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SEMICONJUGACIES TO ANGLE-DOUBLING

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ABSTRACT. A simple consequence of a theorem of Franks says that whenever a continuous map, g, is homotopic to angle-doubling on the circle, it is semiconjugate to it. We show that when this semiconjugacy has one disconnected point inverse, then the typical point in the circle has a point inverse with uncountably many connected components. Further, in this case the topological entropy of g is strictly larger than that of angle-doubling, and the semiconjugacy has unbounded variation. An analogous theorem holds for degree-D circle maps with $D > 2$.

1. INTRODUCTION

The angle-doubling map, d, on the circle, $S^1 := \mathbb{R}/\mathbb{Z}$, is an often cited example of a chaotic dynamical system. If we define the itinerary of $\theta \in S^1$ as the sequence $\underline{s}$ defined by $s_i = 0$ if $0 < d^i(\theta) \leq 1/2$ and $s_i = 1$ if $1/2 < d^i(\theta) \leq 1$, then for *any* sequence of 0's and 1's we can find a θ which has that sequence as its itinerary. Thus the system embeds the randomness of a sequence of coin tosses within its dynamics.

This dynamical complication of angle-doubling is actually topological in character in the sense that it cannot be removed by continuously deforming the system. A theorem of Franks ([6]) shows that any circle map that is homotopic to d has dynamics at least as complicated as those of d in the precise sense given in the next theorem. (Angle-doubling on a circle is a simplest case of a much more general theorem.)

Theorem 1.1 (Franks)**.** *If g is a continuous, circle map that is homotopic to the angle-doubling map d, then there exists a continuous, onto map $\alpha : S^1 \to S^1$ with $\alpha \circ g = d \circ \alpha$.*

An α as in the theorem is called a *semiconjugacy* of g to d. The theorem can be informally understood by noting that whenever g is homotopic to d, the map g^n must of necessity wrap the circle 2^n times around itself, and so iterates of g have an unavoidable topological complication.

A useful interpretation of the theorem considers the point inverses, $\alpha^{-1}(\theta)$, as "fibers" over the points θ. The dynamics of g can be then thought of as a twisted product with the base point θ moved according to d while the fiber over θ is mapped by g to the fiber over $d(\theta)$. Thus all the information about how the dynamics of g

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differ from those of d is contained in the nature of the point inverses of α and in the way in which these point inverses are transformed into each other by g.

If α is homeomorphism, each $\alpha^{-1}(\theta)$ is a single point, and so g and d have the same dynamics. The next simplest case is when each $\alpha^{-1}(\theta)$ is a connected set, and thus is a point or an interval. In this case the essential difference between the dynamics of g and d is contained in the dynamics on intervals, a much studied subject. The case of interest here is when α has at least one disconnected point inverse. In this case the dynamics of g are much more complicated than those of d in the sense that the typical fiber, $\alpha^{-1}(\theta)$, has uncountably many connected components.

Theorem 1.2. *If g is a continuous circle map that is homotopic to the angle-doubling map d and α is its semiconjugacy to d, then the following are equivalent:*

(a) *There exists a point $\theta \in S^1$ with $\alpha^{-1}(\theta)$ disconnected.*
(b) *There exists a full measure, dense, G_δ-set $\Lambda \subset S^1$ so that $\theta \in \Lambda$ implies that $\alpha^{-1}(\theta)$ has uncountably many connected components.*
(c) *The map α is not of bounded variation.*

Further, in this case the topological entropy of g is strictly larger than that of d, $h_{top}(g) > h_{top}(d) = \log(2)$.

Note that the existence of the semiconjugacy yields that $h_{top}(g) \geq h_{top}(d)$, so the content of the last statement of the theorem is the strict inequality. From the point of view developed before the theorem, this conclusion indicates that the action of g in permuting the fibers $\alpha^{-1}(\theta)$ has positive entropy.

We briefly remark on related work. The case not included in the theorem, namely, when the semiconjugacy has connected point inverses, includes the situation where g is a covering map (see the first paragraph of the proof of Theorem 4.1). The semiconjugacies of degree-two covering maps have been widely studied from an analytic point of view (see, for example, section II.2 in [5], and the references therein). Also, there is a theorem in symbolic dynamics concerning a semiconjugacy between two transitive subshifts of finite type which bears a resemblance to Theorems 1.2 and 4.1 (see Remark 5.3). Finally, there are theorems analogous to Theorems 1.2 and 4.1 which hold for homeomorphisms of the two-torus which are isotopic to Anosov diffeomorphisms. These will be the subject of a subsequent paper.

