THE RANK OF ELLIPTIC CURVES WITH RATIONAL 2-TORSION POINTS OVER LARGE FIELDS

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ABSTRACT. Let K be a number field, $\overline{K}$ an algebraic closure of K , G_K the absolute Galois group $\text{Gal}(\overline{K}/K)$, K_{ab} the maximal abelian extension of K and E/K an elliptic curve defined over K . In this paper, we prove that if all 2-torsion points of E/K are K -rational, then for each $\sigma \in G_K$, $E((K_{ab})^\sigma)$ has infinite rank, and hence $E(\overline{K}^\sigma)$ has infinite rank.

Let K be a number field, $\overline{K}$ an algebraic closure of K , and $G_K := \text{Gal}(\overline{K}/K)$ the absolute Galois group of $\overline{K}$ over K . Let E/K be an elliptic curve defined over K .

In [4], M. Larsen proved that for any E/K over K , there exists a nonempty open subset Σ of G_K such that for every $\sigma \in \Sigma$, the Mordell-Weil group $E(\overline{K}^\sigma)$ of E over the fixed field under σ has infinite rank.

It is natural to ask if such an open subset can be the whole Galois group G_K . We have a positive answer for elliptic curves defined over $\mathbb{Q}$ and for elliptic curves with certain rational points over the ground field. In [2], we proved that for any elliptic curve $E/\mathbb{Q}$, the rank of $E(\overline{\mathbb{Q}}^\sigma)$ is infinite for every $\sigma \in G_{\mathbb{Q}}$. Our approach in [2] is arithmetic: taking advantage of the modularity of elliptic curves over $\mathbb{Q}$ and the theory of complex multiplication, and constructing an infinite supply of rational points of E consisting of Heegner points. Also, in [1], we proved that for any elliptic curve E/K over a number field K with a K -rational point which is neither 2-torsion nor 3-torsion, the rank of $E(\overline{K}^\sigma)$ is infinite for every $\sigma \in G_K$. In [1], by a geometric approach, we constructed rational points, essentially by searching for sufficiently rational subvarieties of certain quotients of the n -fold product E^n of E .

This paper has been motivated by [1], [2] and [4]. Extending a method of M. Larsen [4], we prove in this paper that if all 2-torsion points of E/K are K -rational, then there are three hyperelliptic curves X_1, X_2, X_3 defined over K of genus 1, 3, and 3 respectively forming a biquadratic extension of $\mathbb{P}^1$ such that each admits a nonconstant K -morphism to E . By this result, we prove that if all 2-torsion points of E/K are K -rational, then for each $\sigma \in G_K$, $E((K_{ab})^\sigma)$ has infinite rank, where K_{ab} is the maximal abelian extension of K , hence $E(\overline{K}^\sigma)$ has infinite rank. Here, we argue by using Diophantine geometry as in [4], which is a completely different method from the one that we used in [2].

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We will need the following lemmas to prove Proposition 4 and we will use the notation in Lemma 2 in the proof of Proposition 4.

Lemma 1. *If w_1, w_2 and w_3 are distinct elements in K , then there exists an ordered triple (i, j, k) such that $\{i, j, k\} = \{1, 2, 3\}$ as sets, and $w_i + w_j - 2w_k \neq 0$ and $w_i + w_j - w_k \neq 0$.*

Proof. Without loss of generality, suppose $w_1 + w_2 - 2w_3 = 0$. Then if $w_1 + w_3 - 2w_2 = 0$ or $w_2 + w_3 - 2w_1 = 0$, then it implies that $w_3 = w_2$ or $w_3 = w_1$ respectively, which is a contradiction to distinct w_i 's. So if $w_1 + w_2 - 2w_3 = 0$, then $w_1 + w_3 - 2w_2 \neq 0$ and $w_2 + w_3 - 2w_1 \neq 0$.

