REMARKS ON SPECTRA AND L^1 MULTIPLIERS FOR CONVOLUTION OPERATORS

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ABSTRACT. We prove that the spectrum of a convolution operator on a locally compact group G by a self-adjoint L^1 -function f is the same on $L^1(G)$ and $L^2(G)$ and consequently on all L^p spaces, $1 \leq p < \infty$, if and only if a Beurling algebra contains non-analytic functions on $\mathbb{R}$ operating on f into L^1 .

1. INTRODUCTION

Suppose B_1 and B_2 are two Banach spaces such that $B_0 \subset B_1 \cap B_2$ is dense in B_1 and in B_2 . Let

$$T : B_0 \rightarrow B_1 \cap B_2$$

be a linear operator such that

$$\|T\|_{B_1 \rightarrow B_1} < \infty \text{ and } \|T\|_{B_2 \rightarrow B_2} < \infty.$$

Spectral properties of T on B_1 and B_2 have been studied in various context, e.g. [1], [2], [6], [7], [15].

Let G be a locally compact group, and let $L^p(d_r x)$ be the space of p -integrable functions w.r.t. the right invariant Haar measure. Let $f = f^* \in L^1(d_r x)$ and let

$$T_f \xi = \xi * f, \quad \xi \in L^p(G), \quad 1 \leq p < \infty.$$

The spectra of convolution operators $T_f : B_1 \rightarrow B_1$ and $T_f : B_2 \rightarrow B_2$, $B_1 = L^1(d_r x)$ and $B_2 = L^2(d_r x)$ have been discussed for half a century. [23] is an interesting survey article that lists existing literature (as of 1976). Still, there is a host of recent papers on the subject and a number of questions of some importance that are open and seem to be tractable, e.g. [3], [7], [21], [9].

Let $\mathcal{H}$ be a Hilbert space, and let π be a unitary representation of G on $\mathcal{H}$. Suppose that the representation π is faithful on $L^1(d_r x)$ and

$$(1) \quad \text{Sp}_{L^1(d_r x)} T_f = \text{Sp}_{\mathcal{H}} \pi(f).$$

Equality (1) for all $f = f^* \in L^1(d_r x)$, by [25], is equivalent to the definition of what Gelfand called the *symmetry of $L^1(d_r x)$* over fifty years ago.

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Let $g = g^* \in L^1(d_r x)$. Then $\text{Sp}_{\mathcal{H}}\pi(g) \subset [a, b] \subset \mathbb{R}$. By the spectral theorem,

$$(2) \quad \pi(g)\xi = \int_a^b \lambda dE(\lambda)\xi, \quad \xi \in \mathcal{H}.$$

Let $\mathbf{A}$ be the algebra of functions on $[a, b]$ with absolutely convergent Fourier series, and let $\omega : \mathbb{Z} \ni n \rightarrow [1, \infty)$ be a function such that

$$(3) \quad \omega(k+l) \leq \omega(k) \cdot \omega(l).$$

Then

$$(4) \quad A(\omega) = \{F \in \mathbf{A} : \sum_{n \in \mathbb{Z}} |\widehat{F}(n)|\omega(n) = \|F\|_{A(\omega)} < \infty\},$$

where $\widehat{F}(n) = \frac{1}{b-a} \int_a^b F(t) e^{\frac{-2\pi i n t}{b-a}} dt$ is a Banach subalgebra of $\mathbf{A}$. If ω satisfies

$$(5) \quad \lim_{|n| \rightarrow \infty} \omega(n)^{\frac{1}{|n|}} = 1,$$

then $A(\omega)$ contains functions that have no holomorphic extension onto a neighborhood of $[a, b]$ in $\mathbb{C}$. We say that a function $F \in \mathbf{A}$ *operates* on f into $L^1(d_r x)$ if

$$(6) \quad F(f)\xi = \int_a^b F(\lambda) dE(\lambda)\xi = \xi * g, \quad \xi \in \mathcal{H},$$

for some $g \in L^1(d_r x)$. We then say that F is a $L^1(d_r x)$ *spectral multiplier* for f .

The aim of this note is to spell out the following observation.

Theorem 1.1. *Equality (1) is equivalent to the existence of a weight ω which satisfies (5), and every F in $A(\omega)$ is a $L^1(d_r x)$ spectral multiplier for f .*

The proof of this theorem uses more or less standard arguments from Banach $*$ -algebras theory that will be presented in Section 2 and yields similar facts for $L^1(G, \omega)$ discussed in [7].

2. BANACH $*$ -ALGEBRAS

Let X be a Banach $*$ -algebra, and let π be a faithful $*$ -representation of X into the algebra $\mathcal{L}(\mathcal{H})$ of linear bounded operators on a Hilbert space $\mathcal{H}$, and let $\widetilde{X}$ be the C^* -algebra that is the closure in the operator norm of $\pi(X)$, and let $f = f^*$ be an element of X . Of course,

$$(7) \quad \text{Sp}_{\mathcal{H}}\pi(f) = \text{Sp}_{\widetilde{X}}f.$$

A semi-simple Banach $*$ -algebra X is called *symmetric* if for every element $g = g^* \in X$, $\text{Sp}_X g \subset \mathbb{R}$, and this is equivalent ([22], [24], [25]), to the property that for all $g \in X$,

$$(8) \quad \text{Sp}_X g = \text{Sp}_{\widetilde{X}}g.$$

Moreover, if X_0 is a symmetric Banach $*$ -algebra, $X_0 \subset X$ and $g = g^* \in X_0$, then (8) holds.