While we state and prove our results for degree-two maps, it will be clear that virtually identical proofs yield the analogous theorems for degree-D circle maps with $D > 2$.

2. Preliminaries

The circle S^1 has universal cover $\mathbb{R}$, and the phrases *lift* and *projection* always mean lifts to and projections from this cover. A circle map is said to have degree $D \in \mathbb{Z}$ if it is homotopic to $\theta \mapsto D\theta$. In the special case of degree two, we write the angle-doubling map as $d(\theta) = 2\theta$ and for simplicity we choose a preferred lift, $\tilde{d}(x) = 2x$. Note that a map $g : S^1 \to S^1$ has degree two if and only if any lift can be written as

$$\tilde{g} = \tilde{d} + \varphi \tag{2.1}$$

with $\varphi(x+1) = \varphi(x)$. Whenever we consider a degree-two map, we fix a lift once and for all.

Given a degree-two circle map g with lift $\tilde{g}$, for each $D \in \mathbb{Z}$, let B_D be the complete metric space of all lifts of continuous degree-D circle maps with the sup topology, and define $F_D : B_D \to B_D$ by $F_D(\tilde{f}) = (\tilde{f} \circ \tilde{g})/2$. It is easy to see that F_D is a contraction mapping whose fixed point $\tilde{\alpha}_D$ satisfies $\tilde{\alpha}_D \circ \tilde{g} = \tilde{d} \circ \tilde{\alpha}_D$, and so projecting to the circle for any $D \neq 0$, we obtain Theorem 1.1. This proof shows that for each D the semiconjugacy α_D is the unique continuous, degree-D map which satisfies $\alpha_D \circ g = d \circ \alpha_D$. In this paper we will only consider the case $D = 1$, and given a degree-two g by *its semiconjugacy* we always mean α_1, which will henceforth be denoted α. If we begin the iteration of F_1 with the identity map, id, we obtain

$$\frac{\tilde{g}^n}{2^n} = F_1^n(id) \to \tilde{\alpha} \tag{2.2}$$

uniformly.

It is also useful to consider an operator that acts on the periodic parts of the maps. If the given degree-two map is as in (2.1) and C is the Banach space of 1-periodic functions with the sup norm, then $G : C \to C$ defined by $G(\sigma) = (\varphi + \sigma \circ \tilde{g})/2$ is also a contraction mapping, and if its fixed point is γ, then the lift of the semiconjugacy is $\tilde{\alpha} = id + \gamma$. If we begin the iteration of G with the zero map $\mathbf{0}$, we obtain that

$$G^n(\mathbf{0}) = \sum_{i=0}^{n-1} \frac{\varphi \circ \tilde{g}^i}{2^{i+1}} \to \gamma \tag{2.3}$$

uniformly and so

$$\tilde{\alpha} = id + \sum_{i=0}^{\infty} \frac{\varphi \circ \tilde{g}^i}{2^{i+1}},$$

as could have been confirmed directly.

The semiconjugacy gives a uniform bound on the distance between the $\tilde{g}$-orbit of x and the $\tilde{d}$ orbit of $\tilde{\alpha}(x)$. Using the semiconjugacy and $\tilde{\alpha} = id + \gamma$

$$|\tilde{g}^n(x) - 2^n\tilde{\alpha}(x)| = |\tilde{g}^n(x) - \tilde{\alpha}(\tilde{g}^n(x))| \leq \|\gamma\| \leq \|\varphi\|, \tag{2.4}$$

for all n, where for the last inequality we used (2.3). In the language of [7], this says that the orbits $o(x, g)$ and $o(\alpha(x), d)$ globally shadow, where for a given map f, the *orbit* of a point x is $o(x, f) := \{f^n x : n = 0, 1, \ldots\}$. It is worth noting that Theorem 1.1 can also be proved by a slight alteration of the global shadowing proof of the semiconjugacies to pseudoAnosov maps given in [7].

Recall that a map is called *light* if every point preimage is totally disconnected and *monotone* if every point preimage is connected. A theorem of Eilenberg and Whyburn (independently) says that for any continuous map $f : X \to Y$ with X and Y compact metric spaces, there exist a compact metric space Z, a continuous light map $\ell : Z \to Y$ and a continuous monotone map $m : X \to Z$, so that $f = \ell m$. The decomposition is particularly simple in the case at hand, $X = Y = S^1$, for since connected components of point inverses are always closed intervals, $Z = S^1$, and the monotone map m simply collapses certain intervals to points.

