CHARACTERIZATION OF MODULES OF FINITE PROJECTIVE DIMENSION OVER COMPLETE INTERSECTIONS

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ABSTRACT. Let M be a finitely generated module over a local complete intersection R of characteristic $p > 0$. The property that M has finite projective dimension can be characterized by the vanishing of $\text{ext}_R^i(f^n R, M)$ for some $i > 0$ and for some $n > 0$.

Let (R, m, k) be a local ring of characteristic $p > 0$. The Frobenius endomorphism $f_R : R \rightarrow R$ is defined by $f_R(r) = r^p$ for $r \in R$. Each iteration f_R^n defines a new R -module structure on R , denoted by $f^n R$, for which $a \cdot b = a^{p^n} b$. For any R -module M , $F_R^n(M)$ will stand for $M \otimes_R f^n R$ and $\tilde{F}_R^n(M)$ will stand for $\text{Hom}_R(f^n R, M)$. Avramov and Miller [A-M] proved that over a local complete intersection ring, if a finitely generated R -module M satisfies $\text{Tor}_i^R(M, f^n R) = 0$ for some fixed $i, n > 0$, then it is of finite projective dimension. Later, Dutta [D] provided a simple proof of this result without using the notion of “complexity”. Using a similar method as in [D], we obtain another characterization of finitely generated modules of finite projective dimension over complete intersection rings.

Theorem. *Let M be a finitely generated module over a local complete intersection ring R . If for some $i, n > 0$, $\text{Ext}_R^i(f^n R, M) = 0$, then M has finite injective dimension over R .*

Proof. Without loss of generality we can assume that R is complete and the residue field k is perfect. Let $R = S/\mathbf{x}$, where S is a complete regular local ring of characteristic $p > 0$ and $\mathbf{x} = (x_1, \dots, x_r)$ is an ideal generated by an S -sequence $x_1, \dots, x_r$. Write $R_n = S/\mathbf{x}^{p^n}$. Note that with this notation, every R_n -module is also an R_l -module via the natural surjection $R_l \twoheadrightarrow R_n$ for all $l > n$.

Let $\tilde{f}^n = f_S^n \otimes_S R$ (base change of $f_S^n : S \rightarrow S$ along the S -algebra R). We know by Kunz’s Theorem [K, Theorem 3.3] that f_S^n is flat. Since k is perfect, f_S^n is module finite. It follows that $\tilde{f}^n : S/\mathbf{x} \rightarrow S/\mathbf{x}^{p^n}$ is also flat and module finite. Observe that the map $R \xrightarrow{f_R^n} R$ can be factored as

$$(1) \quad S/\mathbf{x} \xrightarrow{\tilde{f}^n} S/\mathbf{x}^{p^n} \xrightarrow{\eta_n} S/\mathbf{x}$$

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or

$$R \xrightarrow{\tilde{f}^n} R_n \xrightarrow{\eta_n} R,$$

where $\eta_n : S/\mathbf{x}^{p^n} \rightarrow S/\mathbf{x}$ is the natural surjection. Thus $f_R^n = \eta_n \cdot \tilde{f}^n$.

It follows from the adjointness of Hom and Tensor that for any R -module T ,

$$(2) \quad \text{Hom}_{R_n}(R, \text{Hom}_R(\tilde{f}^n R_n, T)) \simeq \text{Hom}_R(R \otimes_{R_n} \tilde{f}^n R_n, T) \simeq \text{Hom}_R(f^n R, T).$$

Let $I^\bullet$ be an injective resolution of M over R . Since $\tilde{f}^n$ is flat, $\text{Hom}_R(\tilde{f}^n R_n, I^\bullet)$ is an injective resolution of $\text{Hom}_R(\tilde{f}^n R_n, M)$ over R_n . Therefore, by (2)

$$\begin{aligned} \text{Ext}_R^j(f^n R, M) &= H^j(\text{Hom}_R(f^n R, I^\bullet)) \\ &\simeq H^j(\text{Hom}_{R_n}(R, \text{Hom}_R(\tilde{f}^n R_n, I^\bullet))) \\ &= \text{Ext}_{R_n}^j(R, \text{Hom}_R(\tilde{f}^n R_n, M)), \forall j \geq 0, \end{aligned}$$

but

$$\begin{aligned} \text{Hom}_R(\tilde{f}^n R_n, M) &= \text{Hom}_R(f^n S \otimes_s R, M) \\ &\simeq \text{Hom}_S(f^n S, M) \\ &= \tilde{F}_S^n(M). \end{aligned}$$