If $f = f^*$, the spectrum $\text{Sp}_{\mathcal{H}}\pi(f)$ of the operator $\pi(f)$ is a closed subset of an interval $I \subset \mathbb{R}$. We say that a bounded measurable function $F : I \rightarrow \mathbb{C}$ operates on f into X if there is an element $g \in X$ such that

$$\pi(g)\xi = \int_{\text{Sp}_{\mathcal{H}}\pi(f)} F(\lambda)dE(\lambda)\xi, \quad \xi \in \mathcal{H}.$$

We then write $g = F(f)$. For a fixed element $f = f^* \in X$, the set of functions $F \in \mathbf{A}$ that operate on f forms an algebra under point-wise multiplication, denoted by $\langle f \rangle$.

Thus Theorem 1.1 follows from

Theorem 2.1. *Let X be a semi-simple Banach $*$ -algebra, and let $\tilde{X}$ be a C^* -algebra such that $X \subset \tilde{X}$. Let $f = f^*$ be a fixed element in X . Then the algebra $\langle f \rangle$ contains $A(\omega)$, ω satisfying (5) if and only if*

$$(9) \quad \text{Sp}_X f = \text{Sp}_{\tilde{X}} f.$$

For a weight ω assume that $A(\omega) \subset \langle f \rangle$. Let $\mathcal{A}_f^\omega = \{F(f) : F \in A(\omega)\}$. $\mathcal{A}_f^\omega$ contains $\mathcal{A}_f$ the Banach $*$ -algebra generated by f . Let $\mathfrak{M}$ be the Gelfand space of homomorphisms of $\mathcal{A}_f$ into $\mathbb{C}$, and for $g \in \mathcal{A}_f$ let $\hat{g}$ be the Gelfand transform of g .

Now we are going to prove the following

Lemma 2.2. *The following conditions are equivalent:*

- i) for each $F \in A(\omega)$ we have $F(f) \in X$,
- ii) there exists $C > 0$ such that $\|e^{inf}\|_X \leq C\omega(n)$ for $n \in \mathbb{Z}$.

Proof. i) $\Rightarrow$ ii) Let $T : A(\omega) \ni F \rightarrow F(f) \in X$. We are going to verify that the linear operator T is closed. Let $F_n \rightarrow F$ in $A(\omega)$ and let $g_n = F_n(f) \rightarrow g$ in X . Obviously F_n converges uniformly to F on $[a, b]$ and, by the spectral theorem, $\pi(g_n) = \int_a^b F_n(\lambda) dE(\lambda)$ converges to $\int_a^b F(\lambda) dE(\lambda)$ in the operator norm. On the other hand, since $\pi \rightarrow \mathcal{L}(\mathcal{H})$ is continuous, $\pi(g_n) \rightarrow \pi(g)$ in the operator norm. Hence $\pi(g) = \int_a^b F(\lambda) dE(\lambda)$. By the closed graph theorem, it follows that there exists a constant C , such that $\|T(F)\|_X \leq C\|F\|_{A(\omega)}$. Now, given ω , for every $n \in \mathbb{Z}$ the function $F(x) = e^{inx}$ belongs to $A(\omega)$, so since

$$e^{inf} = \sum_{k=0}^{\infty} \frac{(inf)^k}{k!} \in \mathcal{A}_f \subset \mathcal{A}_f^\omega,$$

$$\|e^{inf}\|_X \leq C\|e^{in\cdot}\|_{A(\omega)} = C\omega(n).$$

ii) $\Rightarrow$ i) Let $F \in A(\omega)$. Then

$$\begin{aligned} \|T(F)\|_X &= \|F(f)\|_X = \left\| \sum_{n \in \mathbb{Z}} \hat{F}(n) e^{inf} \right\|_X \leq \sum_{n \in \mathbb{Z}} |\hat{F}(n)| \cdot \|e^{inf}\|_X \\ &\leq C \cdot \sum_{n \in \mathbb{Z}} |\hat{F}(n)| \omega(n) < \infty. \end{aligned}$$

Hence $F(f) \in X$. □

Now to prove Theorem 2.1, by virtue of (7) and (8), it suffices to show

(a) if $A(\omega) \subset \langle f \rangle$ and ω satisfies (5), then $\mathcal{A}_f$ is symmetric,

and

(b) if $\mathcal{A}_f$ is symmetric, then there exists a weight ω that satisfies (5) and $A(\omega) \subset \langle f \rangle$.

