SEMI-FINITE FORMS OF BILATERAL BASIC HYPERGEOMETRIC SERIES

WILLIAM Y. C. CHEN AND AMY M. FU

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ABSTRACT. We show that several classical bilateral summation and transformation formulas have semi-finite forms. We obtain these semi-finite forms from unilateral summation and transformation formulas. Our method can be applied to derive Ramanujan's ${}_1\psi_1$ summation, Bailey's ${}_2\psi_2$ transformations, and Bailey's ${}_6\psi_6$ summation.

1. INTRODUCTION

We follow the terminology for basic hypergeometric series in [6]. Assuming $|q| < 1$, let

$$(a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \cdots.$$

For any integer n , the q -shifted factorial $(a; q)_n$ is given by

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}.$$

For $n \geq 0$, we have the following relation which is crucial for this paper:

$$(1.1) \quad (a; q)_{-n} = \frac{1}{(aq^{-n}; q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a; q)_n}.$$

For convenience, we employ the following usual notation:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n.$$

The (unilateral) basic hypergeometric series ${}_r\phi_r$ is defined by

$$(1.2) \quad {}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z \right] = \sum_{k=0}^{\infty} A(k),$$

where

$$A(k) = \frac{(a_1, a_2, \dots, a_{r+1}; q)_k}{(b_1, b_2, \dots, b_r, q; q)_k} z^k.$$

The bilateral basic hypergeometric series ${}_s\psi_s$ is defined as follows:

$$(1.3) \quad {}_s\psi_s \left[\begin{matrix} a_1, a_2, \dots, a_s \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right] = \sum_{k=-\infty}^{\infty} B(k),$$

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where

$$B(k) = \frac{(a_1, a_2, \dots, a_s; q)_k}{(b_1, b_2, \dots, b_s; q)_k} z^k.$$

In this paper, we propose the following method for deriving bilateral summation and transformation formulas using *semi-finite forms*. For a bilateral series ${}_s\psi_s$ as given in (1.3), we construct a summand $G(k, m)$ which implies a unilateral series ${}_{r+s+1}\phi_{r+s}$, where r is a nonnegative integer, such that

$$\lim_{m \rightarrow \infty} G(k, m) = B(k)$$

for all k , and the summation

$$(1.4) \quad \sum_{k=-m}^{\infty} G(k, m)$$

can be easily accomplished as a Laurent extension of the summation

$$(1.5) \quad \sum_{k=0}^{\infty} G(k-m, m) = G(-m, m) \sum_{k=0}^{\infty} A(k),$$

where $G(k, m)$ can be written as

$$G(k-m, m) = G(-m, m)A(k)$$

for some $A(k)$. The bilateral series (1.3) is then obtained from (1.4) as $m \rightarrow \infty$, subject to suitable convergence conditions. We apply this procedure to derive bilateral series identities from suitable unilateral ones. The above summation (1.4) is called the *semi-finite form* of the bilateral summation (1.3). A method similar to ours was recently used by Schlosser [9], and Jouhet and Schlosser [8], who derived summations for bilateral series from *finite forms*. We also note that another method, which uses a similar factorization as above, for deriving bilateral series identities from unilateral ones was used by Ismail [7], and Askey and Ismail [2]. Rather than taking limits, they apply analytic continuation as the main ingredient.

In this paper, we present semi-finite forms of several classical bilateral summation and transformation formulas such as Ramanujan's ${}_1\psi_1$ summation, Bailey's ${}_2\psi_2$ transformations, and Bailey's ${}_6\psi_6$ summation.

2. FROM ${}_2\phi_1$ TO ${}_1\psi_1$

Using the well-known Gauss summation formula

$$(2.1) \quad {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; q, c/ab \right] = \frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}},$$

where $|c/ab| < 1$, we get a semi-finite form of Ramanujan's summation of the general ${}_1\psi_1$,

$$(2.2) \quad {}_1\psi_1 \left[\begin{matrix} a \\ b \end{matrix}; q, z \right] = \sum_{k=-\infty}^{\infty} \frac{(a; q)_k}{(b; q)_k} z^k = \frac{(q; q)_{\infty} (b/a; q)_{\infty} (az; q)_{\infty} (q/az; q)_{\infty}}{(b; q)_{\infty} (q/a; q)_{\infty} (z; q)_{\infty} (b/az; q)_{\infty}},$$

where $|b/a| < |z| < 1$.

