ADJUGATES IN BANACH ALGEBRAS

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(Communicated by David R. Larson)

ABSTRACT. A simple formula for the adjugate of a block triangle offers an alternative route to the determinant theory for Banach algebras.

Suppose A is a complex linear algebra, with identity 1 and invertible group A^{-1} . The *radical* of A ,

$$(0.1) \quad \text{Rad}(A) = \{a \in A : 1 - Aa \subseteq A^{-1}\} = \{a \in A : 1 - aA \subseteq A^{-1}\},$$

coincides with the intersection of the maximal left ideals, and also of the maximal right ideals. When

$$(0.2) \quad \text{Rad}(A) = \{0\}$$

the algebra A is described as *semisimple*. The *spectrum* of an element $a \in A$ is given by

$$(0.3) \quad \sigma(a) = \sigma_A(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}\};$$

we shall also write

$$(0.4) \quad \sigma'(a) = \sigma'_A(a) = \sigma(a) \setminus \{0\}$$

for the nonzero spectrum. The nonzero spectrum offers a definition of “rank”: we set ([11], [18])

$$(0.5) \quad \text{rank}(a) = \text{rank}_A(a) = \sup_{x \in A} \# \sigma'(xa) = \sup_{x \in A} \# \sigma'(ax) \leq \infty.$$

If in particular A is semisimple and a Banach algebra, then ([21], [19], [11])

$$(0.6) \quad \text{rank}(a) \leq 1 \iff aAa \subseteq \mathbf{C}a.$$

Thus, if $\text{rank}(a) \leq 1$, then there is a bounded linear functional $\tau_a \in A^*$ for which

$$(0.7) \quad L_a R_a = \tau_a \odot a : x \mapsto \tau_a(x)a.$$

It is clear that if $a \neq 0$, then τ_a is uniquely determined; obviously we take $\tau_0 = 0$. Certainly if (0.7) holds, then

$$(0.8) \quad x \in A \implies \sigma(xa) \cup \sigma(ax) \subseteq \{0, \tau_a(x)\}.$$

When A is semisimple there are three equivalent definitions of the *socle*: the sum of the minimal left ideals, the sum of the minimal right ideals, and the linear subspace generated by the “rank one” elements. These in turn essentially reduce

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Received by the editors December 10, 2004.

2000 *Mathematics Subject Classification*. Primary 46H05; Secondary 15A15.

Key words and phrases. Banach algebras, Fredholm theory, adjugate, determinant, socle.

The first author was supported by Enterprise Ireland Basic Research Grant SC/2002/0266.

to rank one idempotents. The socle $\text{Soc}(A)$ provides the natural basis for Fredholm theory [4] in A , if we declare the “Fredholm” elements $a \in A$ to be those which are invertible modulo the socle:

$$(0.9) \quad A_{ess}^{-1} = \{a \in A : a + \text{Soc}(A) \in (A/\text{Soc}(A))^{-1}\} .$$

Since ([3], [18], [6])

$$(0.10) \quad \text{Soc}(A) \subseteq \overline{A} \equiv \{a \in A : a \in aAa\}$$

the socle is a two-sided ideal consisting entirely of elements which have “generalized inverses”, it turns out [8] that this “algebraic” concept of Fredholm element can also be made to look like the “spatial” version familiar when $A = B(X)$ consists of operators on a Banach space:

$$(0.11) \quad A_{ess}^{-1} = \{a \in \overline{A} : L_a^{-1}(0) \cup R_a^{-1}(0) \subseteq \text{Soc}(A)\} .$$

Aupetit and Mouton [3] have noted how the socle also provides a basis for the extension to Banach algebras of the concepts of *determinant* and *trace*. In this note we offer a rather more pedestrian route to this determinant, using the Aupetit/Mouton formula for finite-dimensional algebras and extending by means of the “adjugate” or *cofactor* of suitable Banach algebra elements, together with a certain block triangle construction. We also extend the trace, and follow Murphy [20] in the use of “Fedosov’s formula” to use it to reach the Fredholm *index* for elements invertible modulo the socle. The reason that the three kinds of socle coincide can be traced to an “orthogonalization” of projections ([4, Lemma F.1.6]):

