ON THE LÉVY CONSTANTS FOR QUADRATIC IRRATIONALS

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ABSTRACT. We prove that the set of Lévy constants for quadratic irrationals is dense in $[\log \frac{\sqrt{5}+1}{2}, \infty)$.

1. INTRODUCTION

Let x be an irrational number and let $[a_0(x); a_1(x), a_2(x), \dots]$ be its regular continued fraction expansion. For any $n \geq 1$, we denote by $p_n(x)/q_n(x) := [a_0(x); a_1(x), a_2(x), \dots, a_n(x)]$ the n th convergent of x .

With the conventions $p_{-1}(x) = 1$, $q_{-1}(x) = 0$, $p_0(x) = a_0(x)$, $q_0(x) = 1$, we have

$$(1.1) \quad p_{n+1}(x) = a_{n+1}(x) \cdot p_n(x) + p_{n-1}(x), \quad n \geq 0,$$

$$(1.2) \quad q_{n+1}(x) = a_{n+1}(x) \cdot q_n(x) + q_{n-1}(x), \quad n \geq 0.$$

The famous Theorem of P. Lévy [6] states that

$$\lim_{n \rightarrow \infty} \frac{\log q_n(x)}{n} = \frac{\pi^2}{12 \log 2}$$

for almost all $x \in \mathbb{R}$ in the sense of Lebesgue.

Definition 1.1. For any irrational number $x \in \mathbb{R}$, if $\lim_{n \rightarrow \infty} \frac{\log q_n(x)}{n}$ exists, we say x has a Lévy constant and denote the limit by $\beta(x)$.

Notice that if x has a Lévy constant, then $\beta(x) \geq \log \frac{\sqrt{5}+1}{2}$.

H. Jager and P. Liardet [4] (see also C. Faivre [2]) proved that every quadratic irrational has a Lévy constant. Let

$$\mathbf{B} = \{\beta(x) : x \text{ is a quadratic irrational}\}.$$

E. P. Golubeva [3] showed that $\frac{\pi^2}{12 \log 2}$ is a limit point of $\mathbf{B}$; she also presented some evidence that $\log \frac{\sqrt{5}+1}{2}$ may be an isolated point of $\mathbf{B}$. However, we prove that:

Theorem 1.2. $\mathbf{B}$ is dense in $[\log \frac{\sqrt{5}+1}{2}, \infty)$.

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Remark 1.3. If $x = [a_0; a_1, a_2, \dots, \overline{a_{r+1}, a_{r+2}, \dots, a_{r+s}}]$, from [4] and [1], we have

$$(1.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log q_n(x) = \frac{1}{s} \log \rho(W)$$

where

$$W = \begin{pmatrix} a_{r+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{r+2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{r+s} & 1 \\ 1 & 0 \end{pmatrix}$$

and $\rho(\cdot)$ denotes the spectral radius of the matrix. Using (1.3), it may be shown that Theorem 1.2 holds, but this nice theorem has not been acknowledged before and the method we used does give a short, simple and self-contained proof of Theorem 1.2.

2. PROOF OF THEOREM 1.2

In this section, we give the proof of Theorem 1.2.

For any $n \geq 1$ and $(a_1, a_2, \dots, a_n) \in \mathbb{N}^n$, let $q_n(a_1, a_2, \dots, a_n)$ be the denominator of the finite continued fraction $[0; a_1, a_2, \dots, a_n]$. The following lemma is proved in [7]; we present it here for completeness.

Lemma 2.1. *For any $n \geq 1$ and $1 \leq k \leq n$, we have*

$$\frac{a_k + 1}{2} \leq \frac{q_n(a_1, a_2, \dots, a_n)}{q_{n-1}(a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n)} \leq a_k + 1 \quad (q_0 := 1).$$

Proof. By (1.2),

$$\begin{aligned} \frac{q_k(a_1, \dots, a_k)}{q_{k-1}(a_1, \dots, a_{k-1})} &= \frac{a_k q_{k-1}(a_1, \dots, a_{k-1}) + q_{k-2}(a_1, \dots, a_{k-2})}{q_{k-1}(a_1, \dots, a_{k-1})} \leq a_k + 1, \\ \frac{q_k(a_1, \dots, a_k)}{q_{k-1}(a_1, \dots, a_{k-1})} &\geq a_k \geq \frac{a_k + 1}{2}, \\ \frac{q_{k+1}(a_1, \dots, a_{k+1})}{q_k(a_1, \dots, a_{k-1}, a_{k+1})} &= \frac{a_{k+1} q_k(a_1, \dots, a_k) + q_{k-1}(a_1, \dots, a_{k-1})}{a_{k+1} q_{k-1}(a_1, \dots, a_{k-1}) + q_{k-2}(a_1, \dots, a_{k-2})} \\ &= \frac{(a_{k+1} a_k + 1) q_{k-1}(a_1, \dots, a_{k-1}) + a_{k+1} q_{k-2}(a_1, \dots, a_{k-2})}{a_{k+1} q_{k-1}(a_1, \dots, a_{k-1}) + q_{k-2}(a_1, \dots, a_{k-2})} \\ &\leq \frac{a_{k+1} a_k q_{k-1}(a_1, \dots, a_{k-1})}{a_{k+1} q_{k-1}(a_1, \dots, a_{k-1})} \\ &\quad + \frac{q_{k-1}(a_1, \dots, a_{k-1}) + a_{k+1} q_{k-2}(a_1, \dots, a_{k-2})}{a_{k+1} q_{k-1}(a_1, \dots, a_{k-1}) + q_{k-2}(a_1, \dots, a_{k-2})} \\ &\leq a_k + 1. \\ \frac{q_{k+1}(a_1, \dots, a_{k+1})}{q_k(a_1, \dots, a_{k-1}, a_{k+1})} &= \frac{(a_{k+1} a_k + 1) q_{k-1}(a_1, \dots, a_{k-1}) + a_{k+1} q_{k-2}(a_1, \dots, a_{k-2})}{a_{k+1} q_{k-1}(a_1, \dots, a_{k-1}) + q_{k-2}(a_1, \dots, a_{k-2})} \\ &\geq \frac{(a_{k+1} a_k + 1) q_{k-1}(a_1, \dots, a_{k-1})}{(a_{k+1} + 1) q_{k-1}(a_1, \dots, a_{k-1})} \\ &= \frac{a_{k+1} a_k + 1}{a_{k+1} + 1} \geq \frac{a_k + 1}{2}. \end{aligned}$$