Proposition 2.1. *For $|z| < 1$, the following identity holds:*

$$(2.3) \quad \sum_{k=-m}^{\infty} \frac{(a; q)_k (bq^m/az; q)_k}{(q^{1+m}; q)_k (b; q)_k} z^k = \frac{(q; q)_m (q/az; q)_m}{(q/a; q)_m (b/az; q)_m} \frac{(b/a; q)_{\infty} (az; q)_{\infty}}{(b; q)_{\infty} (z; q)_{\infty}}.$$

Proof. The left-hand side of (2.3) can be rewritten as

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(a; q)_{k-m} (bq^m/az; q)_{k-m}}{(q^{1+m}; q)_{k-m} (b; q)_{k-m}} z^{k-m} \\
&= z^{-m} \frac{(a; q)_{-m} (bq^m/az; q)_{-m}}{(q^{1+m}; q)_{-m} (b; q)_{-m}} \sum_{k=0}^{\infty} \frac{(aq^{-m}; q)_k (b/az; q)_k}{(q; q)_k (bq^{-m}; q)_k} z^k \\
&\stackrel{(2.1)}{=} z^{-m} \frac{(a; q)_{-m} (bq^m/az; q)_{-m}}{(q^{1+m}; q)_{-m} (b; q)_{-m}} \frac{(b/a; q)_{\infty} (azq^{-m}; q)_{\infty}}{(bq^{-m}; q)_{\infty} (z; q)_{\infty}} \\
&\stackrel{(1.1)}{=} z^{-m} \frac{(q; q)_m (azq^{-m}; q)_m}{(aq^{-m}; q)_m (b/az; q)_m} \frac{(az; q)_{\infty} (b/a; q)_{\infty}}{(b; q)_{\infty} (z; q)_{\infty}},
\end{aligned}$$

which equals the right-hand side of (2.3). $\square$

Taking the limit $m \rightarrow \infty$ in Proposition 2.1 while assuming $|b/az| < 1$, we immediately obtain (2.2).

We remark that our method is different from the method of M. Jackson's elementary proof of (2.2) (see the exposition of Schlosser [9]) in the sense that Jackson's proof does not give a semi-finite form although the Gauss summation is also the basic ingredient. We should also note that a finite form of Ramanujan's ${}_1\psi_1$ summation has been given by Schlosser [10] using the terminating q -Pfaff-Saalschütz summation.

3. FROM ${}_3\phi_2$ TO ${}_2\psi_2$

In this section, we use two ${}_3\phi_2$ summation and transformation formulas to give the semi-finite forms of ${}_2\psi_2$ formulas due to Bailey. We begin with the following ${}_2\psi_2$ transformation formula [6, Ex. 5.20(i)] valid for $|z|, |cd/abz|, |d/a|, |c/b| < 1$,

$$(3.1) \quad {}_2\psi_2 \left[\begin{matrix} a, b \\ c, d \end{matrix}; q, z \right] = \frac{(az, d/a, c/b, dq/abz; q)_{\infty}}{(z, d, q/b, cd/abz; q)_{\infty}} {}_2\psi_2 \left[\begin{matrix} a, abz/d \\ az, c \end{matrix}; q, \frac{d}{a} \right].$$

Using a q -analogue of the Kummer-Thomae-Whipple formula [6, Eq. (3.2.7)],

$$(3.2) \quad {}_3\phi_2 \left[\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc} \right] = \frac{(e/a, de/bc; q)_{\infty}}{(e, de/abc; q)_{\infty}} {}_3\phi_2 \left[\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, \frac{e}{a} \right],$$

where $|de/abc| < 1$ and $|e/a| < 1$, we get a semi-finite form of (3.1).