1. Lemma. *If $p = p^2 \in A$ and if $q = q^2 \in \text{Soc}(A)$, then there is $r = r^2 \in \text{Soc}(A)$ for which*

$$(1.1) \quad pA + qA = pA + rA \text{ with } pr = 0 = rp .$$

Proof. Since $(1-p)q \in \text{Soc}(A) \subseteq \overline{A}$, there is $x \in A$ for which

$$(1.2) \quad (1-p)q = rq \text{ with } r = (1-p)x(1-p)q .$$

Evidently

$$r^2 = r \text{ and } pr = 0 = rp ,$$

and if $a \in A$, then

$$qa = pqa + rqa ; \quad ra = -pqx(1-p)a + qx(1-p)a. \quad \square$$

We are of course being a little disingenuous in Lemma 1, since the orthogonalization is behind the proof ([3, Corollary 2.10]) of the regularity (0.10). We might remark on a curious converse ([18], [6]) to (0.10): any ideal $J \subseteq \overline{A}$ is a subset of the socle. This exactly parallels Smyth’s observation [22] about ideals of algebraic elements. The essential ingredient in the Aupetit/Mouton recipe for the determinant and the trace is their concept of rank (0.5). If $\lambda \notin \text{acc } \sigma(a)$ is at worst an isolated point of the spectrum of $a \in A$, then there is a Riesz projection

$$(1.3) \quad \pi_a(\lambda) = \frac{1}{2\pi i} \oint_{\lambda} (z - a)^{-1} dz .$$

$1 - \pi_a(\lambda)$ can be obtained from $a - \lambda$ by multiplying it by what is essentially its “Drazin inverse”: Koliha [16] has noticed that $q = \pi_a(\lambda)$ can be characterized, with $c = a - \lambda$, by

$$(1.4) \quad q = q^2 ; \quad qc = cq ; \quad c + q \in A^{-1} ; \quad 1 + Ccq \subseteq A^{-1} .$$

The Koliha-Drazin inverse and the spectral projection of course double commute with $a \in A$: using either (1.3) or pure algebra ([7, Lemma 1])

$$(1.5) \quad bv = va \implies \pi_b(\lambda)v = v\pi_a(\lambda) .$$

2. Definition ([3]). If A is a semisimple complex Banach algebra and $a \in \text{Soc}(A)$, then

$$(2.1) \quad \det(1+a) \equiv \det_A(1+a) = \prod_{\lambda \in \sigma(a)} (1+\lambda)^{\nu_a(\lambda)}$$

and

$$(2.2) \quad \text{tr}(a) \equiv \text{tr}_A(a) = \sum_{\lambda \in \sigma(a)} \lambda \nu_a(\lambda) ,$$

where the multiplicity of $\lambda \in \sigma(a)$ is given by

$$(2.3) \quad \nu_a(\lambda) = \text{rank } \pi_a(\lambda) \equiv \text{rank}_A \pi_a(\lambda) .$$

The multiplicity (2.3) of $\lambda \in \sigma(a)$ is well defined, since

$$(2.4) \quad a \in \text{Soc}(A) \implies \sigma(a) = \text{iso } \sigma(a) .$$

Via Jacobson's lemma

$$(2.5) \quad \text{rank}(ab) = \text{rank}(ba)$$

for arbitrary $a, b \in A$, since only the nonzero spectrum σ' contributes to the determinant and the trace. Note that if $a \in A \subseteq B$, then

$$(2.6) \quad \text{iso } \sigma_A(a) \subseteq \partial \sigma_A(a) \subseteq \sigma_B(a) \subseteq \sigma_A(a) .$$

Thus in particular if $a \in \text{Soc}(A)$, then its spectrum is unchanged when referred to a superalgebra $B \supseteq A$. However the formula (0.5) shows only that there is inequality

$$(2.7) \quad \text{rank}_A(a) \leq \text{rank}_B(a) .$$

Thus the determinant and the trace may change with the algebraic environment.