To study semiconjugacies α with disconnected point preimages, it is useful at first to ignore the monotone part of α and assume that α is light. We shall see in the proof of Theorem 1.2 that by collapsing collections of invariant intervals, any degree-two g can be projected to a degree-two map whose semiconjugacy is light.

The next proposition gives various dynamical characterizations of those g whose semiconjugacies are light maps.

Recall that a map f on a space X is *locally eventually onto* (leo) if for any open set U there is an $n > 0$ so that $f^n(U) = X$. A map is *transitive* if it has a dense orbit. A well-known characterization of transitivity on compact metric spaces is that for all open U and V there exists an $n > 0$ so that $f^n(U) \cap V \neq \emptyset$, and so clearly leo implies transitivity. For a one-dimensional system an interval J is *periodic* if there exists an $n > 0$ so that $f^n(J) \subset J$, and J is *wandering* if for all $i \neq j$, $i, j \geq 0$, $f^i(J) \cap f^j(J) = \emptyset$. Here and throughout this paper the terminology *interval* always means a compact, nontrivial interval.

Proposition 2.1. *If g is a continuous degree-two circle map the following are equivalent:*

(a) *The semiconjugacy α of g to d is light.*
(b) *g is locally eventually onto.*
(c) *g is transitive.*
(d) *g is light and has no periodic or wandering intervals.*

Proof. If J is a nontrivial interval and α is light, then there must exist $x_1, x_2 \in \tilde{J}$ with $\tilde{\alpha}(x_2) > \tilde{\alpha}(x_1)$ and $\tilde{J}$ a lift of J. Thus we may find an $n > 0$ with $2^n\tilde{\alpha}(x_2) - 2^n\tilde{\alpha}(x_1) > 1 + 2\|\varphi\|$, where φ is as in (2.1). Thus by (2.4), $\tilde{g}^n(x_2) - \tilde{g}^n(x_1) > 1$, and so $g^n(J) = S^1$. Therefore, (a) implies (b), and as noted above the theorem, (b) implies (c). Now assume that $o(x, g)$ is dense. If α was not light, then for some nontrivial interval J, $\alpha(J) = \theta_0$, a point. Since $o(x, g)$ is dense, there are $i \neq j$ with $g^i(x) \in J$ and $g^j(x) \in J$. Thus $\alpha(g^i(x)) = \alpha(g^j(x)) = \theta_0$ for $i \neq j$, and so by the semiconjugacy, $d^i(\alpha(x)) = d^j(\alpha(x))$, and so $o(\alpha(x), d)$ is eventually periodic. On the other hand, the continuity of the semiconjugacy implies that $o(\alpha(x), d)$ is dense since $o(x, g)$ is, a contradiction, and so (c) implies (a).

Now (b) clearly implies (d). We finish by proving the contrapositive of (d) implies (a), so assume α is not light, and thus there is some nontrivial interval J with $\alpha(J) = \theta_0$. Now if $g^n(J)$ is a point for some $n > 0$ or if J wanders, we are done. So we are left with the case when there is an $i > j$ with $g^i(J) \cap g^j(J) \neq \emptyset$. The semiconjugacy then yields that $d^i(\theta_0) = \alpha(g^i(J)) = \alpha(g^j(J)) = d^j(\theta_0)$. Thus if $\hat{J}$ is the connected component of $\alpha^{-1}(d^j(\theta_0))$ which contains $g^j(J)$, we must have $g^{i-j}(\hat{J}) \subset \hat{J}$, and so g has a periodic interval. $\square$

We shall make frequent use of standard results and techniques of one-dimensional dynamics without mention, but for the reader's convenience we state the following fundamental lemma. Recall that I *covers* J means that $J \subset I$. For more information on one-dimensional dynamics see [2], [3], or [5]. The version of the lemma we give essentially comes from [11].

Lemma 2.2. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous.*

(a) *If $f(J)$ covers I, then there is an interval $J' \subset J$ so that $f(J') = I$ and no interior point of J' maps to the boundary of I under f.*
(b) *If $\{J_i\}$ is a finite collection of intervals such that $f(J_i)$ covers J_{i+1} for all i, then there exists an interval $J' \subset J_0$ with $f^i(J') \subset J_i$ for all i. If $\{J_i\}$ is an countable collection, then there is a $y \in J_0$ with $f^i(y) \in J_i$ for all i.*

3. The main lemmas

The first main lemma locates a copy of the dynamics of d inside the dynamics of g. It makes no assumptions about the lightness or injectivity of the semiconjugacy.