If $w_1 + w_3 - w_2 \neq 0$, the ordered triple $(i, j, k) = (1, 3, 2)$ satisfies the condition. If $w_1 + w_3 - w_2 = 0$, then similarly we show that $w_1 + w_2 - w_3 \neq 0$ and $w_2 + w_3 - w_1 \neq 0$. So the ordered triple $(i, j, k) = (2, 3, 1)$ satisfies the condition. $\square$

Lemma 2. *For given distinct elements α, β and $\gamma \in K$ such that $\beta + \gamma - 2\alpha \neq 0$ and $\beta + \gamma - \alpha \neq 0$, suppose that there exist elements p, q , and s in K satisfying the following conditions (1) through (7):*

- (1) p, q and s are all distinct,
- (2) $(p - q)^3 + (s - q)(p - s)[s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma)] \neq 0$,
- (3) $(p - q)^3(\beta + \gamma - \alpha) + \alpha(s - q)(p - s)[s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma)] \neq 0$,
- (4) $(p - q)^3\alpha + (\beta + \gamma - \alpha)(s - q)(p - s)[s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma)] \neq 0$,
- (5) $(p - q)^3(\gamma - \alpha) + (\alpha - \beta)(s - q)(p - s)[s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma)] \neq 0$,
- (6) $(p - q)^3(\beta - \alpha) + (\alpha - \gamma)(s - q)(p - s)[s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma)] \neq 0$,
- (7) $s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma) \neq 0$.

Then let F and G be the left-hand sides of (2) and (3) respectively, and put

$$\begin{aligned} t &= \frac{G}{F}, \\ A &= (\beta + \gamma - 2\alpha)(s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma)), \\ B_1 &= (t - \alpha)(s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma))(p + q), \\ B_2 &= (t - (\beta + \gamma - \alpha))(s^2(\alpha - \beta) + s(p - q)(2\gamma - \alpha - \beta) + pq(\beta - \alpha)), \\ C_1 &= (t - \alpha)(s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma))pq, \\ C_2 &= (t - (\beta + \gamma - \alpha))(s^2(p(\gamma - \beta) + q(\alpha - \gamma)) + spq(\beta - \alpha)). \end{aligned}$$

Define three polynomials in x :

$$\begin{aligned} h_1(x) &= A(t - \beta)x^2 - ((\gamma - \alpha)B_1 - (\alpha - \beta)B_2)x + ((\gamma - \alpha)C_1 - (\alpha - \beta)C_2), \\ h_2(x) &= A(t - \gamma)x^2 - ((\beta - \alpha)B_1 - (\alpha - \gamma)B_2)x + ((\beta - \alpha)C_1 - (\alpha - \gamma)C_2), \\ h_3(x) &= Ax^2 - (B_1 - B_2)x + (C_1 - C_2). \end{aligned}$$

Then, $A \neq 0$ and $t \neq 0, \beta + \gamma, \beta, \gamma, \beta + \gamma - \alpha, \alpha$. In particular, the h_i are quadratic polynomials in x defined over K .

Proof. Conditions (3) through (6) imply that $t \neq 0, \beta + \gamma, \beta, \gamma$. Since $\beta + \gamma - 2\alpha \neq 0$, conditions (1) and (7) imply that $t \neq \beta + \gamma - \alpha$ and $t \neq \alpha$. By condition (7) and the assumption that $\beta + \gamma - 2\alpha \neq 0$, we see that $A \neq 0$.

Since $A \neq 0$ and $t \neq \beta, \gamma$ and t, A, B_1, B_2, C_1, C_2 are in K , the h_i are quadratic polynomials in x defined over K . $\square$

Lemma 3. *For given distinct elements α, β and $\gamma \in K$ such that $\beta + \gamma - 2\alpha \neq 0$ and $\beta + \gamma - \alpha \neq 0$, there exist elements p, q , and $s \in K$ satisfying the following*

conditions:

- p, q , and s satisfy conditions (1) through (7) in Lemma 2,
- (8) p, q and s are not equal to $\frac{s(p(\gamma - \beta) + q(\alpha - \gamma)) + pq(\beta - \alpha)}{s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma)}$,
- (9) $\alpha(p - s) - \beta(q - s) \neq 0$,
- (10) each quadratic polynomial $h_i(x)$ defined in Lemma 2 has no double zero,
- (11) any two of three quadratic polynomials $h_i(X)$ have no common zero,
- (12) p, q , and s are not zeros of any of $h_i(x)$.