To prove (a) we need to show that $\hat{f}(\lambda)$ is real for every $\lambda \in \mathfrak{M}$. The lemma says that

$$\max_{\lambda \in \mathfrak{M}} |e^{in\hat{f}(\lambda)}| \leq \|e^{inf}\|_{\mathcal{A}_f} \leq \omega(n)$$

so, if ω satisfies (5), $\hat{f}(\lambda)$ is real.

To prove (b) suppose $\mathcal{A}_f$ is symmetric. Then, by the spectral radius formula,

$$\|e^{inf}\|_{\mathcal{A}_f}^{\frac{1}{n}} \rightarrow \sup_{M \in \mathfrak{M}} |e^{i\hat{f}(M)}| = 1.$$

Hence, if we take $\omega(n) = \|e^{inf}\|_{\mathcal{A}_f}$, then ω satisfies (5) and by the lemma, $A(\omega) \subset \langle f \rangle$.

3. REMARKS AND PROBLEMS

Let G be a connected Lie group, let $\mathcal{G}$ be its Lie algebra and let $X_1, \dots, X_k \in \mathcal{G}$ be *left* invariant vector fields on G such that the smallest Lie subalgebra of the Lie algebra $\mathcal{G}$ of G that contains $X_1, \dots, X_k$ is $\mathcal{G}$. Let $\Delta = X_1^2 + \dots + X_k^2$, where Δ is called a sublaplacian. It is the infinitesimal generator of a convolution semigroup

$$p_s * p_t = p_{s+t}, \quad s, t \in \mathbf{R}^+, \quad p_t^* = p_t \in L^1(d_r x).$$

Also

the p_t 's are smooth and rapidly decreasing.

The operator Δ is essentially self-adjoint on $C_c^\infty(G) \subset L^2(d_r x)$; cf. e.g [8]. Also for all $1 \leq p < \infty$,

$$(10) \quad \mathrm{Sp}_{L^p(d_r x)} T_{p_t} = \overline{\exp[t \mathrm{Sp}_{L^p(d_r x)} \Delta]}.$$

For a connected Lie group G let $U = U^{-1}$ be an open neighborhood with compact closure. G is said to be of *polynomial growth* if

$$(11) \quad \int_{U^n} d_r x = O(n^k) \quad \text{for some } k \quad \text{as } n \rightarrow \infty.$$

If a connected Lie group is not of polynomial growth, then it is of *exponential growth*, i.e.

$$(12) \quad e^{\varepsilon n} \leq \int_{U^n} d_r x \leq e^{\varepsilon' n} \quad \text{for some } \varepsilon, \varepsilon' > 0.$$

For a group of polynomial growth there is a k such that C^k functions with compact support operate on T_{p_t} into $L^1(d_r x)$ for *all* sublaplacians [16], and was known somewhat earlier [15] that

$$(13) \quad \mathrm{Sp}_{L^p(G)} \Delta = \mathrm{Sp}_{L^2(G)} \Delta.$$

On the other hand, for exponentially growing groups it may happen that only functions that have holomorphic extensions in a neighborhood in $\mathbb{C}$ of a non-isolated

point of $\mathrm{Sp}_{\mathcal{H}}\Delta$ $\mathbf{R}$ operate on T_{p_t} into $L^1(d_r x)$, and one might think that this is the rule [4], [13]. Then also

$$(14) \quad \mathrm{Sp}_{L^1(d_r x)}\Delta \text{ is not contained in } \mathbf{R}.$$

But still for particular solvable Lie groups of exponential growth G and certain sublaplacians Δ , there exist $L^1(d_r x)$ multipliers of finite smoothness [5], [10], [11]. It is an open problem whether for some other sublaplacians Δ' on G (14) still holds.

At the other end, for discrete groups G , there has been some recent interest in spectral properties of convolution operators on $l^2(G)$, [9]. The classical theorem by H. Kesten [20] states that if (and only if) a group G is amenable, then for every symmetric *non-negative* function f on G such that

$$(15) \quad \bigcup_{n=1}^{\infty} (\mathrm{supp} f)^n = G,$$

we have

$$(16) \quad \max\{|\lambda| : \lambda \in \mathrm{Sp}_{l^2(G)}T_h\} = \max\{|\lambda| : \lambda \in \mathrm{Sp}_{l^1(G)}T_h\}$$

for all $h = f$. It is known that if (16) holds for *all* h in the $*$ -algebra generated by $f = f^* \in l^1(G)$, then

$$(17) \quad \mathrm{Sp}_{l^2(G)}T_f = \mathrm{Sp}_{l^1(G)}T_f$$

(see [15] and [7]). It is an open problem whether there exists an amenable group G and *non-negative* $f \in l^1(G)$ which satisfies (15) such that (17) fails. It seems very probable that there are amenable groups G and functions $f, \mu \in l^1(G)$ such that μ is non-negative, both satisfy (15) with (17) holding for μ , and failing for f . It is also likely that there are μ 's that satisfy (15) on such a group which admits $l^1(G)$ spectral multipliers of finite smoothness. It seems that an old conjecture formulated in [23] that all groups G for which $L^1(G)$ is symmetric must be amenable is still not resolved. In [14], an incomplete “proof” of this conjecture that has never been made correct is presented.

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