Lemma 3.1. *Given a degree-two circle map g with semiconjugacy α, for each $r \in \mathbb{R}$ let $p_r = \min\{\tilde{\alpha}^{-1}(r)\}$.*

(a) *If $x < p_r$, then $\tilde{\alpha}(x) < r$.*
(b) *The map $r \mapsto p_r$ is order preserving.*
(c) *Each p_r satisfies $\tilde{g}(p_r) = p_{2r}$.*
(d) *If $x < p_r$, then $\tilde{g}(x) \leq \tilde{g}(p_r)$.*
(e) *If $s \nearrow r$, then $p_s \nearrow p_r$.*

Proof. If $x < p_r$, then $\tilde{\alpha}(x) \neq r$ by definition. But if $\tilde{\alpha}(x) > r$, then since α is degree one, there is a $y < x < p_r$ with $\tilde{\alpha}(y) = r$, contradicting the definition of p_r, and so we have (a); then (b) follows immediately. Now to prove (c), since $\tilde{\alpha}\tilde{g}(p_r) = 2\tilde{\alpha}(p_r) = 2r$, again by the definition of p_r, we have $\tilde{g}(p_r) \geq p_{2r}$. Now if $x \leq p_r$ and $\tilde{g}(x) > p_{2r}$, there would be a $y < x \leq p_r$ with $\tilde{g}(y) = p_{2r}$. But then $2\tilde{\alpha}(y) = \tilde{\alpha}\tilde{g}(y) = \tilde{\alpha}(p_{2r}) = 2r$, and so $\tilde{\alpha}(y) = r$, contradicting the definition of p_r. Thus $x \leq p_r$ implies $\tilde{g}(x) \leq p_{2r}$, so we have (c), and then immediately (d). Finally, if $s \nearrow r$, by (b), $\{p_s\}$ is increasing in s and is bounded above by p_r. If there was a $z < p_r$ with $p_s \nearrow z$, then by the continuity of $\tilde{\alpha}$, $\tilde{\alpha}(z) = r$, again contradicting the definition of p_r. □

For $r = k/2^n$ with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ we adapt the special notation of $p_{k,n} = p_r$. Conjugation of $\tilde{g}$ by a rigid translation will yield a map $\tilde{g}'$ for which $p_{0,0} = 0$. Now this $\tilde{g}'$ will be the lift of a degree-two g' which is a conjugate of g by a rigid rotation. Since such a conjugation does not change the dynamics of g nor the relevant properties of α, we may assume without loss of generality that $p_{0,0} = 0$. Since α is degree one, this implies that $p_{k,0} = k$ for all k, and so using Lemma 3.1(c), $\tilde{g}^n(p_{k,n}) = k$ for all k, n. The next lemma gives an explicit consequence of a non-injective semiconjugacy in the form of a "fold" in the dynamics of g.

Lemma 3.2. *If $g : S^1 \to S^1$ is a continuous, degree-two circle map which has been conjugated so that $p_{0,0} = 0$ and is such that its semiconjugacy α is light but not injective, then there exists $N, K \in \mathbb{N}$ with $0 \leq K < 2^N$ and $\hat{x} \in \mathbb{R}$ with $p_{K,N} < \hat{x} < p_{K+1,N}$ so that $\tilde{g}^N(\hat{x}) = K - 1$.*

Proof. First note that there exists some $p_{r'}$ and an $x' \in \mathbb{R}$ with $x' > p_{r'}$ and $\tilde{\alpha}(x') < r'$, for otherwise by Lemma 3.1(a), (b), $\tilde{\alpha}$ would be injective. If we fix this x', then the set $\{r : x' > p_r \text{ and } \tilde{\alpha}(x') < r\}$ is nonempty. Let r_0 be its supremum and note that by Lemma 3.1(e), $x' > p_{r_0}$ and $\tilde{\alpha}(x') < r' \leq r_0$. Next we prove that $s > r_0$ implies $x' < p_s$, by assuming to the contrary that $s > r_0$ and $x' \geq p_s$. Now if $x' = p_s$, then $\tilde{\alpha}(x') = s > r_0$, and if $x' > p_s$, by the definition of r_0 we have $\tilde{\alpha}(x') \geq s > r_0$. Thus in either case we have a contradiction to $\tilde{\alpha}(x') < r_0$.