Proof. Note that the set of triples (p, q, s) satisfying each condition from (1) through (9) is a finite number of intersections of nonempty open subsets in $\mathbb{A}^3(K)$ defined by the nontrivial polynomials over K given in each condition. So by the Hilbert irreducibility theorem [3, Chapter 9], the set of $(p, q, s) \in \mathbb{A}^3(K)$ satisfying conditions (1) through (9) is infinite.

For any p, q, s satisfying (1) through (9), Lemma 2 shows that $h_i(x)$ are quadratic polynomials. Let $D(h_i)$ denote the discriminant of the quadratic polynomial h_i . Then,

$$\begin{aligned} D(h_1) &= ((\gamma - \alpha)B_1 - (\alpha - \beta)B_2)^2 - 4A(t - \beta)((\gamma - \alpha)C_1 - (\alpha - \beta)C_2), \\ D(h_2) &= ((\beta - \alpha)B_1 - (\alpha - \gamma)B_2)^2 - 4A(t - \gamma)((\beta - \alpha)C_1 - (\alpha - \gamma)C_2), \\ D(h_3) &= (B_1 - B_2)^2 - 4A(C_1 - C_2). \end{aligned}$$

We can show that the discriminants $D(h_i)$ are nontrivial rational functions in p, q and s . For example, we show¹ that the coefficients of the s^{10} -term in $F^2D(h_1)$ (resp. $F^2D(h_2)$, $F^2D(h_3)$) is $-(\alpha - \beta)^6(\beta + \gamma - 2\alpha)^2$ (resp. $-(\alpha - \beta)^4(\alpha - \gamma)^2(\beta + \gamma - 2\alpha)^2$, $-(\alpha - \beta)^4(\beta + \gamma - 2\alpha)^2$), which is not zero by the assumption on α, β and γ . Also the h_i have the nontrivial relation $h_1 - h_2 = (\gamma - \beta)h_3$. The set of triples (p, q, s) at which $D(h_i)$ for some $i = 1, 2, 3$ vanishes or two of $h_1(x), h_2(x), h_3(x)$ have a common zero is a finite number of unions of proper Zariski closed sets in $\mathbb{A}^3(K)$. Its complementary set is a finite number of intersections of nonempty Zariski open sets, which is infinite by the Hilbert irreducibility theorem. Also, the set of triples (p, q, s) such that p, q , and s are not zeros of any $h_i(x)$ is also an intersection of three nonempty Zariski open sets, which is infinite. So the set of $(p, q, s) \in \mathbb{A}^3(K)$ satisfying conditions (1) through (12) is a finite number of intersections of nonempty Zariski open sets, hence it is nonempty (in fact, it is infinite). $\square$

Proposition 4. *If all 2-torsion points of E/K are K -rational (i.e. $E[2] \subset E(K)$), then there exist three hyperelliptic curves X_1, X_2, X_3 defined over K of genus 1, 3, 3 respectively which map onto E over K and are defined by*

$$(*) \quad \begin{cases} X_1 : y^2 = af(x)g(x), \\ X_2 : y^2 = bf(x)h(x), \\ X_3 : y^2 = abg(x)h(x) \end{cases}$$

for some constants $a, b \in K$, and some polynomials $f, g, h \in K[x]$. Hence the curves X_1, X_2 and X_3 form a biquadratic extension of $\mathbb{P}^1$ over K .

¹This can be shown by a moderate amount of direct calculation and has been verified using **Maple** (full documentation available upon request).