For a specific example embed the upper triangular 2×2 matrices via the left regular representation $a \mapsto L_a$ in the algebra B of all 3×3 matrices: with

$$(2.8) \quad \varphi(a) = u \quad \text{and} \quad \psi(a) = v \quad \text{when} \quad a = \begin{pmatrix} u & w \\ 0 & v \end{pmatrix}$$

we find

$$(2.9) \quad \det_B(L_a) = \varphi(a)\det_A(a) ; \quad \text{tr}_B(L_a) = \varphi(a) + \text{tr}_A(a) .$$

If instead $B = A^2$ and $Ta = (a, a)$, then

$$(2.10) \quad \det_B(Ta) = \det_A(a)^2 ; \quad \text{tr}_B(Ta) = 2\text{tr}_A(a) .$$

In a general semisimple Banach algebra the determinant and the trace are not defined for the same elements; the trace lives on the ideal $\text{Soc}(A)$ while the determinant is defined for elements of the coset $1 + \text{Soc}(A)$. This would appear to rule out anything like a "Cayley-Hamilton theorem". If $a = aba \in \text{Soc}(A)$, then the products $q = ab$ and $p = ba$ are both idempotent and in the socle:

$$(2.11) \quad q = q^2 = ab \in \text{Soc}(A) ; \quad p = p^2 = ba \in \text{Soc}(A) ; \quad a = ap = qa .$$

Moreover if (2.11) holds, then

$$(2.12) \quad qAq \subseteq \overline{A} , \quad \text{hence} \quad qAq \subseteq \overline{qAq} ,$$

which by an old theorem of Kaplansky ([15], [23]) ensures that qAq is finite dimensional:

$$(2.13) \quad \dim(qAq) < \infty .$$

Thus to reach an “adjugate” for elements $a \in 1 + \text{Soc}(A)$ we can begin by doing the job in the finite-dimensional algebra qAq , where the Cayley-Hamilton theorem does apply:

3. Definition. If $a \in A$ for a finite-dimensional algebra A we define

$$(3.1) \quad \text{adj}(a) \equiv \text{adj}_A(a) = \triangle_a^A(a) \equiv \triangle_a(a) ,$$

where, identically on $\mathbf{C}$,

$$(3.2) \quad \det_A(a - z) \equiv \det(a - z) \equiv \det(a) - z\triangle_a(z) .$$

Here of course the determinant is given by the Aupetit/Mouton formula (2.1). Granted the determinant and the adjugate for finite dimensional algebras, it is an easy block triangularization to arrive at the general case:

4. Theorem. If $q = q^2 \in A$, then

$$(4.1) \quad A \leftrightarrow \begin{pmatrix} qAq & qA(1-q) \\ (1-q)Aq & (1-q)A(1-q) \end{pmatrix} .$$

If

$$(4.2) \quad a = qa \in A \text{ with } q = q^2 \in \text{Soc}(A),$$

then

$$(4.3) \quad a \leftrightarrow \begin{pmatrix} aq & a(1-q) \\ 0 & 0 \end{pmatrix} , \quad 1+a \leftrightarrow \begin{pmatrix} q+aq & a(1-q) \\ 0 & 1-q \end{pmatrix} ,$$

$$(4.4) \quad \text{adj}_A(1+a) \leftrightarrow \begin{pmatrix} \text{adj}_{qAq}(q+aq) & -\text{adj}_{qAq}(q+aq) \cdot a(1-q) \\ 0 & \det_{qAq}(q+aq) \cdot (1-q) \end{pmatrix} ,$$

$$(4.5) \quad \det_A(1+a) = \det_{qAq}(q+aq)$$

and

$$(4.6) \quad \text{tr}_A(a) = \text{tr}_{qAq}(aq) .$$

Proof. Generally if

$$(4.7) \quad a = \begin{pmatrix} d & m \\ 0 & b \end{pmatrix} \in A = \begin{pmatrix} D & M \\ 0 & B \end{pmatrix} ,$$

then ([12, Theorem 6])

$$(4.8) \quad \partial(\sigma_D(d) \cup \sigma_B(b)) \subseteq \sigma_A(a) \subseteq \sigma_D(d) \cup \sigma_B(b) ,$$

so that if d and b have spectra consisting of isolated points,

$$(4.9) \quad \sigma_A(a) = \sigma_D(d) \cup \sigma_B(b) \text{ and } \sigma'_A(a) = \sigma'_D(d) \cup \sigma'_B(b) .$$