Letting $s_0 = \tilde{\alpha}(x')$, since $s_0 < r_0$, elementary number theory yields integers K and N with

$$2^N s_0 + 1 + 2\|\varphi\| < K \leq 2^N r_0 < K + 1,$$

with φ as in (2.1). Then since $2^N s_0 = 2^N \tilde{\alpha}(x') = \tilde{\alpha}\tilde{g}^N(x')$, (2.4) says that $|\tilde{g}^N(x') - 2^N s_0| < \|\varphi\|$ and so $\tilde{g}^N(x') < K - 1$. Now since $K/2^N \leq r_0 < (K+1)/2^N$, using the first paragraph of the proof and Lemma 3.1(b), we have $p_{K,N} \leq p_{r_0} < x' <$

$p_{K+1,N}$. By hypothesis $p_{0,0} = 0$, and so by Lemma 3.1(c), $\tilde{g}^N(p_{K,N}) = K$ and $\tilde{g}^N(p_{K+1,N}) = K+1$, and thus $K-1 \in \tilde{g}^N([p_{K,N}, p_{K+1,N}])$. The continuity of $\tilde{g}$ then yields the required $\hat{x}$. Since $\tilde{\alpha}(x+1) = \tilde{\alpha}(x) + 1$, we may assume that $0 \leq K/2^N < 1$. □

4. The main theorem

The main theorem gives a number of conditions which are equivalent to g having a light semiconjugacy that is not injective. It will easily imply Theorem 1.2 of the Introduction.

Theorem 4.1. *If g is a continuous, degree-two circle map with a light semiconjugacy α, then the following are equivalent:*

(a) *The map $\tilde{g}$ is not injective.*
(b) *The map α is not injective.*
(c) *There exists a full measure, dense, G_δ-set $\Lambda \subset S^1$ so that $\theta \in \Lambda$ implies that $\alpha^{-1}(\theta)$ is uncountable, and thus contains a Cantor set.*
(d) *The topological entropy of g satisfies $h_{top}(g) > \log(2)$.*
(e) *For all nontrivial intervals $J \subset S^1$, the map $\alpha|_J$ is not of bounded variation.*

Proof. If α is injective, then so is $\tilde{g} = \tilde{\alpha}\tilde{d}\tilde{\alpha}^{-1}$, and thus (a) implies (b). Since conjugate maps have the same entropy, (d) implies (b). Now if g is injective, then by (2.2), $\tilde{\alpha}$ is nondecreasing, but by hypothesis α is light, and so $\tilde{\alpha}$ is strictly increasing and thus is injective, therefore (b) implies (a). The fact that each of (c) and (e) imply (b) is obvious, so we henceforth assume that $\tilde{\alpha}$ is not injective and show that this implies (c), (d), and (e).

Let K and N be as in Lemma 3.2 and continue to assume that g has been conjugated so that $p_{0,0} = 0$. By Lemma 3.2 and Lemma 2.2(a), we may find intervals I_a, I_b, and I_c in $[p_{K,N}, p_{K+1,N}]$ with disjoint interiors and $I_a \leq I_b \leq I_c$ so that $\tilde{g}^N(I_a) = \tilde{g}^N(I_b) = [K-1, K]$ and $\tilde{g}^N(I_c) = [K, K+1]$. For each $k \neq K$ with $0 \leq k < 2^N$, define intervals $I_k = [p_{k,N}, p_{k+1,N}]$. Define a set of "addresses" as $A = \{0, 1, 2, \ldots, K-1, K+1, \ldots, 2^N - 1, a, b, c\}$, and for $\eta \in A$, let $\phi(\eta)$ be given by $\phi(a) = \phi(b) = K-1$, $\phi(c) = K$, and for $0 \leq k < 2^N$, $\phi(k) = k$. By Lemma 3.1(c) we now have that for all $\eta \in A$, $\tilde{g}^N(I_\eta)$ covers $[0,1] + \phi(\eta)$. Projecting the collection $\{I_\eta\}$ to the circle we see that g^N has a (2^N+2)-fold horseshoe and so (see Theorem 4.3.2 in [2]) $h_{top}(g^N) \geq \log(2^N+2)$ and therefore $h_{top}(g) \geq \log(2^N+2)/N > \log(2)$, yielding (d).