Proposition 3.1. *For $|z| < 1$ and $|d/a| < 1$, we have*

$$\begin{aligned}
(3.3) \quad \sum_{k=-m}^{\infty} \frac{(a, b; q)_k (cdq^m/abz; q)_k}{(c, d; q)_k (q^{1+m}; q)_k} z^k &= \frac{(az, d/a; q)_{\infty}}{(z, d; q)_{\infty}} \frac{(c/b, dq/abz; q)_m}{(q/b, cd/abz; q)_m} \\
&\quad \cdot \sum_{k=-m}^{\infty} \frac{(a, cq^m/b, abz/d; q)_k}{(c, q^{1+m}, az; q)_k} (d/a)^k.
\end{aligned}$$

Proof. The left-hand side of (3.3) equals

$$\begin{aligned}
 & z^{-m} \frac{(a, b, cdq^m/abz; q)_{-m}}{(c, d, q^{1+m}; q)_{-m}} \sum_{k=0}^{\infty} \frac{(aq^{-m}, bq^{-m}, cd/abz; q)_k}{(cq^{-m}, dq^{-m}, q; q)_k} z^k \\
 & \stackrel{(3.2)}{=} z^{-m} \frac{(a, b, cdq^m/abz; q)_{-m}}{(c, d, q^{1+m}; q)_{-m}} \frac{(d/a, azq^{-m}; q)_{\infty}}{(dq^{-m}, z; q)_{\infty}} \\
 & \quad \cdot \sum_{k=0}^{\infty} \frac{(aq^{-m}, c/b, abzq^{-m}/d; q)_k}{(q, cq^{-m}, azq^{-m}; q)_k} \left(\frac{d}{a}\right)^k \\
 & \stackrel{(1.1)}{=} \frac{(d/a, az; q)_{\infty}}{(d, z; q)_{\infty}} \frac{(c/b, abzq^{-m}/d; q)_m}{(bq^{-m}, cd/abz; q)_m} \left(\frac{d}{az}\right)^m \\
 & \quad \cdot \sum_{k=0}^{\infty} \frac{(a, cq^m/b, abz/d; q)_{k-m}}{(c, q^{1+m}, az; q)_{k-m}} (d/a)^{k-m},
 \end{aligned}$$

which can be rewritten in the form of the right-hand side of (3.3). $\square$

The next ${}_2\psi_2$ transformation formula we consider is the following [6, Ex. 5.20(ii)]:

$$(3.4) \quad {}_2\psi_2 \left[\begin{matrix} a, b \\ c, d \end{matrix}; q, z \right] = \frac{(az, bz, cq/abz, dq/abz; q)_{\infty}}{(q/a, q/b, c, d; q)_{\infty}} {}_2\psi_2 \left[\begin{matrix} abz/c, abz/d \\ az, bz \end{matrix}; q, \frac{cd}{abz} \right].$$

Using a summation of Hall [6, Eq. (3.2.10)],

$$(3.5) \quad {}_3\phi_2 \left[\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc} \right] = \frac{(b, de/ab, de/bc; q)_{\infty}}{(d, e, de/abc; q)_{\infty}} {}_3\phi_2 \left[\begin{matrix} d/b, e/b, de/abc \\ de/ab, de/bc \end{matrix}; q, b \right],$$

where $|de/abc| < 1$ and $|b| < 1$, we obtain the following semi-finite form of (3.4).

Proposition 3.2. For $|z| < 1$ and $|cd/abz| < 1$, we have

$$\begin{aligned}
 \sum_{k=-m}^{\infty} \frac{(a, b; q)_k (cdq^m/abz; q)_k}{(c, d; q)_k (q^{1+m}; q)_k} z^k &= \frac{(az, bz, cd/abz; q)_{\infty}}{(c, d, z; q)_{\infty}} \frac{(cq/abz, dq/abz, z; q)_m}{(q/a, q/b, cd/abz; q)_m} \\
 &\quad \cdot \sum_{k=-m}^{\infty} \frac{(abz/c, abz/d, zq^m; q)_k}{(az, bz, q^{1+m}; q)_k} (cd/abz)^k.
 \end{aligned}$$

