implies that Q_0 is also an orthogonal projection on $\mathcal{H}_2 \oplus \mathcal{H}_3$. That is, $Q_0 = Q_0^2$. Hence we have

$$\begin{aligned} Q_0^2 &= \begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix} \\ &= \begin{pmatrix} Q_{11}^2 + Q_{11}^{\frac{1}{2}} D_1 Q_{22} D_1^* Q_{11}^{\frac{1}{2}} & Q_{11}^{\frac{3}{2}} D_1 Q_{22}^{\frac{1}{2}} + Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{3}{2}} \\ Q_{22}^{\frac{3}{2}} D_1^* Q_{11}^{\frac{1}{2}} + Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{3}{2}} & Q_{22}^2 + Q_{22}^{\frac{1}{2}} D_1^* Q_{11} D_1 Q_{22}^{\frac{1}{2}} \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} \\ Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} \end{pmatrix} = \begin{pmatrix} Q_{11}^2 + Q_{11}^{\frac{1}{2}} D_1 Q_{22} D_1^* Q_{11}^{\frac{1}{2}} & Q_{11}^{\frac{3}{2}} D_1 Q_{22}^{\frac{1}{2}} + Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{3}{2}} \\ Q_{22}^{\frac{3}{2}} D_1^* Q_{11}^{\frac{1}{2}} + Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{3}{2}} & Q_{22}^2 + Q_{22}^{\frac{1}{2}} D_1^* Q_{11} D_1 Q_{22}^{\frac{1}{2}} \end{pmatrix}.$$

Comparing the two sides of the above equation, we have

$$\begin{cases} Q_{11} = Q_{11}^2 + Q_{11}^{\frac{1}{2}} D_1 Q_{22} D_1^* Q_{11}^{\frac{1}{2}}, \\ Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} = Q_{11}^{\frac{3}{2}} D_1 Q_{22}^{\frac{1}{2}} + Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{3}{2}}, \\ Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} = Q_{22}^{\frac{3}{2}} D_1^* Q_{11}^{\frac{1}{2}} + Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{3}{2}}, \\ Q_{22} = Q_{22}^2 + Q_{22}^{\frac{1}{2}} D_1^* Q_{11} D_1 Q_{22}^{\frac{1}{2}}. \end{cases}$$

Observing that Q_{11} , $I_{\mathcal{H}_2} - Q_{11}$, Q_{22} and $I_{\mathcal{H}_3} - Q_{22}$ are injective, we get

$$(3) \quad \begin{cases} I_{\mathcal{H}_2} = Q_{11} + D_1 Q_{22} D_1^*, \\ D_1 = Q_{11} D_1 + D_1 Q_{22}, \\ D_1^* = Q_{22} D_1^* + D_1^* Q_{11}, \\ I_{\mathcal{H}_3} = Q_{22} + D_1^* Q_{11} D_1, \end{cases}$$

where $I_{\mathcal{H}_i}$ denotes the identity on $\mathcal{H}_i$, $i = 2, 3$. From the last of equations (3), we see that $I_{\mathcal{H}_3} - Q_{22} = D_1^* Q_{11} D_1$, and hence $Q_{11} D_1$ is injective. From the second of equations (3) we see that

$$(I_{\mathcal{H}_2} - Q_{11}) D_1 = D_1 Q_{22}.$$

On the other hand, from the first of equations (3) we get

$$(I_{\mathcal{H}_2} - Q_{11}) D_1 = D_1 Q_{22} D_1^* D_1.$$

Equating these two expressions for $(I_{\mathcal{H}_2} - Q_{11}) D_1$ and using the injectivity of $D_1 Q_{22}$ then leaves us with

$$I_{\mathcal{H}_3} = D_1^* D_1.$$

In a similar vein, from the second of equations (3) we have $D_1(I_{\mathcal{H}_3} - Q_{22}) = Q_{11} D_1$, while from the last of equations (3) we have $D_1(I_{\mathcal{H}_3} - Q_{22}) = D_1 D_1^* Q_{11} D_1$. Equating these two expressions for $D_1(I_{\mathcal{H}_3} - Q_{22})$ and using the injectivity of $Q_{11} D_1$ then gives