Returning to the covering space $\mathbb{R}$, since g is a degree-two map, for any integer m, $\tilde{g}^N(I_\eta + m)$ covers $[0,1] + \phi(\eta) + 2^N m$. Thus by Lemma 2.2(b) for any sequence $\underline{s} \in A^{\mathbb{N}}$ we may find a $y \in [0,1]$ with

$$\tilde{g}^{Nj}(y) \in I_{s_j} + \sum_{i=0}^{j-1} 2^{N(j-i-1)} \phi(s_i) \tag{4.1}$$

for all $j \in \mathbb{N}$. Now a given y can represent two or more sequences, but that can only happen if for some i, $\tilde{g}^{Ni}(y)$ is contained in two intervals and so must be in the boundary of some I_η. However, then by construction of the I_η, $\tilde{g}^{N(i+1)}(y) \in \mathbb{Z}$, and since $p_{0,0} = 0$ as noted above in Lemma 3.2 we have that for all $j > i$, $\tilde{g}^{Nj}(y) \in \mathbb{Z}$. If we assume initially that $K \neq 0, 2^N - 1$, then for any integer m, $(I_{2^N-1} + m - 1) \cap (I_0 + m) = \{m\}$. Thus a point y can represent two sequences $\underline{s}$ and $\underline{s}'$ only if s_j and s'_j are contained in $\{2^N - 1, 0\}$ for all sufficiently large j.

Therefore, if we say a sequence has a *nontrivial tail* if there exist arbitrarily large j with $s_j \notin \{2^N - 1, 0\}$, we see that when $\underline{s}$ has a nontrivial tail, $\underline{s} \neq \underline{s}'$ implies that the corresponding $y's$ are distinct. To make this true when $K = 0$ the definition of nontrivial tail must be altered to require arbitrarily large j with $s_j \notin \{2^N - 1, a\}$, and when $K = 2^N - 1$ to require arbitrarily large j with $s_j \notin \{c, 0\}$.

Now note that (2.2) implies that a y which satisfies (4.1) will have

$$\tilde{\alpha}(y) = \lim_{j\to\infty} \frac{1}{2^{Nj}} \sum_{i=0}^{j-1} 2^{N(j-i-1)} \phi(s_i) = \sum_{i=0}^{\infty} \frac{\phi(s_i)}{2^{N(i+1)}}.$$

Since $\phi(a) = \phi(b) = \phi(K-1) = K-1$, whenever $\phi(s_i) = K-1$ in this sum, there are three possible choices of s_i which give the same value of $\tilde{\alpha}(y)$. Thus if $\underline{t} \in \{1, 2, \ \ldots, \ 2^N - 1\}^{\mathbb{N}}$ is a sequence with $t_i = K-1$ for infinitely many i, the sum

$$r = \sum_{i=0}^{\infty} \frac{t_i}{2^{N(i+1)}} \tag{4.2}$$

is equal to the sum in (4.1) for uncountably many sequences $\underline{s}$. If uncountably many of these sequences $\underline{s}$ have a nontrivial tail, then for such an r the set $\tilde{\alpha}^{-1}(r)$ is uncountable. We will prove that the collection of all such r is as in (c).

It is well known that when a map is ergodic with respect to a smooth measure on a compact manifold, the collection of x whose orbits are dense is a dense, G_δ, full measure set, and that the angle-doubling map d is ergodic with respect to Lebesgue measure. Thus (c) is proven after we show that whenever θ has a dense orbit, its lift to an $r \in [0, 1)$ is as described at the end of the previous paragraph.

The proof of this proceeds by repeating the construction that gave rise to (4.1) in the easier case of $\tilde{d}$. For $0 \leq k < 2^N$, let $\hat{I}_k = [k/2^N, (k+1)/2^N]$, and so for any integer m, $\tilde{d}^N(\hat{I}_k + m)$ covers $[0, 1] + k + 2^N m$. Thus for any sequence $\underline{t} \in \{1, 2, \ \ldots \ , 2^N - 1\}^{\mathbb{N}}$, we may find an $r \in [0, 1)$ with

$$\tilde{d}^{Nj}(r) \in \hat{I}_{t_j} + \sum_{i=0}^{j-1} 2^{N(j-i-1)} t_i \tag{4.3}$$