4. FROM NONTERMINATING ${}_8\phi_7$ TO ${}_6\psi_6$

In this section, we give a semi-finite form of Bailey's ${}_6\psi_6$ summation formula [4] by using Bailey's 3-term transformation formula for a nonterminating, very well-

poised ${}_8\phi_7$ series [6, Eq. (2.11.1)]:

$$\begin{aligned}
 (4.1) \quad & {}_8\phi_7 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, f \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix}; q, \frac{a^2 q^2}{bcdef} \right] \\
 &= \frac{(aq, aq/de, aq/df, aq/ef, eq/c, fq/c, b/a, bef/a; q)_{\infty}}{(aq/d, aq/e, aq/f, aq/def, q/c, eq/c, be/a, bf/a; q)_{\infty}} \\
 &\quad \cdot {}_8\phi_7 \left[\begin{matrix} ef/c, q(ef/c)^{\frac{1}{2}}, -q(ef/c)^{\frac{1}{2}}, aq/bc, aq/cd, ef/a, e, f \\ (ef/c)^{\frac{1}{2}}, -(ef/c)^{\frac{1}{2}}, bef/a, def/a, aq/c, fq/c, eq/c \end{matrix}; q, \frac{bd}{a} \right] \\
 &\quad + \frac{b}{a} \frac{(aq, bq/a, bq/c, bq/d, bq/e, bq/f, d, e, f; q)_{\infty}}{(aq/b, aq/c, aq/d, aq/e, aq/f, bd/a, be/a, bf/a, def/a; q)_{\infty}} \\
 &\quad \cdot \frac{(aq/bc, bdef/a^2, a^2 q/bdef; q)_{\infty}}{(aq/def, q/c, b^2 q/a; q)_{\infty}} \\
 &\quad \cdot {}_8\phi_7 \left[\begin{matrix} b^2/a, qba^{-\frac{1}{2}}, -qba^{-\frac{1}{2}}, b, bc/a, bd/a, be/a, bf/a \\ ba^{-\frac{1}{2}}, -ba^{-\frac{1}{2}}, bq/a, bq/c, bq/d, bq/e, bq/f \end{matrix}; q, \frac{a^2 q^2}{bcdef} \right],
 \end{aligned}$$

where $|bd/a| < 1$ and $|a^2 q^2 / bcdef| < 1$.