$$I_{\mathcal{H}_2} = D_1 D_1^*.$$

With this identity, the first of equations (3) can then be rewritten as

$$Q_{11} = D_1(I_{\mathcal{H}_3} - Q_{22}) D_1^*.$$

We have thus arrived at the system of equations

$$\begin{cases} D_1 D_1^* = I_{\mathcal{H}_2}, \\ D_1^* D_1 = I_{\mathcal{H}_3}, \\ Q_{11} = D_1(I_{\mathcal{H}_3} - Q_{22}) D_1^*. \end{cases}$$

Denote $\hat{Q} = Q_{11}$. Then

$$Q_0 = \begin{pmatrix} \hat{Q} & \hat{Q}^{\frac{1}{2}}(I - \hat{Q})^{\frac{1}{2}} D_1 \\ D_1^* \hat{Q}^{\frac{1}{2}}(I - \hat{Q})^{\frac{1}{2}} & D_1^*(I - \hat{Q}) D_1 \end{pmatrix}.$$

Now $\lambda_1 P + \lambda_2 Q$ has the following operator matrix form:

$$\begin{aligned} \lambda_1 P + \lambda_2 Q &= \lambda_1 \begin{pmatrix} I & 0 & 0 & P_{11} & P_{12} \\ 0 & I & 0 & P_{21} & P_{22} \\ 0 & 0 & I & P_{31} & P_{32} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &\quad + \lambda_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} & 0 \\ 0 & 0 & Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 I & 0 & 0 & \lambda_1 P_{11} & \lambda_1 P_{12} \\ 0 & (\lambda_1 + \lambda_2) I & 0 & \lambda_1 P_{21} & \lambda_1 P_{22} \\ 0 & 0 & \lambda_1 I + \lambda_2 \hat{Q} & \lambda_1 P_{31} + \lambda_2 \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{\frac{1}{2}} D_1 & \lambda_1 P_{32} \\ 0 & 0 & \lambda_2 D_1^* \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{\frac{1}{2}} & \lambda_2 D_1^* (I - \hat{Q})^{\frac{1}{2}} D_1 & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \end{aligned}$$

with respect to the space decomposition $\mathcal{H} = \bigoplus_{i=0}^4 \mathcal{H}_i$.

Denote

$$\bar{Q}_0 = \begin{pmatrix} \lambda_1 I + \lambda_2 \hat{Q} & \lambda_1 P_{31} + \lambda_2 \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{\frac{1}{2}} D_1 \\ \lambda_2 D_1^* \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{\frac{1}{2}} & \lambda_2 D_1^* (I - \hat{Q})^{\frac{1}{2}} D_1 \end{pmatrix}.$$

Obviously, if $\lambda_1 + \lambda_2 \neq 0$, the invertibility of the operator $\lambda_1 P + \lambda_2 Q$ on $\mathcal{H}$ is equivalent to the invertibility of $\bar{Q}_0$ on $\mathcal{H}_2 \oplus \mathcal{H}_3$. Moreover, by Lemma 3, that $\bar{Q}_0$ is invertible if and only if $I - \hat{Q}$ and

$$\lambda_1 I + \lambda_2 \hat{Q} - (\lambda_1 P_{31} + \lambda_2 \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{\frac{1}{2}} D_1) \lambda_2^{-1} D_1^* (I - \hat{Q})^{-1} D_1 \lambda_2 D_1^* \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{\frac{1}{2}}$$

are invertible. Noting that $D_1 D_1^* = I_{\mathcal{H}_2}$ and $\lambda_1 \lambda_2 \neq 0$, we have

$$\begin{aligned} &\lambda_1 I + \lambda_2 \hat{Q} - (\lambda_1 P_{31} + \lambda_2 \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{\frac{1}{2}} D_1) \lambda_2^{-1} D_1^* (I - \hat{Q})^{-1} D_1 \lambda_2 D_1^* \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{\frac{1}{2}} \\ &= \lambda_1 (I - P_{31} D_1^* \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{-\frac{1}{2}}). \end{aligned}$$