Hence, $\text{Ext}_R^j(f^n R, M) \simeq \text{Ext}_{R_n}^j(S/\mathbf{x}, \tilde{F}_S^n(M)), \forall j \geq 0$.

Next, we show $\text{Ext}_R^i(f^n R, M) = 0$ for some $i > 0$ implies $\text{Ext}_R^i(f^n R, M) = 0$ for all $i > 0$. Consider a filtration of $S/\mathbf{x}^{p^n}$ of the form

$$\begin{aligned} 0 \rightarrow K_1 \rightarrow S/\mathbf{x}^{p^n} \rightarrow S/\mathbf{x} \rightarrow 0, \\ 0 \rightarrow K_2 \rightarrow K_1 \rightarrow S/\mathbf{x} \rightarrow 0, \\ \vdots \\ 0 \rightarrow K_{t_n} \rightarrow K_{t_n-1} \rightarrow S/\mathbf{x} \rightarrow 0, \end{aligned}$$

where $K_{t_n} = S/\mathbf{x}$. By applying $\text{Hom}_{R_n}(-, \tilde{F}_S^n(M))$ to the above short exact sequences, we get the long exact sequences of Ext's. Since $\text{Ext}_{R_n}^i(S/\mathbf{x}, \tilde{F}_S^n(M)) = 0$, working from the last long exact sequence up to the first long exact sequence, we obtain that $\text{Ext}_{R_n}^{i+1}(S/\mathbf{x}, \tilde{F}_S^n(M)) = 0$. Similarly, one can also obtain that $\text{Ext}_{R_n}^{i-1}(S/\mathbf{x}, \tilde{F}_S^n(M)) = 0$ if $i > 1$. Repeating these processes, we get

$$\text{Ext}_{R_n}^i(S/\mathbf{x}, \tilde{F}_S^n(M)) = 0$$

for all $i > 0$.

Then, we show $\text{Ext}_R^i(f^n R, M) = 0$ implies $\text{Ext}_R^i(f^{n+1} R, M) = 0, \forall i > 0$. We need the following lemma.

Lemma. *Let $A \rightarrow B$ be a ring homomorphism such that B is a finitely generated free module over A . Let M, N be A -modules. Then, as A -modules,*

$$\text{Ext}_B^i(\text{Hom}_A(B, M), \text{Hom}_A(B, N)) \simeq \text{Hom}_A(B, \text{Ext}_A^i(M, N)).$$

Proof of the Lemma. Note that, as A -modules,

$$\begin{aligned} & \operatorname{Hom}_B(\operatorname{Hom}_A(B, M), \operatorname{Hom}_A(B, N)) \\ & \simeq \operatorname{Hom}_A(\operatorname{Hom}_A(B, M) \otimes_B B, N) \quad (\text{adjointness}) \\ & \simeq \operatorname{Hom}_A(\operatorname{Hom}_A(B, M), N) \\ & \simeq \operatorname{Hom}_A(M \otimes_A B, N) \quad (\text{since } B \text{ is } A\text{-free}) \\ & \simeq \operatorname{Hom}_A(B, \operatorname{Hom}_A(M, N)). \end{aligned}$$