Using (1.2) and induction, we get the desired result. $\square$

The following is the proof of Theorem 1.2.

Proof. For any $\log \frac{\sqrt{5}+1}{2} < \lambda < \infty$ and any $0 < \epsilon < \lambda - \log \frac{\sqrt{5}+1}{2}$, choose $N \in \mathbb{N}$ large enough such that

$$e^{N(\lambda + \epsilon - \log \frac{\sqrt{5}+1}{2}) + \log \frac{\sqrt{5}+1}{2}} - 1 \geq 2 \cdot e^{N(\lambda - \epsilon - \log \frac{\sqrt{5}+1}{2}) + \log \frac{\sqrt{5}+1}{2}}.$$

Choose $b \in \mathbb{N}$ such that

$$(2.1) \quad 2 \cdot e^{N(\lambda - \epsilon - \log \frac{\sqrt{5}+1}{2}) + \log \frac{\sqrt{5}+1}{2}} - 1 \leq b \leq e^{N(\lambda + \epsilon - \log \frac{\sqrt{5}+1}{2}) + \log \frac{\sqrt{5}+1}{2}} - 1.$$

Let $x \in (0, 1)$ such that $a_n(x) = b$ if $n = kN$ for some $k \in \mathbb{N}$, and $a_n(x) = 1$ otherwise. x has a purely periodic continued fraction expansion, thus by Lagrange's theorem (see [5], page 56), x is a quadratic irrational.

For any $n \geq N$, there exists $k \in \mathbb{N}$ such that $kN \leq n < (k+1)N$. By Lemma 2.1,

$$(2.2) \quad q_n(x) \leq q_{n-k}(1, 1, \dots, 1)(b+1)^k \leq c_1 \left(\frac{\sqrt{5}+1}{2} \right)^{n-k} (b+1)^k,$$

$$(2.3) \quad q_n(x) \geq q_{n-k}(1, 1, \dots, 1) \left(\frac{b+1}{2} \right)^k \geq c_2 \left(\frac{\sqrt{5}+1}{2} \right)^{n-k} \left(\frac{b+1}{2} \right)^k,$$

where c_1, c_2 in (2.2) and (2.3) are positive constants which do not depend on n . Thus by (2.1), we have

$$\begin{aligned} \beta(x) &\leq \limsup_{n \rightarrow \infty} \frac{\log \left(c_1 \left(\frac{\sqrt{5}+1}{2} \right)^{n-k} (b+1)^k \right)}{n} \\ &= \limsup_{n \rightarrow \infty} \frac{k \left(\log(b+1) - \log \frac{\sqrt{5}+1}{2} \right)}{n} + \log \frac{\sqrt{5}+1}{2} \\ &\leq \frac{1}{N} \left(\log(b+1) - \log \frac{\sqrt{5}+1}{2} \right) + \log \frac{\sqrt{5}+1}{2} \leq \lambda + \epsilon, \\ \beta(x) &\geq \liminf_{n \rightarrow \infty} \frac{\log \left(c_2 \left(\frac{\sqrt{5}+1}{2} \right)^{n-k} \left(\frac{b+1}{2} \right)^k \right)}{n} \\ &= \liminf_{n \rightarrow \infty} \frac{k \left(\log(b+1) - \log \frac{\sqrt{5}+1}{2} \right)}{n} + \log \frac{\sqrt{5}+1}{2} \\ &\geq \liminf_{k \rightarrow \infty} \frac{k \cdot \left(\log \frac{b+1}{2} - \log \frac{\sqrt{5}+1}{2} \right)}{(k+1)N} + \log \frac{\sqrt{5}+1}{2} \\ &= \frac{1}{N} \left(\log \frac{b+1}{2} - \log \frac{\sqrt{5}+1}{2} \right) + \log \frac{\sqrt{5}+1}{2} \geq \lambda - \epsilon. \end{aligned}$$

Therefore

$$\beta(x) \in [\lambda - \epsilon, \lambda + \epsilon].$$

□

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