This means (0.5) that

$$(4.10) \quad \text{rank}_A(a) = \text{rank}_D(d) + \text{rank}_B(b) .$$

This gives (4.5) and (4.6), and hence also (4.4) provided $1+a \in A$ is not a zero divisor. As observed by Kovacs/Silver/Williams ([17], [13]) we can embed the algebra A in the ring $A[z] = A \otimes \text{Poly}$ of polynomials with A coefficients, or consider the “Zarisky topology”, to obtain (4.4) generally. $\square$

Generally when (4.7) holds and $d \in 1 + \text{Soc}(D) \subseteq D$, $b \in 1 + \text{Soc}(B) \subseteq B$, then, writing

$$\text{adj}_D(d) = d^\sim, \det_D(d) = |d|, \text{adj}_B(b) = b^\sim, \det_B(b) = |b|,$$

we find

$$(4.11) \quad \text{adj}_A(a) = \begin{pmatrix} |b|d^\sim & -d^\sim mb^\sim \\ 0 & |d|b^\sim \end{pmatrix} \in A = \begin{pmatrix} D & M \\ N & B \end{pmatrix}.$$

If $a \in \text{Soc}(A)$, then by (0.10) there is a projection $q = q^2$ for which (4.2) holds, namely $q = ab$ as in (2.11). It is clear from Theorem 4 that the right-hand sides of (4.4), (4.5) and (4.6) are independent of the particular $q = q^2$ satisfying (4.2). A direct proof uses the formula (4.11) twice:

5. Theorem. *Suppose*

$$(5.1) \quad a = qa \text{ with } q = q^2 = rq \in \text{Soc}(A) \text{ and } r = r^2 \in \text{Soc}(A).$$

Then, with } B = rAr \text{ and } D = qAq,

$$(5.2) \quad \begin{pmatrix} \text{adj}_D(q + aq) & -\text{adj}_D(q + aq) \cdot a(1 - q) \\ 0 & \det_D(q + aq) \cdot (1 - q) \end{pmatrix} \\ \leftrightarrow \begin{pmatrix} \text{adj}_B(r + ar) & -\text{adj}_B(r + ar) \cdot a(1 - r) \\ 0 & \det_B(r + ar) \cdot (1 - r) \end{pmatrix},$$

$$(5.3) \quad \det_D(q + aq) = \det_B(r + ar)$$

and

$$(5.4) \quad \text{tr}_D(aq) = \text{tr}_B(ar).$$

Proof. If, generalizing (4.7),

$$a = \begin{pmatrix} d & m & k \\ 0 & c & \ell \\ 0 & 0 & b \end{pmatrix} \in \begin{pmatrix} D & M & K \\ M' & C & L \\ K' & L' & B \end{pmatrix},$$

then in two different ways, applications of (4.11) give, writing $\text{adj}_B(b) = b^\sim$ and $\det_D(d) = |d|$, etc.,

$$(5.5) \quad \text{adj}_A \begin{pmatrix} d & m & k \\ 0 & c & \ell \\ 0 & 0 & b \end{pmatrix} = \begin{pmatrix} |c||b|d^\sim & -d^\sim m|b|c^\sim & d^\sim(mc^\sim\ell - k|c|)d^\sim \\ 0 & |a||b|c^\sim & -|d|c^\sim\ell b^\sim \\ 0 & 0 & |d||c|b^\sim \end{pmatrix}.$$