This proves the case when $i = 0$. For $i > 0$, let $J^\bullet$ be an injective resolution of N . Then $\operatorname{Hom}_A(B, J^\bullet)$ is an injective resolution of $\operatorname{Hom}_A(B, N)$ over B since B is free over A . So, as A -modules,

$$\begin{aligned} & \operatorname{Ext}_B^i(\operatorname{Hom}_A(B, M), \operatorname{Hom}_A(B, N)) \\ & = H^i(\operatorname{Hom}_B(\operatorname{Hom}_A(B, M), \operatorname{Hom}_A(B, J^\bullet))) \\ & \simeq H^i(\operatorname{Hom}_A(B, \operatorname{Hom}_A(M, J^\bullet))) \\ & \simeq \operatorname{Hom}_A(B, H^i(\operatorname{Hom}_A(M, J^\bullet))) \quad (B \text{ is } A\text{-free}) \\ & = \operatorname{Hom}_A(B, \operatorname{Ext}_A^i(M, N)). \end{aligned} \quad \square$$

Now suppose for some fixed $i > 0$ (therefore for all $i > 0$ by the first part of the proof) that

$$\operatorname{Ext}_R^i(f^n R, M) = 0,$$

i.e.

$$\operatorname{Ext}_{R_n}^i(S/\mathbf{x}, \tilde{F}_S^n(M)) = 0.$$

This implies that

$$\operatorname{Hom}_{R_n}(\hat{f}R_{n+1}, \operatorname{Ext}_{R_n}^i(S/\mathbf{x}, \tilde{F}_S^n(M))) = 0,$$

where $\hat{f}$ denotes $f_S \otimes_S R_n : R_n \rightarrow R_{n+1}$ (base change). Since $\hat{f}R_{n+1}$ is free over R_n , by the Lemma, we have

$$\operatorname{Ext}_{R_{n+1}}^i(\operatorname{Hom}_{R_n}(\hat{f}R_{n+1}, S/\mathbf{x}), \operatorname{Hom}_{R_n}(\hat{f}R_{n+1}, \tilde{F}_S^n(M))) = 0,$$

i.e.

$$(3) \quad \operatorname{Ext}_{R_{n+1}}^i(\tilde{F}_S(S/\mathbf{x}), \tilde{F}_S^{n+1}(M)) = 0.$$

We claim that, as $S/\mathbf{x}^p$ -modules,

$$\tilde{F}_S(S/\mathbf{x}) \simeq S/\mathbf{x}^p.$$

To see this, first note that by adjointness

$$\begin{aligned} \tilde{F}_S(S/\mathbf{x}) &= \operatorname{Hom}_S(f_S, S/\mathbf{x}) \\ &\simeq \operatorname{Hom}_{S/\mathbf{x}}(S/\mathbf{x} \otimes_S f_S, S/\mathbf{x}) \\ &= \operatorname{Hom}_{S/\mathbf{x}}(\tilde{f}(S/\mathbf{x}^p), S/\mathbf{x}), \end{aligned}$$

where $\tilde{f}$ is the map $\tilde{f}^n$ in the factorization (1) with $n = 1$. Then observe that since $S/\mathbf{x}$ and $S/\mathbf{x}^p$ are complete intersections and $S/\mathbf{x}^p$ is a finitely generated $S/\mathbf{x}$ -module via $\tilde{f}$, $\operatorname{Hom}_{S/\mathbf{x}}(\tilde{f}(S/\mathbf{x}^p), S/\mathbf{x})$ is a canonical module for $S/\mathbf{x}^p$ and hence is isomorphic to $S/\mathbf{x}^p$ as an $S/\mathbf{x}^p$ -module.

It follows that $\tilde{F}_S(S/\mathbf{x})$ is isomorphic to $S/\mathbf{x}^p$ as an R_{n+1} -module via the natural map $R_{n+1} \twoheadrightarrow R_1$ (i.e. the natural map $S/\mathbf{x}^{p^{n+1}} \twoheadrightarrow S/\mathbf{x}^p$).