Proposition 4.1. When $|bd/a| < 1$ and $|a^2 q^2 / bcdef| < 1$, we have

$$\begin{aligned}
 (4.2) \quad & \sum_{k=-m}^{\infty} \frac{(q^{m-k+1}/a, fq^m; q)_k}{(q^{m-k}f/a, q^{1+m}; q)_k} \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_k}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_k} \left(\frac{qa^2}{bcde} \right)^k \\
 &= \frac{1 - efq^{2m}/c}{1 - efq^m/c} \frac{(q/a, df/a, ef/a, aq/bc, aq/cd, efq^m/a; q)_m}{(f/a, q/b, q/c, q/d, def/a, fq^{1+m}/c; q)_m} \\
 &\quad \times \frac{(aq, aq/de, aq/df, aq/ef, eq^{1+m}/c, fq^{1+m}/c, b/a, befq^m/a; q)_{\infty}}{(aq/d, aq/e, aq/f, aq/def, q^{1+m}/c, eq^{1+m}/c, be/a, bfq^m/a; q)_{\infty}} \\
 &\quad \sum_{k=-m}^{\infty} \frac{(efq^m/c, q^{1+m}(ef/c)^{\frac{1}{2}}, -q^{1+m}(ef/c)^{\frac{1}{2}}, aq^{1+m}/bc; q)_k}{(q^{1+m}, q^m(ef/c)^{\frac{1}{2}}, -q^m(ef/c)^{\frac{1}{2}}, befq^m/a; q)_k} \\
 &\quad \cdot \frac{(aq^{1+m}/cd, efq^{2m}/a, e, fq^m; q)_k}{(defq^m/a, aq/c, fq^{1+2m}/c, eq^{1+m}/c; q)_k} \left(\frac{bd}{a} \right)^k \\
 &\quad + \frac{b}{a} \frac{1 - b^2 q^{2m}/a}{1 - b^2 q^m/a} \left(\frac{a^2 q}{bcde} \right)^m \frac{(q/a, bc/a; q)_m}{(f/a; q)_m} \frac{(aq, bq^{1+2m}/a, bq^{1+m}/c; q)_{\infty}}{(aq/b, aq/c, aq/d; q)_{\infty}} \\
 &\quad \times \frac{(bq^{1+m}/d, bq^{1+m}/e, bq/f, d, e, fq^m, aq/bc, bdef/a^2, a^2 q/bdef; q)_{\infty}}{(aq/e, aq/f, bdq^m/a, beq^m/a, bfq^{2m}/a, def/a, aq/def, q/c, b^2 q^{1+m}/a; q)_{\infty}} \\
 &\quad \times \sum_{k=-m}^{\infty} \frac{(b^2 q^m/a, q^{1+m}ba^{-\frac{1}{2}}, -q^{1+m}ba^{-\frac{1}{2}}; q)_k}{(q^{1+m}, q^m ba^{-\frac{1}{2}}, -q^m ba^{-\frac{1}{2}}; q)_k} \\
 &\quad \cdot \frac{(b, bcq^m/a, bdq^m/a, beq^m/a, bfq^{2m}/a; q)_k}{(bq^{1+2m}/a, bq^{1+m}/c, bq^{1+m}/d, bq^{1+m}/e, bq/f; q)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k.
 \end{aligned}$$

Proof. The left-hand side of (4.2) equals

$$\begin{aligned}
& \sum_{k=-m}^{\infty} \frac{(aq^{-m}, fq^m; q)_k}{(aq^{1-m}/f, q^{1+m}; q)_k} \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_k}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_k} \left(\frac{q^2 a^2}{bcdef} \right)^k \\
& \stackrel{(1.5)}{=} \frac{(aq^{-m}, fq^m, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_{-m}}{(aq^{1-m}/f, q^{1+m}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_{-m}} \left(\frac{q^2 a^2}{bcdef} \right)^{-m} \\
& \quad \times \sum_{k=0}^{\infty} \frac{(aq^{-2m}, q^{1-m}a^{\frac{1}{2}}, -q^{1-m}a^{\frac{1}{2}}; q)_k}{(q, q^{-m}a^{\frac{1}{2}}, -q^{-m}a^{\frac{1}{2}}; q)_k} \\
& \quad \cdot \frac{(bq^{-m}, cq^{-m}, dq^{-m}, eq^{-m}, f; q)_k}{(aq^{1-m}/b, aq^{1-m}/c, aq^{1-m}/d, aq^{1-m}/e, aq^{1-2m}/f; q)_k} \left(\frac{q^2 a^2}{bcdef} \right)^k \\
& \stackrel{(4.1)}{=} \frac{(aq^{-m}, fq^m, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_{-m}}{(aq^{1-m}/f, q^{1+m}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_{-m}} \left(\frac{q^2 a^2}{bcdef} \right)^{-m} \\
& \quad \times \frac{(aq^{1-2m}, aq/de, aq^{1-m}/df, aq^{1-m}/ef, eq/c, fq^{1+m}/c, bq^m/a, bef/a; q)_{\infty}}{(aq^{1-m}/d, aq^{1-m}/e, aq^{1-2m}/f, aq/def, q^{1+m}/c, efq/c, be/a, bfq^m/a; q)_{\infty}} \\
& \quad \times 8\phi_7 \left[\begin{matrix} ef/c, q(ef/c)^{\frac{1}{2}}, -q(ef/c)^{\frac{1}{2}}, aq/bc, aq/cd, efq^m/a, eq^{-m}, f \\ (ef/c)^{\frac{1}{2}}, -(ef/c)^{\frac{1}{2}}, bef/a, def/a, aq^{1-m}/c, fq^{1+m}/c, eq/c \end{matrix}; q, \frac{bd}{a} \right] \\
& \quad + \frac{(aq^{-m}, fq^m, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_{-m}}{(aq^{1-m}/f, q^{1+m}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_{-m}} \left(\frac{q^2 a^2}{bcdef} \right)^{-m} \\
& \quad \times \frac{bq^m}{a} \frac{(aq^{1-2m}, bq^{1+m}/a, bq/c, bq/d, bq/e, bq^{1-m}/f, dq^{-m}; q)_{\infty}}{(aq^{1-m}/b, aq^{1-m}/c, aq^{1-m}/d, aq^{1-m}/e, aq^{1-2m}/f, bd/a, be/a; q)_{\infty}} \\
& \quad \times \frac{(eq^{-m}, f, aq/bc, bdefq^m/a^2, a^2q^{1-m}/bdef; q)_{\infty}}{(bfq^m/a, def/a, aq/def, q^{1+m}/c, b^2q/a; q)_{\infty}} \\
& \quad \times 8\phi_7 \left[\begin{matrix} b^2/a, qba^{-\frac{1}{2}}, -qba^{-\frac{1}{2}}, bq^{-m}, bc/a, bd/a, be/a, bfq^m/a \\ ba^{-\frac{1}{2}}, -ba^{-\frac{1}{2}}, bq^{1+m}/a, bq/c, bq/d, bq/e, bq^{1-m}/f \end{matrix}; q, \frac{a^2 q^2}{bcdef} \right],
\end{aligned}$$