for all $j \in \mathbb{N}$. This implies that r is given by (4.2). Conversely, because $\tilde{d}$ is expanding and for all k, m, $\tilde{d}^N(I_k + m) = [0, 1] + k + 2^N m$, it follows that any $r \in [0, 1)$ with $d^{Nj}(r) \notin \mathbb{Z}$ for all j will be the unique $r \in [0, 1]$ satisfying (4.3) for a sequence $\underline{t}$ with $t_i \notin \{0, 2^N - 1\}$ for arbitrarily large i. In particular, if $\theta \in S^1$ has a dense orbit under d, then its orbit lands infinitely often in the projection to the circle of every interval $\hat{I}_k$, and thus its lift $r \in [0, 1)$ yields a sequence $\underline{t}$ for which $t_i = K-1$ infinitely often, and any $\underline{s}$ with $\phi(t_i) = s_i$ for all i must have a nontrivial tail. Thus for such r, $\tilde{\alpha}^{-1}(r)$ is uncountable and thus $\alpha^{-1}(\theta)$ is uncountable also, proving (c).

Now to prove (e), say that an interval $J \subset \mathbb{R}$ *unit covers* if for some integer M, $[M, M+1] \subset J$. By construction for each $\eta \in A$, $\tilde{g}^N(I_\eta)$ unit covers. Since there are $2^N + 2$ such intervals I_η, using $\tilde{\alpha} = (\tilde{\alpha} \circ \tilde{g}^N)/2^N$ we obtain that the variation of $\tilde{\alpha}$ on the interval $[0, 1]$ satisfies $\mathrm{var}(\alpha, [0, 1]) \geq (2^N + 2)/2^N$. Now since each $\tilde{g}^N(I_\eta)$ unit covers and each unit interval $[M, M+1]$ contains $I_\eta + M$ for all η, using Lemma 2.2 there are $2^N + 2$ intervals $I_{\eta,j}$ in each I_η so that each $\tilde{g}^{2N}(I_{\eta,j})$ unit covers, so $\mathrm{var}(\alpha, [0, 1]) \geq (2^N + 2)^2/2^{2N}$. An obvious induction then yields that for all j,

$$\operatorname{var}(\alpha, [0,1]) \geq \frac{(2^N+2)^j}{2^{Nj}}, \tag{4.4}$$

which goes to infinity as $j \to \infty$, and so $\tilde{\alpha}$ has unbounded variation on $[0,1]$.

Now as noted at the beginning of the proof of Proposition 2.1, for any interval $J \subset \mathbb{R}$ there is a $w \in \mathbb{N}$ so that $\tilde{g}^w(J)$ unit covers. Then using Lemma 2.2 and the intervals of the previous paragraph we get that

$$\operatorname{var}(\tilde{\alpha}, J) \geq \frac{(2^N+2)^j}{2^{Nj+w}} \to \infty,$$

proving (e). □

Proof of Theorem 1.2. Assume that the semiconjugacy of g to d has monotone-light decomposition, $\alpha = \ell m$. Now if J is an interval such that $m(J)$ is a point, then certainly $d\ell m(J) = \ell m g(J)$ is also a point, and since ℓ is light, this says that $mg(J)$ must also be a point. Thus the formula $\hat{g} = mgm^{-1}$ unambiguously defines a continuous degree-two map with light conjugacy ℓ. Now if α has a disconnected preimage, then ℓ must also, and so by Theorem 4.1(c), (a) implies (b), while the converse is trivial. The graph of ℓ differs from that of α only by the insertion of perhaps a countable number of horizontal intervals, and so assuming (a), by Theorem 4.1(c), (c) follows, and the converse is also clear. Finally, since g is semiconjugate to $\hat{g}$, $h_{top}(g) \geq h_{top}(\hat{g})$. Assuming (a), Theorem 4.1(d) gives $h_{top}(\hat{g}) > \log(2)$, finishing the proof. □

5. Remarks and questions

Remark 5.1. The primary distinction between the general case of Theorem 1.2 and the light semiconjugacy case of Theorem 4.1 is that a general g can have have an arbitrary amount of dynamical complications and thus entropy in, say, a periodic interval. In this case, one can have $h_{top}(g) > \log(2)$, which clearly implies that α is not injective, but it does not necessarily imply that α is not monotone.

Remark 5.2. If g is piecewise monotone with a finite number of turning points, then it follows from a theorem of Misiurewicz and Szlenk ([10]) that the variation estimate on $\tilde{g}^{Nj}$ that gives rise to (4.4) is equivalent to the entropy result.