Replacing $\tilde{F}_S(S/\mathbf{x})$ in (3) by $S/\mathbf{x}^p$, we obtain

$$(4) \quad \text{Ext}_{R_{n+1}}^i(S/\mathbf{x}^p, \tilde{F}_S^{n+1}(M)) = 0.$$

Consider the following short exact sequences:

$$(5) \quad 0 \rightarrow S/(x_1, x_2^p, \dots, x_r^p) \xrightarrow{\lambda=x_1^p} S/(x_1^{p+1}, x_2^p, \dots, x_r^p) \rightarrow S/\mathbf{x}^p \rightarrow 0$$

and

$$(6) \quad 0 \rightarrow S/\mathbf{x}^p \xrightarrow{x_1} S/(x_1^{p+1}, x_2^p, \dots, x_r^p) \xrightarrow{\mu} S/(x_1, x_2^p, \dots, x_r^p) \rightarrow 0.$$

Apply $\text{Hom}_{R_{n+1}}(-, \tilde{F}_S^{n+1}(M))$ to (5). From the associated long exact sequence and (4), we obtain an isomorphism (induced by λ)

$$(7) \quad \begin{aligned} & \text{Ext}_{R_{n+1}}^i(S/(x_1, x_2^p, \dots, x_r^p), \tilde{F}_S^{n+1}(M)) \\ & \simeq \text{Ext}_{R_{n+1}}^i(S/(x_1^{p+1}, x_2^p, \dots, x_r^p), \tilde{F}_S^{n+1}(M)). \end{aligned}$$

Similarly, applying $\text{Hom}_{R_{n+1}}(-, \tilde{F}_S^{n+1}(M))$ to (6), we get a map (induced by μ)

$$(8) \quad \begin{aligned} & \text{Ext}_{R_{n+1}}^i(S/(x_1, x_2^p, \dots, x_r^p), \tilde{F}_S^{n+1}(M)) \\ & \rightarrow \text{Ext}_{R_{n+1}}^i(S/(x_1^{p+1}, x_2^p, \dots, x_r^p), \tilde{F}_S^{n+1}(M)), \end{aligned}$$

which is an isomorphism for $i \geq 2$ and a surjection for $i = 1$.

The composition of maps λ in (5) and μ in (6) is the following multiplication:

$$S/(x_1, x_2^p, \dots, x_r^p) \xrightarrow{x_1^p} S/(x_1, x_2^p, \dots, x_r^p),$$

which is a 0-map. By (7) and (8), it induces

$$\text{Ext}_{R_{n+1}}^i(S/(x_1, x_2^p, \dots, x_r^p), \tilde{F}_S^{n+1}(M)) \xrightarrow{x_1^p} \text{Ext}_{R_{n+1}}^i(S/(x_1, x_2^p, \dots, x_r^p), \tilde{F}_S^{n+1}(M)),$$

which is an isomorphism for $i \geq 2$ and a surjection for $i = 1$. Hence

$$\text{Ext}_{R_{n+1}}^i(S/(x_1, x_2^p, \dots, x_r^p), \tilde{F}_S^{n+1}(M)) = 0.$$

Repeating this process $(r-1)$ times, we obtain

$$\text{Ext}_{R_{n+1}}^i(S/\mathbf{x}, \tilde{F}_S^{n+1}(M)) = 0,$$

i.e.

$$\text{Ext}_R^i(f^{n+1}R, M) = 0$$

for all $i > 0$.

Therefore by induction, $\text{Ext}_R^i(f^k R, M) = 0$ for all $k \geq n$ and all $i > 0$.

Finally, the assertion in the Theorem follows immediately from the following result due to Herzog ([H], Theorem 5.2): If M is a finitely generated module over R , then M is of finite injective dimension if and only if $\text{Ext}_R^i(f^n R, M) = 0$ for all $i > 0$ and infinitely many n . $\square$

Remark. The above theorem actually characterizes finitely generated modules with finite projective dimension also, since it is well known that over Gorenstein ring, a finitely generated module is of finite projective dimension if and only if it is of

finite injective dimension. To sum up, if M is a finitely generated module over a local complete intersection R , then the following are equivalent:

- (i) $\text{proj dim}_R M < \infty$,
- (ii) $\text{Tor}_i^R(M, f^n R) = 0$ for some fixed (all) $i, n > 0$,
- (iii) $\text{Ext}_R^i(f^n R, M) = 0$ for some fixed (all) $i, n > 0$,
- (iv) $\text{inj dim}_R M < \infty$.

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