Proof. Let $y^2 = (x - \alpha)(x - \beta)(x - \gamma)$ be a Weierstrass equation of E . Here, α, β , and γ are distinct elements in K , since the 2-torsion points of E are K -rational. Then, by Lemma 1, we may assume that

$$(13) \quad \beta + \gamma - 2\alpha \neq 0 \text{ and } \beta + \gamma - \alpha \neq 0.$$

The main strategy for proving the proposition is to construct three curves defined by

$$\begin{aligned} X_1 : y^2 &= a(x - p)(x - q)(x - s)(x - r), \\ X_2 : y^2 &= b(x - p)(x - q)(x - x_1)(x - x_2)(x - x_3)(x - y_1)(y - y_2)(y - y_3), \\ X_3 : y^2 &= ab(x - s)(x - r)(x - x_1)(x - x_2)(x - x_3)(x - y_1)(y - y_2)(y - y_3) \end{aligned}$$

together with three dominant morphisms $\phi_i : X_i \rightarrow E$ for $i = 1, 2, 3$ defined by K -rational functions which map the set of ramification points of each X_i over $\mathbb{P}^1$ onto the set of ramification points of E over $\mathbb{P}^1$ as follows:

$$(14) \quad \phi_1 : (p, 0) \mapsto (\alpha, 0), \quad (q, 0) \mapsto (\beta, 0), \quad (r, 0) \mapsto (\gamma, 0), \text{ and } (s, 0) \mapsto \infty,$$

$$(15) \quad \phi_2 : (p, 0), (q, 0) \mapsto (\alpha, 0), \quad (x_1, 0), (x_2, 0) \mapsto (\beta, 0), \\ (y_1, 0), (y_2, 0) \mapsto (\gamma, 0), \text{ and } (x_3, 0), (y_3, 0) \mapsto \infty,$$

$$(16) \quad \phi_3 : (r, 0), (s, 0) \mapsto (\alpha, 0), \quad (x_1, 0), (x_2, 0) \mapsto (\gamma, 0), \\ (y_1, 0), (y_2, 0) \mapsto (\beta, 0), \text{ and } (x_3, 0), (y_3, 0) \mapsto \infty.$$

First, under the assumption (13), by Lemma 3, we can choose $p, q, s \in K$ satisfying conditions (1) through (12).

Then we find X_1 and a dominant morphism $\phi_1 : X_1 \rightarrow E$ satisfying (14) in terms of the given p, q, s, α, β and γ . Since X_1 has genus 1 and ϕ_1 sends $(s, 0) \in X_1$ to $\infty \in E$ as a dominant map, we try to set $\phi_1(x, y) = \left(\frac{c_1x + c_0}{(x - s)}, \frac{c_3}{(x - s)^2}y \right)$ as rational functions for some constants c_i and try to find constants r and c_i in terms of p, q, s, α, β and γ . For ϕ_1 to satisfy condition (14), we need three relations,

$$(17) \quad c_1p + c_0 = \alpha(p - s), \quad c_1q + c_0 = \beta(q - s), \quad c_1r + c_0 = \gamma(r - s).$$

Also, for ϕ_1 to be a dominant morphism to E , the following must be valid:

$$(18) \quad Z(\phi_1(x, y)) = 0, \text{ where } Z(x, y) = y^2 - (x - \alpha)(x - \beta)(x - \gamma).$$

Next, we find X_2 and X_3 as well as dominant morphisms ϕ_2 and ϕ_3 satisfying conditions (15) and (16). Since $(x_3, 0)$ and $(y_3, 0) \in X_i$ are mapped to ∞ under ϕ_i for $i = 2, 3$ and two distinct points $(x_1, 0)$ and $(x_2, 0)$ (and $(y_1, 0)$ and $(y_2, 0)$) are mapped onto the same point in E , ϕ_i needs to be defined by rational functions with polynomials of degree at least 2 as numerators and with $(x - x_3)(x - y_3)$ as denominators. We let for some constants d_i and e_i ,

$$\phi_2(x, y) = \left(\frac{d_2x^2 + d_1x + d_0}{(x - x_3)(x - y_3)}, \frac{d_3}{((x - x_3)(x - y_3))^2}y \right), \text{ and}$$

$$\phi_3(x, y) = \left(\frac{e_2x^2 + e_1x + e_0}{(x - x_3)(x - y_3)}, \frac{e_3}{((x - x_3)(x - y_3))^2}y \right).$$