This shows that the invertibility of $\bar{Q}_0$ is only dependent on the invertibility of the operator $I - P_{31} D_1^* \hat{Q}^{\frac{1}{2}} (I - \hat{Q})^{-\frac{1}{2}}$ if both λ_1 and λ_2 are not zero; that is, the invertibility of $\bar{Q}_0$ is independent of the choice of λ_1 and λ_2 if both λ_1 and λ_2 are not zero.

In other words, the invertibility of $\lambda_1 P + \lambda_2 Q$ is independent of the choice of λ_1 and λ_2 if both λ_1 and λ_2 are not zero and $\lambda_1 + \lambda_2 \neq 0$. $\square$

Remark. By the proof of Theorem 1, if $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$, the invertibility of $\lambda_1 P + \lambda_2 Q$ is independent of the choice of λ_1 and λ_2 if both λ_1 and λ_2 are not zero.

The following consequence is immediate.

Corollary 5. *Let P and Q be two idempotents and $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$. Then the following statements hold:*

- (1) $P + Q$ is invertible if and only if $P - Q$ is invertible;
- (2) In addition, $\mathcal{R}(\tilde{P}) \cap \mathcal{R}(\tilde{Q}) = \{0\}$, and $P + Q$ is invertible if and only if $\tilde{P} + \tilde{Q}$ is invertible, where $\tilde{K} = I - K$.

Proof. (1) It is clear from the Remark above.

(2) Obviously, $\tilde{P}$ and $\tilde{Q}$ are idempotents. Since $\mathcal{R}(\tilde{P}) \cap \mathcal{R}(\tilde{Q}) = \{0\}$, $\tilde{P} + \tilde{Q}$ is invertible if and only if $\tilde{P} - \tilde{Q}$ is invertible by (1). But

$$\tilde{P} - \tilde{Q} = I - P - (I - Q) = -(P - Q),$$

so $\tilde{P} - \tilde{Q}$ is invertible if and only if $P - Q$ is invertible. By (1), $P - Q$ is invertible if and only if $P + Q$ is invertible. $\square$

Corollary 6. *Let P and Q be two orthogonal projections such that $P + Q$ is invertible and $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$. Then $P - Q, 1 - PQ, P + Q - PQ$ are all invertible.*

Proof. If $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$, then $\mathcal{H}_1 = \{0\}$. Observe that in the proof of Theorem 1, $P + Q$ has the following operator matrix:

$$\begin{aligned} P + Q &= \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Q_{11} & Q_{11}^{\frac{1}{2}} D_1 Q_{22}^{\frac{1}{2}} & 0 \\ 0 & Q_{22}^{\frac{1}{2}} D_1^* Q_{11}^{\frac{1}{2}} & Q_{22} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I + Q_{11} & Q_{11}^{\frac{1}{2}} (I - Q_{11})^{\frac{1}{2}} D_1 & 0 \\ 0 & D_1^* Q_{11}^{\frac{1}{2}} (I - Q_{11})^{\frac{1}{2}} & D_1^* (I - Q_{11})^{\frac{1}{2}} D_1 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \end{aligned}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{H}_0 \oplus (\bigoplus_{i=2}^4 \mathcal{H}_i)$. Then the invertibility of $P + Q$ implies that $I - Q_{11}$ is invertible. A direct calculation can show that $1 - PQ$ and $P + Q - PQ$ are all invertible. $\square$

ACKNOWLEDGMENT

A part of this paper was written while the first author visited the Department of Mathematics and Statistics of the University of New Hampshire during the summer of 2004. He would like to thank the faculty and administration of those units for their warm hospitality. In particular, the first author wants to express his gratitude to Professor Liming Ge for his useful suggestions and comments.

The authors wish to thank the referees for their valuable comments and suggestions.

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