which equals the right-hand side of (4.2). $\square$

The above proposition can be viewed as a semi-finite form of Bailey's ${}_6\psi_6$ summation formula. By taking $f = b$ and $m \rightarrow \infty$ in (4.2) while assuming $|a^2 q/bcde| < 1$, we get

$$\begin{aligned}
& {}_6\psi_6 \left[\begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e \end{matrix}; q, \frac{a^2 q}{bcde} \right] \\
& = \frac{(q/a, bd/a, aq/bc, aq/cd, aq, aq/de, aq/bd, aq/be; q)_{\infty}}{(q/b, q/c, q/d, aq/b, aq/d, aq/e, aq/bde, bde/a; q)_{\infty}} \\
& \quad \times \sum_{k=-\infty}^{\infty} \frac{(e; q)_k}{(aq/c; q)_k} (bd/a)^k \\
& \stackrel{(2.2)}{=} \frac{(q/a, bd/a, aq/bc, aq/cd, aq, aq/de, aq/bd, aq/be; q)_{\infty}}{(q/b, q/c, q/d, aq/b, aq/d, aq/e, aq/bde, bde/a; q)_{\infty}} \\
& \quad \times \frac{(q, aq/ce, bde/a, aq/bde; q)_{\infty}}{(aq/c, q/e, bd/a, a^2 q/bcde; q)_{\infty}}
\end{aligned}$$

$$= \frac{(aq, aq/bc, aq/bd, aq/be, aq/ce, aq/cd, aq/de, q, q/a; q)_{\infty}}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_{\infty}}.$$

Many proofs of the above identity have been found; see, for example, Slater and Lakin [11], Andrews [1], Askey and Ismail [2], Askey [3], Chen and Liu [5], Schlosser [9] and Jouhet and Schlosser [8]. Our proof shows that the semi-finite form of the ${}_6\psi_6$ summation is in essence a shifted version of the ${}_8\phi_7$ summation. This proof utilizes Ramanujan's ${}_1\psi_1$ summation (2.2). It would be interesting to find a proof using a semi-finite (or even finite) form which yields Bailey's ${}_6\psi_6$ summation in a direct limit, without the need to invoke another summation formula as above.

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CENTER FOR COMBINATORICS, LPMC, NANKAI UNIVERSITY, TIANJIN 300071, PEOPLE'S REPUBLIC OF CHINA

E-mail address: chen@nankai.edu.cn

CENTER FOR COMBINATORICS, LPMC, NANKAI UNIVERSITY, TIANJIN 300071, PEOPLE'S REPUBLIC OF CHINA

E-mail address: fu@nankai.edu.cn