In this setting, for ϕ_2 and ϕ_3 to satisfy conditions (15) and (16) respectively, the following six equations must hold:

$$(**) \quad \begin{cases} (d_2 - \alpha)x^2 + (d_1 + \alpha(x_3 + y_3))x + d_0 - \alpha x_3 y_3 = (d_2 - \alpha)(x - p)(x - q), \\ (d_2 - \beta)x^2 + (d_1 + \beta(x_3 + y_3))x + d_0 - \beta x_3 y_3 = (d_2 - \beta)(x - x_1)(x - x_2), \\ (d_2 - \gamma)x^2 + (d_1 + \gamma(x_3 + y_3))x + d_0 - \gamma x_3 y_3 = (d_2 - \gamma)(x - y_1)(x - y_2), \\ (e_2 - \alpha)x^2 + (e_1 + \alpha(x_3 + y_3))x + e_0 - \alpha x_3 y_3 = (e_2 - \alpha)(x - r)(x - s), \\ (e_2 - \beta)x^2 + (e_1 + \beta(x_3 + y_3))x + e_0 - \beta x_3 y_3 = (e_2 - \beta)(x - y_1)(x - y_2), \\ (e_2 - \gamma)x^2 + (e_1 + \gamma(x_3 + y_3))x + e_0 - \gamma x_3 y_3 = (e_2 - \gamma)(x - x_1)(x - x_2). \end{cases}$$

Then we find constants d_j , e_j , x_j , and y_j by equating the coefficients of constant terms, x and x^2 -terms in each equation in (**) and requiring that

$$(19) \quad Z(\phi_i(x, y)) = 0, \text{ where } Z(x, y) = y^2 - (x - \alpha)(x - \beta)(x - \gamma) \text{ and } i = 2, 3.$$

Now we solve for $r, x_i, y_i, c_i, d_i, e_i$ and give a solution to all equations from (17) to (**) and (19) listed in the above for X_i and ϕ_i as follows.

A moderate amount of direct calculation from (17) gives the following in terms of p, q, s, α, β , and γ :

$$(20) \quad r = \frac{s(p(\gamma - \beta) + q(\alpha - \gamma)) + pq(\beta - \alpha)}{s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma)} \in K,$$

$$(21) \quad c_0 = \frac{(\beta - \alpha)pq + s(\alpha q - \beta p)}{(p - q)} \in K,$$

$$(22) \quad c_1 = \frac{\alpha(p - s) - \beta(q - s)}{(p - q)} \in K \setminus \{0\}.$$

Conditions (1), (7) and (8) imply that r, c_0 and c_1 are well-defined and that p, q, s and r are all distinct. Also, c_1 is nonzero by condition (9).

If we let

$$(23) \quad c_3 = (\alpha - \beta) \in K \setminus \{0\},$$

$$(24) \quad a = \frac{(s - q)(p - s)[s(\alpha - \beta) + p(\gamma - \alpha) + q(\beta - \gamma)]}{(p - q)^3} \in K \setminus \{0\},$$

then $a \neq 0$ by conditions (1) and (7) and ϕ_1 is a nonconstant map since $c_3 \neq 0$. We can verify² that ϕ_1 is a dominant morphism over K satisfying condition (14) and that (18) holds.

Next we let

$$b = (t - \alpha)(t - \beta)(t - \gamma) \in K. \text{ (Then } b \neq 0 \text{ by Lemma 2.)}$$

We can show² that a defined in (24) equals $\frac{\beta + \gamma - \alpha - t}{t - \alpha}$. Hence we get

$$ab = (\beta - t)(\gamma - t)(\beta + \gamma - \alpha - t) \in K \setminus \{0\}.$$

Then, for the quadratic polynomials $h_i(x)$ defined in Lemma 2, let x_1 and x_2 be zeros of $h_1(x)$, y_1 and y_2 zeros of $h_2(x)$, and x_3 and y_3 zeros of $h_3(x)$. Then p, q, s, r, x_i, y_j are all distinct by conditions (1), (10), (11) and (12).