This would give (5.2) provided there was inclusion $qAq \subseteq rAr$, which would need commutivity $qr = rq$. Note however that if (5.1) holds, then qr and $(1 - q)r$ are both idempotent; thus

$$(5.6) \quad a \in qrAq \subseteq rAr \cap qAq.$$

We may therefore apply (5.5) twice, with $D = qrAq$ and $\begin{pmatrix} D & M \\ M' & C \end{pmatrix}$ either qAq or rAr . $\square$

If $a \in A$ is Fredholm with respect to the socle, in the sense (0.9), then there are $b', b'' \in A$ for which

$$(5.7) \quad \{1 - b'a, 1 - ab''\} \subseteq \text{Soc}(A);$$

necessarily $b'' - b' \in \text{Soc}(A)$. In view of (0.10) we can arrange that $b'' = b' = b$ with $a = aba$. Following Murphy [20] we can therefore make the following definition.

6. Definition. If $a \in A$ is Fredholm and b', b'' satisfy (5.7), then

$$(6.1) \quad \text{index}(a) = \text{tr}(ab'' - b'a) .$$

To see that (6.1) is independent of the particular choice of b', b'' satisfying (5.7), we need to know that

$$(6.2) \quad a \in A, d \in \text{Soc}(A) \implies \text{tr}(ad - da) = 0 .$$

While this probably follows from the proof of Theorem 3.3 of Aupetit/Mouton [3], we can see it here first for finite-dimensional algebras and then generally, using the argument of Theorem 3 above. Alternatively it is the Jacobson lemma that $\sigma'(ad) = \sigma'(da)$, and we just have to check (cf. (1.5); [6], Lemma 2)) that for each point $\lambda \in \sigma'(ad)$ there is equal multiplicity

$$(6.3) \quad \nu_{da}(\lambda) = \nu_{ad}(\lambda) .$$

As noted by Murphy [20], the index enjoys familiar properties:

7. Theorem. If $a \in A$ is Fredholm, then

$$(7.1) \quad \text{index}(a + d) = \text{index}(a) \text{ if } d \in \text{Soc}(A),$$

and, if b', b'' satisfy (5.7),

$$(7.2) \quad \text{index}(b') = \text{index}(b'') = -\text{index}(a) .$$

If a, a' are Fredholm, then the logarithmic law holds:

$$(7.3) \quad \text{index}(a'a) = \text{index}(a') + \text{index}(a) .$$

The index is locally constant: if $\{1 - ba, 1 - ab\} \subseteq \text{Soc}(A)$, then

$$(7.4) \quad 1 + (a' - a)b \in A^{-1} \implies \text{index}(a') = \text{index}(a) .$$

Proof. Most of this is word for word as in Murphy [20]. For (7.1) we argue ([20], Theorem 2.3) that

$$\text{index}(a + d) = \text{tr}((a + d)b - b(a + d)) = \text{tr}(ab - ba) = \text{index}(a) + 0 .$$

For (7.3) we argue ([20], Theorem 2.4) that

$$\begin{aligned} \text{index}(a'a) &= \text{tr}(a'abb' - bb'a'a) \\ &= \text{tr}(a'(1 + d)b' - b'a' + b'a' - bb'a'a) \\ &= \text{tr}(a'b' - b'a') + \text{tr}(a'db') + \text{tr}(b'a' - bb'a'a) \\ &= \text{index}(a') + \text{tr}(b'a'(1 + d) - bb'a'a) \\ &= \text{index}(a') + \text{tr}((b'a'a)b - b(b'a'a)) \\ &= \text{index}(a') + \text{index}(b'a'a) \\ &= \text{index}(a') + \text{index}((1 + d')a) \end{aligned}$$

with $d = 1 - ab$ and $d' = 1 - b'a'$ in $\text{Soc}(A)$. For (7.4) we argue ([20], Theorem 2.5) that

$$c = 1 + (a' - a)b \in A^{-1} \implies 0 = \text{index}(a'b - d) = \text{index}(a'b) = \text{index}(a') + \text{index}(b),$$

giving $\text{index}(a') = -\text{index}(b) = \text{index}(a)$. Finally (7.4) also gives (7.2). $\square$