Remark 5.3. As noted in the Introduction, there is a theorem in symbolic dynamics which has similarities with Theorems 1.2 and 4.1. This theorem says that if (Σ, σ) and (Σ', σ') are transitive subshifts of finite type with $h_{top}(\sigma) > 0$, and α is a surjective semiconjugacy, then there is a dichotomy: either $h_{top}(\sigma) = h_{top}(\sigma')$ and there exists an integer N so that the cardinality of every $\alpha^{-1}(\underline{s})$ is at most N, or $h_{top}(\sigma) > h_{top}(\sigma')$ and for the topologically generic point $\underline{s}' \in \Sigma'$, the point inverse $\alpha^{-1}(\underline{s}')$ is uncountable. See Corollary 4.1.8 in [8] as well as Section 6 in [1].

The proof of Theorem 4.1 given here has much of the flavor of this symbolic dynamics result, basically showing the existence of a diamond in the semiconjugacy. In fact, parts of the result could have been reduced to the symbolic dynamics theorem, but doing so would have resulted in a longer, less self-contained proof.

Remark 5.4. Parts of Theorem 4.1 can also be obtained by more topological methods. From Proposition 2.1 it follows that any g with a light conjugacy is locally eventually onto, and from this is follows fairly easily that if α is not injective in one open set, then it is not injective in any open set. Such an α is called nowhere locally injective. Block, Oversteegen and Tymchatyn have shown that any light, nowhere locally injective map between manifolds has the property that the topologically generic point has a Cantor set as its point inverse ([4]).

Remark 5.5. This paper has dealt primarily with combinatorial/topological aspects of degree-two circle maps. It would also be of interest to study quantitative/analytic aspects. For example, for a g with a light semiconjugacy, give an explicit relationship between properties of its semiconjugacy α, say the fractal dimensions of the graph of α, and the difference in entropy, $h_{top}(g) - \log(2)$. In this regard we note then when g has a finite number of turning points, its semiconjugacy can be treated in the context of fractal functions. In particular, if g is piecewise linear with expanding pieces, then α is an affine fractal function, and its graph is the attractor of a planar iterated function system (see [9]). Also, in analogy to the degree-one case, it would also be interesting to study the transition to a nonmonotone semiconjugacy in parameterized families, for example in the standard degree-two family $f_{b,\omega}(x) = 2x + \omega + b\sin(2\pi x)$.

References

[1] Roy L. Adler, *Symbolic dynamics and Markov partitions*, Bull. Amer. Math. Soc. (N.S.) **35** (1998), no. 1, 1–56. MR1477538 (98j:58081)

[2] Lluís Alsedà, Jaume Llibre, and Michał Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, Advanced Series in Nonlinear Dynamics, vol. 5, World Scientific Publishing Co., Inc., River Edge, NJ, 2000. MR1807264 (2001j:37073)

[3] L. S. Block and W. A. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics, vol. 1513, Springer-Verlag, Berlin, 1992. MR1176513 (93g:58091)

[4] A. Blokh, L. Oversteegen, and E. Tymchatyn, *On almost one-to-one maps*, Trans. Amer. Math. Soc., to appear.

[5] Welington de Melo and Sebastian van Strien, *One-dimensional dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 25, Springer-Verlag, Berlin, 1993. MR1239171 (95a:58035)

[6] John Franks, *Anosov diffeomorphisms*, Global Analysis (Proc. Sympos. Pure Math., vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, RI, 1970, pp. 61–93. MR0271990 (42:6871)

[7] Michael Handel, *Entropy and semi-conjugacy in dimension two*, Ergodic Theory Dynam. Systems **8** (1988), no. 4, 585–596. MR0980798 (90g:58096)

[8] Bruce P. Kitchens, *Symbolic dynamics*, Universitext, Springer-Verlag, Berlin, 1998. MR1484730 (98k:58079)

[9] Peter R. Massopust, *Fractal functions, fractal surfaces, and wavelets*, Academic Press, Inc., San Diego, CA, 1994. MR1313502 (96b:28007)

[10] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. **67** (1980), no. 1, 45–63. MR0579440 (82a:58030)

[11] Michał Misiurewicz and Zbigniew Nitecki, *Combinatorial patterns for maps of the interval*, Mem. Amer. Math. Soc. **94** (1991), no. 456, vi+112. MR1086562 (92h:58105)

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