²This can be shown by a moderate amount of direct calculation and has been verified using **Maple** (full documentation available upon request).

Using the same notation as in Lemma 2, we define the curves X_i and the maps $\phi_i : X_i \rightarrow E$ for $i = 1, 2, 3$ as follows (note that X_1 and ϕ_1 have been found in (20) through (24)):

$$\begin{aligned}
X_1 : y^2 &= a(x-p)(x-q)(x-s)\left(x - \frac{s(p(\gamma-\beta)+q(\alpha-\gamma))+pq(\beta-\alpha)}{s(\alpha-\beta)+p(\gamma-\alpha)+q(\beta-\gamma)}\right), \\
X_2 : y^2 &= \frac{(t-\alpha)}{A^3}(x-p)(x-q)h_1(x)h_2(x)h_3(x) \\
&\quad (= b(x-p)(x-q)(x-x_1)(x-x_2)(x-y_1)(x-y_2)(x-x_3)(x-y_3)), \\
X_3 : y^2 &= \frac{(\beta+\gamma-\alpha-t)}{A^3}(x-s)\left(x - \frac{s(p(\gamma-\beta)+q(\alpha-\gamma))+pq(\beta-\alpha)}{s(\alpha-\beta)+p(\gamma-\alpha)+q(\beta-\gamma)}\right)h_1(x)h_2(x)h_3(x) \\
&\quad (= ab(x-s)(x-r)(x-x_1)(x-x_2)(x-y_1)(x-y_2)(x-x_3)(x-y_3)), \\
\phi_1(x, y) &= \left(\frac{(\alpha(p-s)-\beta(q-s))x+(\beta-\alpha)pq+s(\alpha q-\beta p)}{(p-q)(x-s)}, \quad \frac{(\alpha-\beta)}{(x-s)^2}y \right), \\
\phi_2(x, y) &= \left(\frac{h_1(x)+\beta h_3(x)}{h_3(x)}, \quad \frac{A^2}{h_3(x)^2}y \right) \\
&\quad \left(= \left(\frac{Atx^2-((\beta+\gamma-\alpha)B_1-\alpha B_2)x+((\beta+\gamma-\alpha)C_1-\alpha C_2)}{Ax^2-(B_1-B_2)x+(C_1-C_2)}, \right. \right. \\
&\quad \left. \left. \frac{A^2}{(Ax^2-(B_1-B_2)x+(C_1-C_2))^2}y \right) \right), \\
\phi_3(x, y) &= \left(\frac{-h_2(x)+\beta h_3(x)}{h_3(x)}, \quad \frac{A^2}{h_3(x)^2}y \right) \\
&\quad \left(= \left(\frac{A(\beta+\gamma-t)x^2-(\alpha B_1-(\beta+\gamma-\alpha)B_2)x+(\alpha C_1-(\beta+\gamma-\alpha)C_2)}{Ax^2-(B_1-B_2)x+(C_1-C_2)}, \right. \right. \\
&\quad \left. \left. \frac{A^2}{(Ax^2-(B_1-B_2)x+(C_1-C_2))^2}y \right) \right).
\end{aligned}$$

Then, since p, q, s, r, x_i, y_i are all distinct and $a \neq 0, b \neq 0, A \neq 0, t \neq 0, \alpha, \beta + \gamma - \alpha, \beta + \gamma$, X_1, X_2, X_3 are nonsingular curves of genus 1, 3, 3 respectively and ϕ_i are nonconstant maps. And since $A, a, b \in K$ and $h_i(x)$ are defined over K , X_i and ϕ_i are defined over K .