Since evidently the socle and its scalar perturbations are contained in the closure of the invertibles, it is clear ([9], [10], (7.3.4.4)) that

$$(7.7) \quad A^{-1} + \text{Socle}(A) \subseteq \overline{A} \equiv \{a \in A : a \in aA^{-1}a\} .$$

We thus obtain a familiar decomposition:

8. Theorem. *If $a \in A$ the following two conditions are equivalent:*

$$(8.1) \quad a \in A^{-1} + \text{Soc}(A),$$

$$(8.2) \quad a \in \overline{A} \cap A_{ess}^{-1} .$$

Each of them imply

$$(8.3) \quad a \in A_{ess}^{-1} \text{ with } \text{index}(a) = 0 .$$

Proof. For implication (8.1) $\implies$ (8.2) argue

$$a = c^{-1} + d \in A^{-1} + \text{Soc}(A) \implies ca \in 1 + \text{Soc}(A) \subseteq \overline{A} \cap A_{ess}^{-1} .$$

Conversely if (8.2) holds, then ([8] Theorem 2.2; [10] Theorem 7.8.2), using (0.11), there is $b \in A$ for which

$$b \in A^{-1} ; 1 - ba \in \text{Soc}(A) ; 1 - ab \in \text{Soc}(A) .$$

Now $b^{-1} - a = b^{-1}(1 - ba) \in \text{Soc}(A)$. For implication (8.2) $\implies$ (8.3) argue

$$\begin{aligned} a &= aba \in A_{ess}^{-1} \text{ with } b \in A^{-1} \\ \implies \text{index}(a) &= \text{index}(a) + \text{index}(b) + \text{index}(a) = 2 \text{index}(a) . \quad \square \end{aligned}$$

The condition (8.3) is in general weaker than (8.1) and (8.2): for example take

$$(8.4) \quad a = \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix} \in \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \text{ with } \text{index}(b') = 1 = -\text{index}(b) .$$

The socle can be extended to the larger ideal of “inessential” elements,

$$(8.5) \quad \text{Hull Soc}(A) = \{d \in A : d + \text{Soc}(A) \in \text{Rad}(A/\text{Soc}(A))\} ,$$

which gives the same “Fredholm theory”:

9. Theorem. *If $a \in A$, then*

$$(9.1) \quad a + \text{Hull Soc}(A) \in (A/\text{Hull Soc}(A))^{-1} \implies a \in A_{ess}^{-1} .$$

There is inclusion

$$(9.2) \quad A^{-1} + \text{Hull Soc}(A) \subseteq \overline{A}$$

and implication

$$(9.3) \quad d \in \text{Hull Soc}(A) \implies \text{index}(a + d) = \text{index}(a) .$$

Proof. The implication (9.1) holds ([10], Theorem 7.2.6; [8], (2.5.11)) separately for left and for right invertibility: if $a \in A$ is left invertible modulo the Hull of the socle, then there is $b \in A$ for which $1 - ba \in \text{Hull Soc}(A)$ and hence $c \in A$ for which

$$1 - cba = 1 - c(1 - (1 - ba)) \in \text{Soc}(A) .$$

Inclusion (9.2) follows easily ([10], Theorem 7.3.3; [8], (2.5.12)), using lemmas of Atkinson. For (9.3) suppose $d \in \text{Hull Soc}(A)$. Then $1 - Ad \cup 1 - dA \subseteq A_{ess}^{-1}$, and if $\{1 - ab, 1 - ba\} \subseteq \text{Soc}(A)$, then there are $b', b'' \in A$ for which

$$\{b'(1 - bd), (1 - db)b', b''(1 + b'bd), (1 + dbb')b''\} \subseteq \text{Soc}(A) ,$$

giving

$$\text{index}(a+d) = \text{tr}((a+d)b - b(a+d)) = \text{tr}(ab - ba) + \text{tr}(db - bd) = \text{index}(a) + \text{tr}(b'' - b').$$

□

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