Before we show that the maps ϕ_2 and ϕ_3 defined above satisfy (15) and (16) respectively (i.e. satisfy the equations in (**)) and satisfy (19), we verify the following identities³:

$$(25) \quad h_1(x) + (\beta - \alpha)h_3(x) = A(t - \alpha)(x - p)(x - q),$$

$$(26) \quad h_1(x) + (\beta - \gamma)h_3(x) = h_2(x),$$

$$(27) \quad -h_2(x) + (\beta - \alpha)h_3(x) = A(\beta + \gamma - \alpha - t)(x - s)(x - r),$$

$$(28) \quad -h_2(x) + (\beta - \gamma)h_3(x) = -h_1(x).$$

For the validity of (15) for ϕ_2 , we show that

$$\begin{aligned}
\phi_2(p, 0) &= \left(\frac{h_1(p) + \beta h_3(p)}{h_3(p)}, 0 \right) = \left(\frac{(\alpha - \beta)h_3(p) + \beta h_3(p)}{h_3(p)}, 0 \right) = (\alpha, 0) \text{ by (25),} \\
\phi_2(q, 0) &= \left(\frac{h_1(q) + \beta h_3(q)}{h_3(q)}, 0 \right) = \left(\frac{(\alpha - \beta)h_3(q) + \beta h_3(q)}{h_3(p)}, 0 \right) = (\alpha, 0) \text{ by (25),} \\
\text{for } i = 1, 2, \phi_2(x_i, 0) &= \left(\frac{h_1(x_i) + \beta h_3(x_i)}{h_3(x_i)}, 0 \right) = (\beta, 0), \text{ since } h_1(x_i) = 0,
\end{aligned}$$

³This can be shown by a moderate amount of direct calculation and has been verified using **Maple** (full documentation available upon request).

$\phi_2(y_i, 0) = \left(\frac{h_1(y_i) + \beta h_3(y_i)}{h_3(y_i)}, 0 \right) = \left(\frac{(\gamma - \beta)h_3(y_i) + \beta h_3(y_i)}{h_3(y_i)}, 0 \right) = (\gamma, 0)$ by (26), since $h_2(y_i) = 0$, and $\phi_2(x_3, 0) = \infty = \phi_2(x_3, 0)$, since $h_3(x_3) = 0 = h_3(y_3)$.

Similarly, (16) can be verified for ϕ_3 .

For the validity of (19) for $i = 2$, by using (25) and (26), we show that

$$\begin{aligned}
 \frac{A^4}{h_3(x)^4} y^2 &= \left(\frac{h_1(x) + \beta h_3(x)}{h_3(x)} - \alpha \right) \left(\frac{h_1(x) + \beta h_3(x)}{h_3(x)} - \beta \right) \left(\frac{h_1(x) + \beta h_3(x)}{h_3(x)} - \gamma \right) \\
 &= \frac{(h_1(x) + (\beta - \alpha)h_3(x))(h_1(x))(h_1(x) + (\beta - \gamma)h_3(x))}{h_3(x)^3} \\
 &= \frac{A(t - \alpha)(x - p)(x - q)h_1(x)h_2(x)}{h_3(x)^3}
 \end{aligned}$$

which reduces to $y^2 = \frac{(t - \alpha)}{A^3}(x - p)(x - q)h_1(x)h_2(x)h_3(x)$ and defines X_2 .

Similarly, for the validity of (19) for $i = 3$, by using (27) and (28), we show that

$$\begin{aligned}
 \frac{A^4}{h_3(x)^4} y^2 &= \left(\frac{-h_2(x) + \beta h_3(x)}{h_3(x)} - \alpha \right) \left(\frac{-h_2(x) + \beta h_3(x)}{h_3(x)} - \beta \right) \left(\frac{-h_2(x) + \beta h_3(x)}{h_3(x)} - \gamma \right) \\
 &= \frac{(-h_2(x) + (\beta - \alpha)h_3(x))(-h_2(x))(-h_2(x) + (\beta - \gamma)h_3(x))}{h_3(x)^3} \\
 &= \frac{A(\beta + \gamma - \alpha - t)(x - s)(x - r)(-h_2(x))(-h_1(x))}{h_3(x)^3}
 \end{aligned}$$

which reduces to $y^2 = \frac{(\beta + \gamma - \alpha - t)}{A^3}(x - s)(x - r)h_1(x)h_2(x)h_3(x)$ and defines X_3 .

Note that all X_i and ϕ_i are defined over K . We have shown that there exist three nonsingular hyperelliptic curves X_i over K which map onto E via dominant morphisms $\phi_i : X_i \rightarrow E$ over K for $i = 1, 2, 3$, and this completes the proof. $\square$

Remark 5. We note that for an arbitrary elliptic curve E/K and for any integer $g \geq 1$, there exist three hyperelliptic curves X_1, X_2, X_3 of genus 1, g, g respectively over $\bar{K}$ which are defined as in (*) of Proposition 4. But, in particular, Proposition 4 shows that when $g = 3$ we can find three such curves X_i defined over K under the assumption that all 2-torsion points of E/K are K -rational.

Lemma 6. *Let K be an infinite field of finite type and $P_1(x), P_2(x), \dots, P_k(x)$ a sequence of polynomials in $K[x]$ each of which has a zero of odd multiplicity. If L/K is a finite separable extension of K , then there exists $a \in K$ such that $P_i(a)$ is not a perfect square in L for $i = 1, \dots, k$.*

Proof. See [4, Lemma 4]. $\square$

Theorem 7. *Let K be a number field, K_{ab} the maximal abelian extension of K , and E/K an elliptic curve with all K -rational 2-torsion points. Then for each $\sigma \in G_K$, the rank of $E((K_{ab})^\sigma)$ is infinite, therefore $E(\bar{K}^\sigma)$ is infinite.*

Proof. By Proposition 4 there are three hyperelliptic curves (*) X_1, X_2, X_3 defined over K of genus 1, 3, and 3, respectively forming a biquadratic extension of $\mathbb{P}^1$ such that each admits a nonconstant K -morphism ϕ_i onto E .

We inductively construct quadratic extensions K_n of K and points $P_n \in E(K_n)$ as follows: for each n , we apply Lemma 6 to find $x_n \in K$ such that

$$K_1 K_2 \cdots K_{n-1}(\sqrt{af(x_n)g(x_n)}, \sqrt{bf(x_n)h(x_n)})$$

is a biquadratic extension of $K_1 K_2 \cdots K_{n-1}$. Then define L_n to be the biquadratic extension of K generated by the two square roots $\sqrt{af(x_n)g(x_n)}, \sqrt{bf(x_n)h(x_n)}$ above. Note that L_n contains $\sqrt{abg(x_n)h(x_n)}$ as well.

Then any element $\sigma \in G_K$ induces an action on L_n which must fix at least one of the three quadratic K -subfields of L_n . We choose K_n to be one of these fixed quadratic subfields of L_n over K under σ . Then one of the three curves X_i has a rational point Q_n over $K_n \subset (K_{ab})^\sigma \subset \overline{K}^\sigma$ with x -coordinate x_n , hence for the curve X_i containing Q_n , the image of Q_n under the corresponding K -morphism ϕ_i from X_i onto E is a rational point P_n of $E(K_n)$ fixed under σ .

By [5, Lemma], we may assume P_n is not a torsion point. Then by the linear disjointness of the quadratic extensions K_n over K , we can show that the points P_n are linearly independent in $E(K_{ab}^\sigma) \otimes \mathbb{Q}$ as in [4, Theorem 5]. Hence, they generate a submodule of $E(\prod_{n \geq 1} K_n) \otimes \mathbb{Q}$ of infinite rank. Since K_n are abelian extensions of

K and fixed under σ , we conclude that $E((K_{ab})^\sigma)$ has infinite rank, so $E(\overline{K}^\sigma)$ has infinite rank as well. $\square$

Remark 8. For a number field K and an elliptic curve E/K , if we let L be a finite extension of K over which all 2-torsion points of E/K are defined, then Theorem 7 implies that for every σ in the open subset $\text{Gal}(\overline{K}/L)$ of G_K , $E(\overline{K}^\sigma)$ has infinite rank, which implies [4, Theorem 5]. Moreover, since $[L : K] \leq 6$, Theorem 7 gives a bound of 6 on the index of the open subset.

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